# Classical Enumerative Geometry and Quantum Cohomology

James M$^{c}$Kernan

UCSB

# Amazing Equation

It is the purpose of this talk to convince the listener that the following formula is truly amazing. end

$$N_d(a,b,c) = \sum_{\substack{d_1+d_2=d\\ a_1+a_2=a-1\\ b_1+b_2=b\\ c_1+c_2=c}} N_{d_1}(a_1,b_1,c_1)N_{d_2}(a_2,b_2,c_2)\left[d_1^2d_2^2\binom{a-3}{a_1-1} - d_1^3d_2\binom{a-3}{a_1}\right]\binom{b}{b_1}\binom{c}{c_1}$$

$$+\,2\cdot \sum_{\substack{d_1+d_2=d\\ a_1+a_2=a\\ b_1+b_2=b-1\\ c_1+c_2=c}} N_{d_1}(a_1,b_1,c_1)N_{d_2}(a_2,b_2,c_2)\left[d_1^2d_2\binom{a-3}{a_1-1} - d_1^3\binom{a-3}{a_1}\right]\binom{b}{b_1\; b_2\; 1}\binom{c}{c_1}$$

$$+\,4\cdot \sum_{\substack{d_1+d_2=d\\ a_1+a_2=a+1\\ b_1+b_2=b-2\\ c_1+c_2=c}} N_{d_1}(a_1,b_1,c_1)N_{d_2}(a_2,b_2,c_2)\left[d_1d_2\binom{a-3}{a_1-2} - d_1^2\binom{a-3}{a_1-1}\right]\binom{b}{b_1\; b_2\; 2}\binom{c}{c_1}$$

$$+\,2\cdot \sum_{\substack{d_1+d_2=d\\ a_1+a_2=a+1\\ b_1+b_2=b\\ c_1+c_2=c-1}} N_{d_1}(a_1,b_1,c_1)N_{d_2}(a_2,b_2,c_2)\left[d_1d_2\binom{a-3}{a_1-2} - d_1^2\binom{a-3}{a_1-1}\right]\binom{b}{b_1}\binom{c}{c_1\; c_2\; 1}.$$

# Projective Space

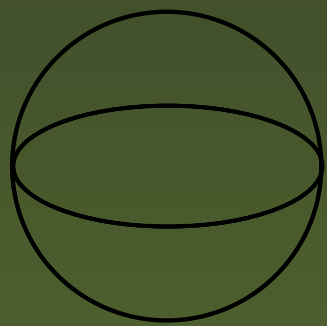
- $\mathbb{P}^n$ is the set of lines in $\mathbb{C}^{n+1}$.

# Projective Space

- $\mathbb{P}^n$ is the set of lines in $\mathbb{C}^{n+1}$.
- Suppose $n = 1$. Pick $p = (x, y) \in \mathbb{C}^2$. The line through $p$ is represented by its slope, that is the ratio $z = y/x$.

# Projective Space

- $\mathbb{P}^n$ is the set of lines in $\mathbb{C}^{n+1}$.
- Suppose $n = 1$. Pick $p = (x, y) \in \mathbb{C}^2$. The line through $p$ is represented by its slope, that is the ratio $z = y/x$.
- We get the classical Riemann sphere, $\mathbb{C}$ compactified by adding a point at infinity.



$$z \in \mathbb{C} \cup \{\infty\}$$

# Automorphisms of $\mathbb{P}^1$

- $\mathrm{Aut}(\mathbb{P}^1) = \frac{az+b}{cz+d}$, the group of Möbius transformations.

# Automorphisms of $\mathbb{P}^1$

- $\mathrm{Aut}(\mathbb{P}^1) = \frac{az+b}{cz+d}$, the group of Möbius transformations.
- Suppose we are given four points $p$, $q$, $r$ and $s$. Then there is a unique Möbius transformation, sending $p \longrightarrow 0$, $q \longrightarrow 1$, $r \longrightarrow \infty$.

# Automorphisms of $\mathbb{P}^1$

- $\mathrm{Aut}(\mathbb{P}^1) = \frac{az+b}{cz+d}$, the group of Möbius transformations.
- Suppose we are given four points $p$, $q$, $r$ and $s$. Then there is a unique Möbius transformation, sending $p \longrightarrow 0$, $q \longrightarrow 1$, $r \longrightarrow \infty$.
- The image $s \longrightarrow \lambda \in \mathbb{C} - \{0, 1\}$ is called the cross-ratio.

# Automorphisms of $\mathbb{P}^1$

- $\mathrm{Aut}(\mathbb{P}^1) = \frac{az+b}{cz+d}$, the group of Möbius transformations.
- Suppose we are given four points $p$, $q$, $r$ and $s$. Then there is a unique Möbius transformation, sending $p \longrightarrow 0$, $q \longrightarrow 1$, $r \longrightarrow \infty$.
- The image $s \longrightarrow \lambda \in \mathbb{C} - \{0, 1\}$ is called the cross-ratio.
- link. In fact

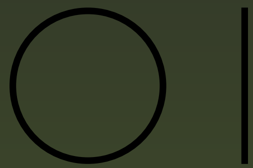
$$\lambda = \frac{(r-q)(p-s)}{(p-q)(r-s)}.$$

# Some Consequences

Our definition of $\mathbb{P}^n$ has some interesting consequences.

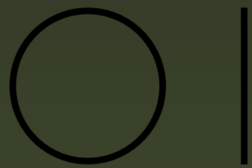
# Some Consequences

Our definition of $\mathbb{P}^n$ has some interesting consequences.
In fact these two curves



# Some Consequences

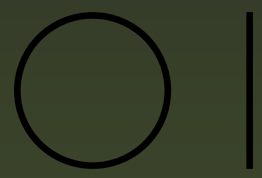
Our definition of $\mathbb{P}^n$ has some interesting consequences. In fact these two curves



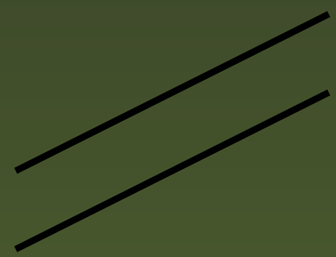
intersect at two points

# Some Consequences

Our definition of $\mathbb{P}^n$ has some interesting consequences.
In fact these two curves

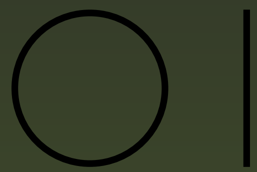


intersect at two points
and these two lines

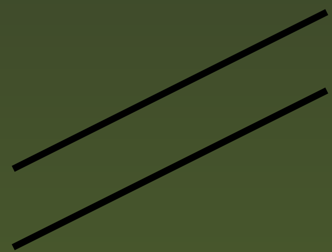


# Some Consequences

Our definition of $\mathbb{P}^n$ has some interesting consequences.
In fact these two curves



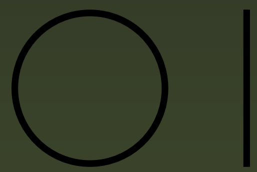
intersect at two points
and these two lines



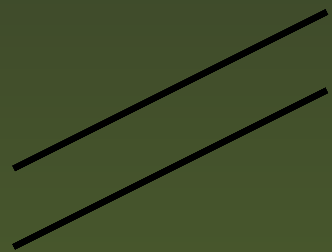
meet at a point.

# Some Consequences

Our definition of $\mathbb{P}^n$ has some interesting consequences.
In fact these two curves



intersect at two points
and these two lines



meet at a point.

Principle of Continuity The number of intersection points is an invariant of a continuous family of curves.

# Bézout's Theorem

**Theorem.** *(Bezout's Theorem) Given two plane curves* $C$ *and* $D$ *defined by polynomials* $F$ *and* $G$ *of degree* $d$ *and* $e$*, then*

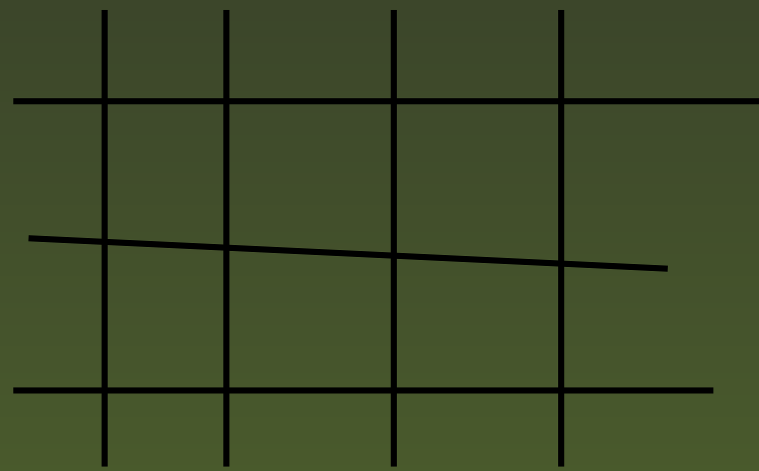
$$\#C \cap D = de.$$

# Bézout's Theorem

**Theorem.** *(Bezout's Theorem) Given two plane curves $C$ and $D$ defined by polynomials $F$ and $G$ of degree $d$ and $e$, then*

$$\#C \cap D = de.$$

Indeed the result is obviously true when our two curves are the union of $d$ and $e$ lines

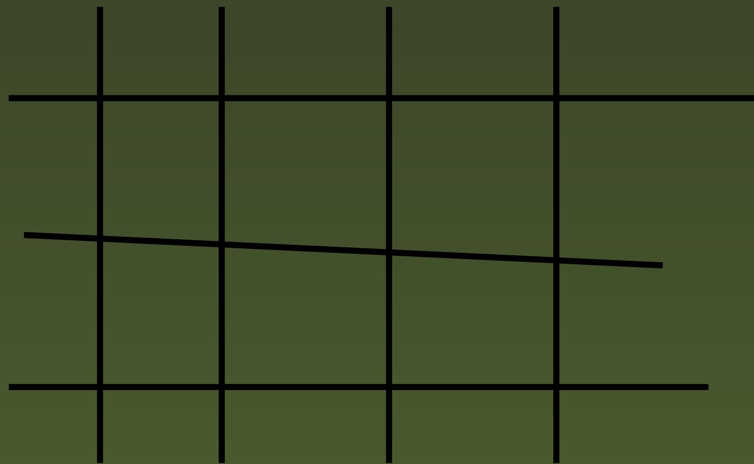


# Bézout's Theorem

**Theorem.** *(Bezout's Theorem) Given two plane curves $C$ and $D$ defined by polynomials $F$ and $G$ of degree $d$ and $e$, then*

$$\#C \cap D = de.$$

Indeed the result is obviously true when our two curves are the union of $d$ and $e$ lines



*Proof.* Let $F_\infty$ and $G_\infty$ be the product of linear forms. Then $F + tF_\infty$ and $G + tG_\infty$ define continuous families and when $t = \infty$, the answer is obviously $de$. $\square$

# The degree

- The degree $d$ of a variety $X^k \subset \mathbb{P}^n$ is the number of points $X \cap \Lambda$, where $\Lambda$ is a general linear subspace of dimension $n - k$.

# The degree

- The degree $d$ of a variety $X^k \subset \mathbb{P}^n$ is the number of points $X \cap \Lambda$, where $\Lambda$ is a general linear subspace of dimension $n-k$.
- Bézout's Theorem: If we have $n$ hypersurfaces $X_1, X_2, \dots, X_n$ of degrees $d_1, d_2, \dots, d_n$, then the number of common intersection points is

$$d_1 d_2 \dots d_n.$$

# The degree

- The degree $d$ of a variety $X^k \subset \mathbb{P}^n$ is the number of points $X \cap \Lambda$, where $\Lambda$ is a general linear subspace of dimension $n-k$.
- Bézout's Theorem: If we have $n$ hypersurfaces $X_1, X_2, \ldots, X_n$ of degrees $d_1, d_2, \ldots, d_n$, then the number of common intersection points is

$$d_1 d_2 \ldots d_n.$$

- Indeed, the same proof applies.

# Conics in $\mathbb{P}^2$

- A conic is given as the zero locus of

$$aX^2 + bY^2 + cZ^2 + dXY + eYZ + fXZ,$$

where $[a : b : c : d : e : f] \in \mathbb{P}^5$.

# Conics in $\mathbb{P}^2$

- A conic is given as the zero locus of

$$aX^2 + bY^2 + cZ^2 + dXY + eYZ + fXZ,$$

where $[a : b : c : d : e : f] \in \mathbb{P}^5$.

- How many conics pass through five points, $p_1$, $p_2$, $p_3$, $p_4$ and $p_5$?

# Conics in $\mathbb{P}^2$

- A conic is given as the zero locus of

$$aX^2 + bY^2 + cZ^2 + dXY + eYZ + fXZ,$$

where $[a : b : c : d : e : f] \in \mathbb{P}^5$.

- How many conics pass through five points, $p_1$, $p_2$, $p_3$, $p_4$ and $p_5$?
- The condition that a conic contains a point $p$ is a linear condition on the coefficients.

# Correspondence

- So we have a correspondence between

$$\{\, C \mid C \text{ contains } p_i \,\} \qquad \text{and} \qquad H_i \subset \mathbb{P}^5,$$

where $H_i$ is a hyperplane.

# Correspondence

- So we have a correspondence between

$$\{\, C \mid C \text{ contains } p_i \,\} \qquad \text{and} \qquad H_i \subset \mathbb{P}^5,$$

  where $H_i$ is a hyperplane.
- Thus the set of conics passing through five points, corresponds to the intersection of five hyperplanes.

# Correspondence

- So we have a correspondence between

$$\{\, C \mid C \text{ contains } p_i \,\} \qquad \text{and} \qquad H_i \subset \mathbb{P}^5,$$

where $H_i$ is a hyperplane.

- Thus the set of conics passing through five points, corresponds to the intersection of five hyperplanes.
- So the answer is one.

# Lines in $\mathbb{P}^3$

How many lines in $\mathbb{P}^3$ meet four given lines?

# Lines in $\mathbb{P}^3$

How many lines in $\mathbb{P}^3$ meet four given lines?
Schubert Calculus: Let

- $g_l$ represent the condition to meet a line,
- $g_P$ be the condition to meet a point, and
- $g_\pi$ be the condition to meet a plane.

# Lines in $\mathbb{P}^3$

How many lines in $\mathbb{P}^3$ meet four given lines?
Schubert Calculus: Let

- $g_l$ represent the condition to meet a line,
- $g_P$ be the condition to meet a point, and
- $g_\pi$ be the condition to meet a plane.

I claim

$$g_l^2 = g_P + g_\pi .$$

# Lines in $\mathbb{P}^3$

How many lines in $\mathbb{P}^3$ meet four given lines?
Schubert Calculus: Let

- $g_l$ represent the condition to meet a line,
- $g_P$ be the condition to meet a point, and
- $g_\pi$ be the condition to meet a plane.

I claim

$$g_l^2 = g_P + g_\pi.$$

Indeed, fix two lines $l$ and $m$, degenerate them until they are concurrent, and use the principle of continuity.

# A little Algebra

So

$$\begin{aligned} g_l^4 &= (g_P + g_\pi)^2 \\ &= g_P^2 + 2g_P g_\pi + g_\pi^2 \\ &= 1 + 2 \cdot 0 + 1 = 2. \end{aligned}$$

# A little Algebra

So

$$
\begin{aligned}
g_l^4 &= (g_P + g_\pi)^2 \\
&= g_P^2 + 2g_P g_\pi + g_\pi^2 \\
&= 1 + 2 \cdot 0 + 1 = 2.
\end{aligned}
$$

In fact we are working in the cohomology ring.

# A little Algebra

So

$$g_l^4 = (g_P + g_\pi)^2 = g_P^2 + 2g_P g_\pi + g_\pi^2 = 1 + 2 \cdot 0 + 1 = 2.$$

In fact we are working in the cohomology ring.

$$H^*(\mathbb{P}^2) = \frac{\mathbb{Z}[x]}{\langle x^3 \rangle}$$

where $x$ is the class of a line, and $C \sim dx$ and $D \sim ex$, so that $C \cdot D = dex^2 = de$.

# A classical Problem

- How many conics tangent to five given conics?

# A classical Problem

- How many conics tangent to five given conics?
- Fact: The set of conics tangent to a given conic corresponds to a hypersurface of degree $6$, so by Bézout the answer ought to be $6^5$.

# A classical Problem

- How many conics tangent to five given conics?
- Fact: The set of conics tangent to a given conic corresponds to a hypersurface of degree $6$, so by Bézout the answer ought to be $6^5$.
- Wrong!

# A classical Problem

- How many conics tangent to five given conics?
- Fact: The set of conics tangent to a given conic corresponds to a hypersurface of degree $6$, so by Bézout the answer ought to be $6^5$.
- Wrong!
- Conics tangent to five lines? The set of conics tangent to one line corresponds to a hypersurface of degree two. Bézout predicts the answer is $2^5$. But the actual answer is $1$.

# Use duality

- $\hat{\mathbb{P}}^2$ is the dual space to $\mathbb{P}^2$. It consists of the set of lines in $\mathbb{P}^2$.

# Use duality

- $\hat{\mathbb{P}}^2$ is the dual space to $\mathbb{P}^2$. It consists of the set of lines in $\mathbb{P}^2$.
- Given a conic $C$ in $\mathbb{P}^2$, we associate a dual conic $\hat{C}$ in $\hat{\mathbb{P}}^2$, simply by sending a point of $p$ of $C$ to its tangent line, a point of $\hat{\mathbb{P}}^2$.

# Use duality

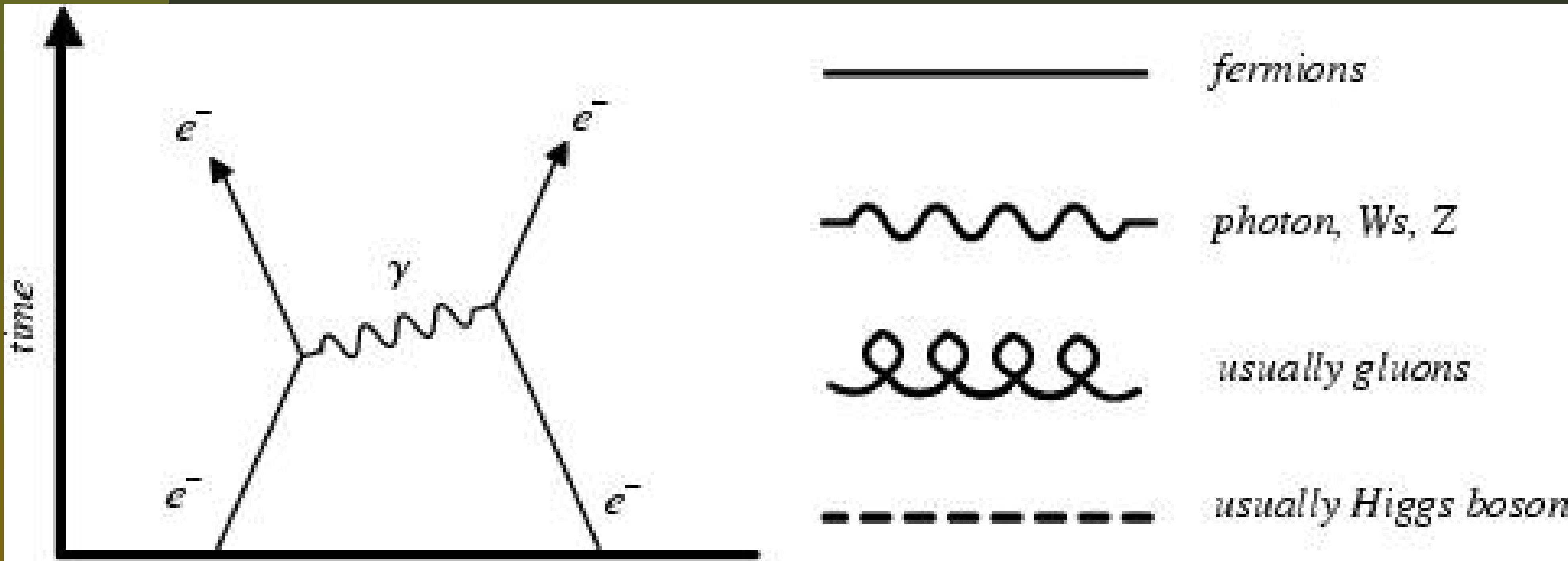
- $\hat{\mathbb{P}}^2$ is the dual space to $\mathbb{P}^2$. It consists of the set of lines in $\mathbb{P}^2$.
- Given a conic $C$ in $\mathbb{P}^2$, we associate a dual conic $\hat{C}$ in $\hat{\mathbb{P}}^2$, simply by sending a point of $p$ of $C$ to its tangent line, a point of $\hat{\mathbb{P}}^2$.
- So the number of conics tangent to five given lines, is equal, by duality, to the number of conics through five given points which, as we have seen, is one.

# Use duality

- $\hat{\mathbb{P}}^2$ is the dual space to $\mathbb{P}^2$. It consists of the set of lines in $\mathbb{P}^2$.
- Given a conic $C$ in $\mathbb{P}^2$, we associate a dual conic $\hat{C}$ in $\hat{\mathbb{P}}^2$, simply by sending a point of $p$ of $C$ to its tangent line, a point of $\hat{\mathbb{P}}^2$.
- So the number of conics tangent to five given lines, is equal, by duality, to the number of conics through five given points which, as we have seen, is one.
- (Hwk) What is wrong?

# Physics

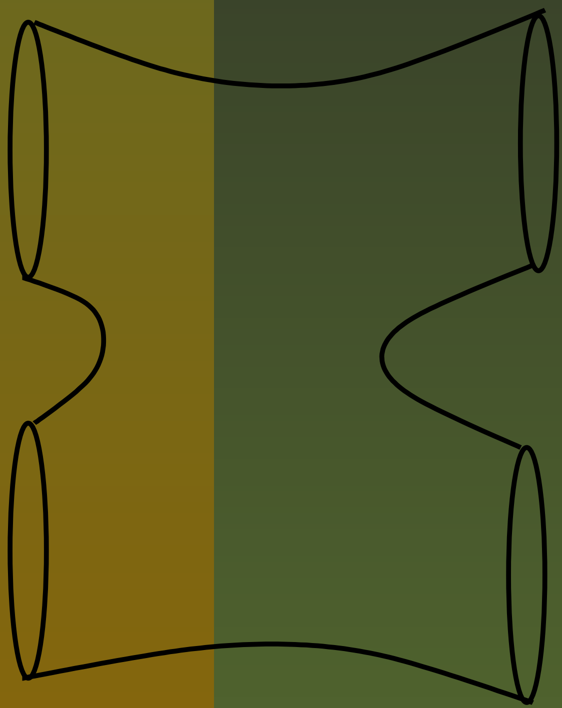
Here is a typical Feynmann diagram.




Feynmann diagrams are used to encode the complicated interactions which particles undergo.

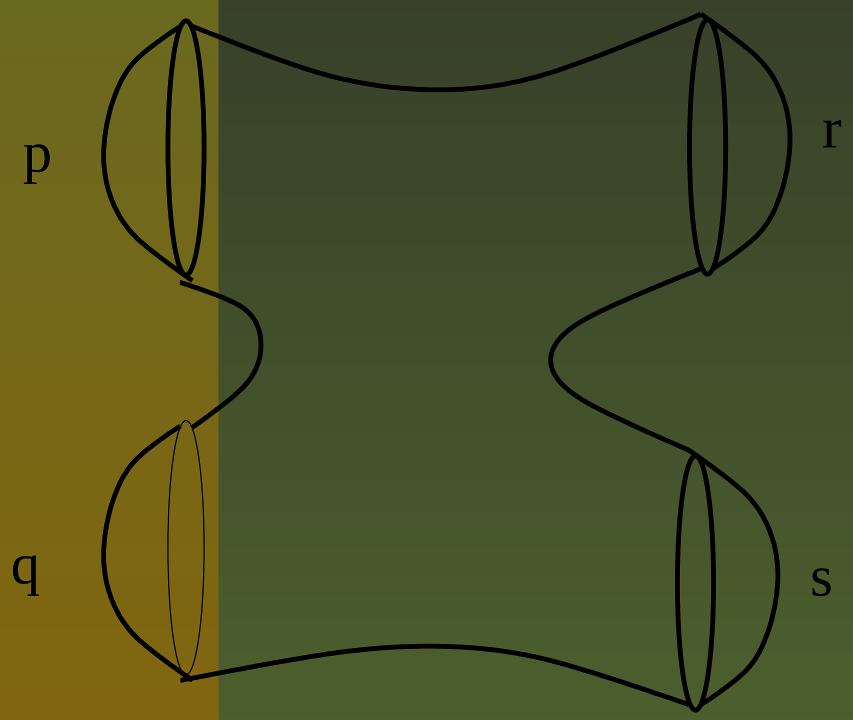
# String Theory

One of the ideas of string theory, is that a string is the basic object and not particles. Replacing a point by a string, means replacing a line by a tube and our Feynmann diagram becomes:



# Points at Infinity

Now take this picture and extend the tubes to infinity. Adding the points at infinity is topologically equivalent to adding caps.



# Back to the Riemann sphere

- Topologically, the resulting surface is a sphere.

# Back to the Riemann sphere

- Topologically, the resulting surface is a sphere.
- So now we have the Riemann sphere, that is a copy of $\mathbb{P}^1$, with four marked points.

# Back to the Riemann sphere

- Topologically, the resulting surface is a sphere.
- So now we have the Riemann sphere, that is a copy of $\mathbb{P}^1$, with four marked points.
- More generally, we will get a Riemann surface, together with a collection of marked points.

# Rational curves in $\mathbb{P}^2$

- Note that plane curves of degree $d$ correspond to polynomials of degree $d$, modulo scalars, which in turn corresponds to a $\mathbb{P}^N$, for some $N$.

# Rational curves in $\mathbb{P}^2$

- Note that plane curves of degree $d$ correspond to polynomials of degree $d$, modulo scalars, which in turn corresponds to a $\mathbb{P}^N$, for some $N$.
- For example, if $d = 1$ we get $N = 2$ (in fact $\hat{\mathbb{P}}^2$) and if $d = 2$, we get $N = 5$, $\mathbb{P}^5$.

# Rational curves in $\mathbb{P}^2$

- Note that plane curves of degree $d$ correspond to polynomials of degree $d$, modulo scalars, which in turn corresponds to a $\mathbb{P}^N$, for some $N$.
- For example, if $d = 1$ we get $N = 2$ (in fact $\hat{\mathbb{P}}^2$) and if $d = 2$, we get $N = 5$, $\mathbb{P}^5$.
- Some plane curves are rational, that is to say there is a map $\mathbb{P}^1 \longrightarrow \mathbb{P}^2$, $[S : T] \longrightarrow [F : G : H]$, where $F$, $G$ and $H$ are polynomials of degree $d$, in $S$ and $T$.

# Rational curves in $\mathbb{P}^2$

- Note that plane curves of degree $d$ correspond to polynomials of degree $d$, modulo scalars, which in turn corresponds to a $\mathbb{P}^N$, for some $N$.
- For example, if $d = 1$ we get $N = 2$ (in fact $\hat{\mathbb{P}}^2$) and if $d = 2$, we get $N = 5$, $\mathbb{P}^5$.
- Some plane curves are rational, that is to say there is a map $\mathbb{P}^1 \longrightarrow \mathbb{P}^2$, $[S : T] \longrightarrow [F : G : H]$, where $F$, $G$ and $H$ are polynomials of degree $d$, in $S$ and $T$.
- Let $X \subset \mathbb{P}^N$, be the locus of these rational curves. Basic question: what is the degree of $X \subset \mathbb{P}^N$?

# The Grade

- Call this degree $N_d$. Classically this number is known as the grade.

# The Grade

- Call this degree $N_d$. Classically this number is known as the grade.
- $N_1 = 1$. Indeed $X_1 = \mathbb{P}^2$, as every line is rational. Similarly every conic is rational, so $X_2 = \mathbb{P}^5$ and $N_2 = 1$.

# The Grade

- Call this degree $N_d$. Classically this number is known as the grade.
- $N_1 = 1$. Indeed $X_1 = \mathbb{P}^2$, as every line is rational. Similarly every conic is rational, so $X_2 = \mathbb{P}^5$ and $N_2 = 1$.
- In general to calculate the degree of $X_d$ we need to cut by hyperplanes. As the dimension of $X_d$ is $3d - 1$, we want to cut by $3d - 1$ hyperplanes.

# The Grade

- Call this degree $N_d$. Classically this number is known as the grade.
- $N_1 = 1$. Indeed $X_1 = \mathbb{P}^2$, as every line is rational. Similarly every conic is rational, so $X_2 = \mathbb{P}^5$ and $N_2 = 1$.
- In general to calculate the degree of $X_d$ we need to cut by hyperplanes. As the dimension of $X_d$ is $3d - 1$, we want to cut by $3d - 1$ hyperplanes.
- Now imposing the condition that a curve passes through a point is one linear condition. So we want to count the number of rational curves of degree $d$ that pass through $3d - 1$ points.

# An argument due to Kontsevich

- Fix $3d - 2$ points $p_1, p_2, \ldots, p_{3d-2}$ in $\mathbb{P}^2$. Then we get a 1-dimensional family of rational curves $C_t$ of degree $d$, which contain these points.

# An argument due to Kontsevich

- Fix $3d-2$ points $p_1, p_2, \ldots, p_{3d-2}$ in $\mathbb{P}^2$. Then we get a 1-dimensional family of rational curves $C_t$ of degree $d$, which contain these points.
- Pick two auxiliary lines $l_1$ and $l_2$ and set $p_{3d-1} = l_1 \cap l_2$.

# An argument due to Kontsevich

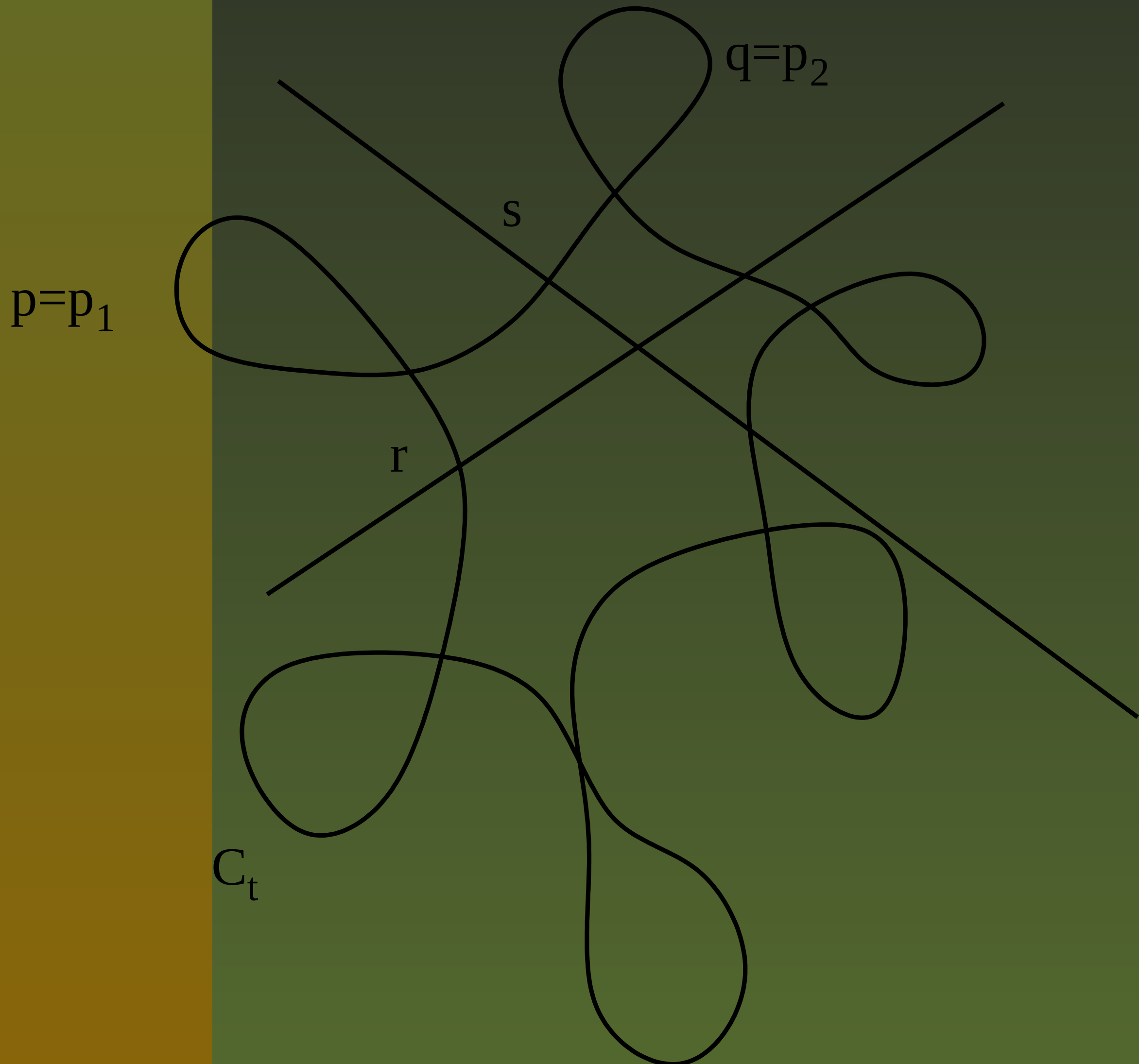
- Fix $3d - 2$ points $p_1, p_2, \ldots, p_{3d-2}$ in $\mathbb{P}^2$. Then we get a 1-dimensional family of rational curves $C_t$ of degree $d$, which contain these points.
- Pick two auxiliary lines $l_1$ and $l_2$ and set $p_{3d-1} = l_1 \cap l_2$.
- Pick four points of $C_t$, $p = p_1$, $q = p_2$, $r$ a point of $l_1$ and $s$ a point of $l_2$.

# An argument due to Kontsevich

- Fix $3d-2$ points $p_1, p_2, \ldots, p_{3d-2}$ in $\mathbb{P}^2$. Then we get a 1-dimensional family of rational curves $C_t$ of degree $d$, which contain these points.
- Pick two auxiliary lines $l_1$ and $l_2$ and set $p_{3d-1} = l_1 \cap l_2$.
- Pick four points of $C_t$, $p = p_1$, $q = p_2$, $r$ a point of $l_1$ and $s$ a point of $l_2$.
- Observe that $C_t$ passes through iff $r = s$.

# Picture

Here is a picture of what is going on back :



# Keeping track of the cross-ratio

- Note that there is a map $B \longrightarrow \mathbb{P}^1$ which assigns to a point $t \in B$, the cross-ratio of the four points $p$, $q$, $r$ and $s$.

# Keeping track of the cross-ratio

- Note that there is a map $B \longrightarrow \mathbb{P}^1$ which assigns to a point $t \in B$, the cross-ratio of the four points $p$, $q$, $r$ and $s$.
- Note that the cross-ratio is infinity if $r = s$.

# Keeping track of the cross-ratio

- Note that there is a map $B \longrightarrow \mathbb{P}^1$ which assigns to a point $t \in B$, the cross-ratio of the four points $p$, $q$, $r$ and $s$.
- Note that the cross-ratio is infinity if $r = s$.
- By the principle of continuity, the number of times the cross-ratio is zero, is equal to the number of times the cross-ratio is infinity.

# Keeping track of the cross-ratio

- Note that there is a map $B \longrightarrow \mathbb{P}^1$ which assigns to a point $t \in B$, the cross-ratio of the four points $p$, $q$, $r$ and $s$.
- Note that the cross-ratio is infinity if $r = s$.
- By the principle of continuity, the number of times the cross-ratio is zero, is equal to the number of times the cross-ratio is infinity.
- So, when is the cross-ratio zero, and when is it infinity? cross-ratio

# Keeping track of the cross-ratio

- Note that there is a map $B \longrightarrow \mathbb{P}^1$ which assigns to a point $t \in B$, the cross-ratio of the four points $p$, $q$, $r$ and $s$.
- Note that the cross-ratio is infinity if $r = s$.
- By the principle of continuity, the number of times the cross-ratio is zero, is equal to the number of times the cross-ratio is infinity.
- So, when is the cross-ratio zero, and when is it infinity? cross-ratio
- It is zero when $p = s$ or $r = q$ and it is infinity if $r = s$ or $p = q$.

# Picture of family $C_t$

# Bubbling off

- From the previous picture, it would seem that $p = r$ and $q = r$ can occur.

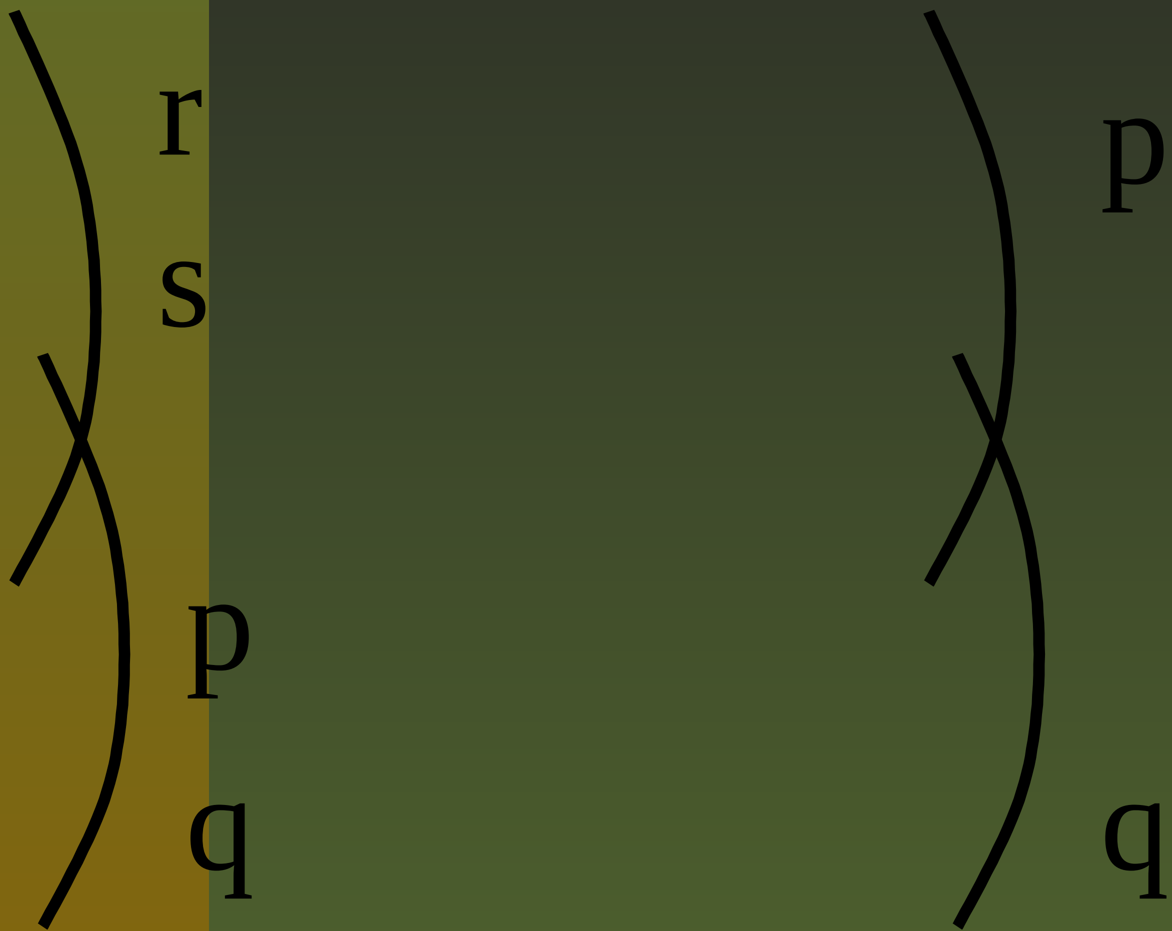
# Bubbling off

- From the previous picture, it would seem that $p = r$ and $q = r$ can occur.
- But looking at this picture, picture , in fact it would seem this cannot occur (and nor can $p = q$).

# Bubbling off

- From the previous picture, it would seem that $p = r$ and $q = r$ can occur.
- But looking at this picture, picture , in fact it would seem this cannot occur (and nor can $p = q$).
- In fact what is happening, is that a copy of $\mathbb{P}^1$ is bubbling off. $C_t$ is forced to break into two curves, one of degree $d_1$ and $d_2$, where $d = d_1 + d_2$.

# Singular fibre



# Choices

- $d_1$ and $d_2$.

# Choices

- $d_1$ and $d_2$.
- $C_1$ passes through $3d_1 - 1$ points. Choose these points $3d_1 - 1$ points, from amongst $3d - 1 - 3$ points.

# Choices

- $d_1$ and $d_2$.
- $C_1$ passes through $3d_1 - 1$ points. Choose these points $3d_1 - 1$ points, from amongst $3d - 1 - 3$ points.
- Choose which nodes to smooth.

# Choices

- $d_1$ and $d_2$.
- $C_1$ passes through $3d_1 - 1$ points. Choose these points $3d_1 - 1$ points, from amongst $3d - 1 - 3$ points.
- Choose which nodes to smooth.
- Choose two curves $C_1$ and $C_2$ through given points.

# Choices

- $d_1$ and $d_2$.
- $C_1$ passes through $3d_1 - 1$ points. Choose these points $3d_1 - 1$ points, from amongst $3d - 1 - 3$ points.
- Choose which nodes to smooth.
- Choose two curves $C_1$ and $C_2$ through given points.
- Choose the points $r$ and $s$.

# Recursive Formula

Putting all this together we get

$$N_d = \sum_{d_1+d_2=d} N_{d_1} N_{d_2} \left[ d_1^2 d_2^2 \binom{3d-4}{3d_1-1} - d_1^3 d_2 \binom{3d-4}{3d_1-2} \right],$$

where $N_1 = 1$ and $N_2 = 1$.
In fact $N_3 = 12, \ldots$.

# Quantum Cohomology

- Set up a ring to count these objects,

$$QH^*(\mathbb{P}^2) = \frac{\mathbb{Z}[x]}{\langle x^3 = q \rangle}.$$

This is the (small) quantum cohomology ring.

# Quantum Cohomology

- Set up a ring to count these objects,

$$QH^*(\mathbb{P}^2) = \frac{\mathbb{Z}[x]}{\langle x^3 = q \rangle}.$$

  This is the (small) quantum cohomology ring.

- We have a new product, a deformation of the old product. Instead of counting number of intersection points, it counts the number of rational curves meeting given cycles (ie Gromov-Witten invariants).

# Quantum Cohomology

- Set up a ring to count these objects,

$$QH^*(\mathbb{P}^2) = \frac{\mathbb{Z}[x]}{\langle x^3 = q \rangle}.$$

  This is the (small) quantum cohomology ring.
- We have a new product, a deformation of the old product. Instead of counting number of intersection points, it counts the number of rational curves meeting given cycles (ie Gromov-Witten invariants).
- Recursion formula corresponds to associativity of quantum product.

# Tangency condition

- For example, set $N_d(a, b, c)$ to be the number of curves of degree $d$ through $a$ general points, tangent to $b$ lines, and tangent to $c$ lines, at specified general points.

# Tangency condition

- For example, set $N_d(a, b, c)$ to be the number of curves of degree $d$ through $a$ general points, tangent to $b$ lines, and tangent to $c$ lines, at specified general points.
- Formula .

# Tangency condition

- For example, set $N_d(a, b, c)$ to be the number of curves of degree $d$ through $a$ general points, tangent to $b$ lines, and tangent to $c$ lines, at specified general points.
- Formula .
- In particular, we derive $N_2(0, 5, 0) = 3264$, the correct answer to the question, how many conics tangent to five given lines?

# Further Work

**Theorem.** *(Beauville, Yau-Zaslow) Let $S$ be a general K3 surface in $\mathbb{P}^g$. Then the number $n(g)$ of rational curves on $S$ which are hyperplane sections is equal to*

$$\sum_{g=1}^{\infty} n(g) q^g = \frac{q}{q \prod_{n=1}^{\infty} (1-q^n)^{24}}.$$

*provided every such curve is nodal.*

# Further Work

**Theorem.** *(Beauville, Yau-Zaslow) Let $S$ be a general K3 surface in $\mathbb{P}^g$. Then the number $n(g)$ of rational curves on $S$ which are hyperplane sections is equal to*

$$\sum_{g=1}^{\infty} n(g)q^g = \frac{q}{q\prod_{n=1}^{\infty}(1-q^n)^{24}}.$$

*provided every such curve is nodal.*

**Theorem.** *(Xi Chen) Let $S$ be a general K3 surface in $\mathbb{P}^n$, such that $\mathcal{O}_S(1)$ is not a multiple of another line bundle.*

*Then every rational curve which is a hyperplane section is nodal.*