Finite generation of the canonical ring

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- Locally, in coords $z_1, z_2, \ldots, z_n$, a global n-form ω is of the form $f(z)dz_1 \wedge dz_2 \wedge dz_3 \ldots dz_n$, where $f(z)$ is a holomorphic function of the coordinates.

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- If we choose another coordinate system, then $f(z)$ changes according to the Jacobian rule. A pluricanonical n-form of weight m is something that transforms to the mth power of the Jacobian.

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- The space of all holomorphic n-forms is denoted $H^0(X, \mathcal{O}_X(K_X))$.
- The space of all holomorphic pluricanonical n-forms of weight m is denoted $H^0(X, \mathcal{O}_X(mK_X))$.

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I would like to spend the rest of the talk giving some idea of how one might prove this result and perhaps more importantly why one might be interested in the canonical ring.

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- There are then no global holomorphic 1-forms on $\mathbb{P}^1$ (any meromorphic 1-form has at least two poles).
- Thus the canonical ring

$$R(\mathbb{P}^1, K_{\mathbb{P}^1}) = \bigoplus R_m = \mathbb{C},$$

is concentrated in degree zero, and it is surely finitely generated.

Elliptic Curve

- $X = E = \mathbb{C}/\Lambda$ is an elliptic curve (a complex torus), the quotient of $\mathbb{C}$ by a two dimensional lattice $\mathbb{Z}^2 \simeq \Lambda \subset \mathbb{C}$. Any such curve can be represented as a smooth plane cubic.

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$$R(E, K_E) = \bigoplus H^0(E, \mathcal{O}_E(mK_E)) = \bigoplus \mathbb{C},$$

and this is generated in degree one by ω.

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- In this example it is now not so obvious why the canonical ring is finitely generated.

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- The homogeneous coordinate ring $\mathbb{C}[X]$ of X is the quotient $\mathbb{C}[X_0, X_1, \ldots, X_n]/I$.
- Clearly $\mathbb{C}[X]$ is a graded ring, which is a finitely generated $\mathbb{C}$-algebra.

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- In fact $3K_C = f^*H$ for any hyperplane H in $\mathbb{P}^{5g-5}$.
- That is to get the canonical divisor of C, just intersect C with a hyperplane and divide by three.
- In fact, more often than not, there is an embedding of C into $\mathbb{P}^g$, such that $R = \mathbb{C}[C]$.

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- In the hardest case, we have to exhibit an embedding of C into projective space, and realise the canonical ring as a coordinate ring.
- Hopefully it is becoming clearer that to prove finite generation, one needs to understand the geometry of X.
- Further the geometry of X is closely related to the behaviour of the canonical divisor and the canonical ring.

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  - The Kodaira dimension is two.

Kodaira dimension one

- Suppose that $X = S = C \times E$, where E is an elliptic curve. As K_E is the trivial divisor,

$$R(S, K_S) = R(C, K_C),$$

which as we have seen is finitely generated.

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- It is too much to expect that S is always a product.
- In general, there is a morphism $f\colon S \longrightarrow C$ to a curve C, where the general fibre is an elliptic curve.
- Kodaira proved that, after suitably modifying S,

$$K_S = f^*(K_C + \Delta),$$

  where Δ is a divisor with positive rational coeffs.

Kodaira's formula

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  where the rounddown is taken componentwise.
- Finite generation then follows, as before.

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- This case is therefore much more subtle.
- In fact, based on Zariski's example, Fujita gave examples of smooth complex manifolds of dimension four whose canonical ring is not finitely generated.
- The key point is to use the fact that X is projective and to exhibit the canonical ring as a coordinate ring, under some embedding.

Nef divisors

- If we can find a map $f\colon X \longrightarrow \mathbb{P}^k$ such that $D = f^*H$, where H is a hyperplane, then we say that D is semiample. In this case $R(X, D) = \mathbb{C}[X]$, up to truncation, and so $R(X, D)$ is finitely generated.

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- Thus if $D = f^*H$ is semiample then D is nef, that is $D \cdot C \geq 0$ (we get equality iff C is contracted by f).
- The converse is far from being true in general. E.g. divisors of degree zero on elliptic curves which are not torsion.

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In particular if K_X is big and nef then $R(X, K_X)$ is finitely generated.

Base point theorem II

Let X be a variety and let Δ be a divisor. We say that the pair (X, Δ) is kawamata log terminal, if $K_X + \Delta$ is $\mathbb{Q}$-Cartier, the coefficients of Δ lie between zero and one, and this continues to hold on any resolution.

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Then the base point free theorem applies to $K_X + \Delta$ if it is big and nef. In fact it applies even if Δ is big.

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Then the base point free theorem applies to $K_X + \Delta$ if it is big and nef. In fact it applies even if Δ is big.

Without going into technical details, it is very important that the coefficients of Δ are all less than one.

Kodaira's formula revisited

Applying the ideas of Iitaka's program, we have:

Theorem (Fujino, Mori). *Let* X *be a smooth projective variety.*
Then there is a kawamata log terminal pair (Y, Γ) *such that*

$$R(X, K_X) = R(Y, K_Y + \Gamma),$$

where $K_Y + \Gamma$ *is big.*

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Summary: we may assume that K_X (or more generally $K_X + \Delta$) is big and the whole problem turns on making K_X (or $K_X + \Delta$) nef.

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- Zen type question: what is the slope of the line connecting the origin to the origin?

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- If coordinates on $\mathbb{C}^2$ are (x, y) and on $\mathbb{P}^1$ are $[S : T]$, then the equation for Γ is $xT = yS$.

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- If coordinates on $\mathbb{C}^2$ are (x, y) and on $\mathbb{P}^1$ are $[S : T]$, then the equation for Γ is $xT = yS$.
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- Then $p^{-1}(x, y)$ is a unique point, unless $(x, y) = (0, 0)$, when we get a copy of $\mathbb{P}^1$.
- Thus the map p is an isomorphism outside the origin, but it replaces the origin by a whole copy E of $\mathbb{P}^1$. E is called the exceptional divisor.

General case

- Given any smooth surface S, we can always pick local coordinates (x, y) about any point, and we can then always blow up the point p, $f: T \longrightarrow S$, using these local coordinates.

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- For example, let $S = \mathbb{C}^2/\Lambda$, where $\mathbb{Z}^4 \simeq \Lambda \subset \mathbb{C}^2$ is a lattice. Then S is a compact algebraic group, an abelian variety.
- But if $f : T \longrightarrow S$ is the blow up of S at the identity, then T is no longer an algebraic group. Indeed S contains no copies of $\mathbb{P}^1$, so that T is not even homogeneous.

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- As $E \cdot E = -1$, $K_T \cdot E = -1$. E is called a -1-curve.
- Note that K_T is not nef and we are looking for curves E such that $K_T \cdot E < 0$.

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- Z is a surface. f is a blow up, so that f contracts a copy E of $\mathbb{P}^1$ to a point p. Replace S by Z and go back to (2).

Notes on the MMP

- As with any algorithm, it is important to know that the MMP terminates. In the case of surfaces this is clear, since every time we contract a curve the second betti number drops by one (we replace a copy of S^2 by a point).

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- Note that if $\pi\colon T \longrightarrow S$ is a blow up, then the pluricanonical forms of S and T may be naturally identified. Thus the canonical rings of S and T are naturally isomorphic.

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- $\dim Z = \dim X$. There are two cases.
  - The exceptional locus is a divisor. Replace X by Z and go back to (2).
  - The exceptional locus has codimension at least two. We cannot replace X by Z as Z is too singular.

New operation

Instead of contracting C, we try to replace X by another birational model X^+, $X \dashrightarrow X^+$, such that $\pi^+\colon X^+ \longrightarrow Y$ is K_{X^+}-ample.

$$\begin{tikzcd} X \arrow[rr, dashed, "\phi"] \arrow[dr, "\pi"'] & & X^+ \arrow[dl, "\pi^+"] \\ & Z. & \end{tikzcd}$$

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- Even supposing we can perform a flip, how do know that this process terminates?
- It is clear that we cannot keep contracting divisors, but why could there not be an infinite sequence of flips?

Brief History of the MMP

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- The steps of the MMP were developed in the eighties, by Kawamata, Kollár, Mori, Reid, Shokurov and many others.
- Mori proved the existence of flips in dimension three, and recently Shokurov proved existence of flips in dimension four.
- Alexeev, Hacon, Kawamata and Shokurov proved termination of flips in dimensions three and four.

Existence and termination

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Theorem. *[Birkar,Cascini,Hacon,-] Let X be a smooth projective variety. If $K_X \cdot C < 0$ for some covering family of curves, then X is birational to a Mori fibre space $\pi\colon Y \longrightarrow Z$.*

Existence of flips

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- The existence of the flip is local, so that we may assume that $Z = \operatorname{Spec} A$ is affine.
- In fact it suffices to prove that the canonical ring

$$R(X, K_X) = \bigoplus_{m\in\mathbb{N}} H^0(X, \mathcal{O}_X(mK_X)),$$

is a finitely generated A-algebra.

Reduction to pl-flips

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- Let R_S be the restricted algebra in $R(S, K_S + C)$, the direct sum of all the images.

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- If the natural restriction maps are surjective, then $R_S = R(S, K_S + C)$ and we would be done by induction.
- In fact it is a very natural question to ask which sections can one lift. Fortunately Siu, using ideas from PDE's (multiplier ideal sheaves) and then Kawamata gave some important partial answers to this question.

Limiting algebras

- In fact we show that in every degree, one can find a divisor Θ_m on some fixed higher model of S, such that

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