Symmetries of Varieties

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- If $d = 3$, then C is an elliptic curve. C acts on itself by translation, and $\mathrm{Aut}(C)$ is a finite extension of C.
- The dimension of $\mathrm{Aut}(C)$ is one.
- More generally, $\mathrm{Aut}(\mathbb{P}^n) = \mathrm{PGL}_{n+1}(\mathbb{C})$, of dimension $(n+1)^2 - 1$ and $\mathrm{Aut}(A)$ is a finite extension of itself, so $\dim \mathrm{Aut}(A) = \dim A$.

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- The Wiman sextic C, given by

$$10x^3y^3 + 9(x^5 + y^5)z - 45x^2y^2z^2 - 135xyz^4 + 27z^6.$$

$$\mathrm{Aut}(C) = A_6.\ |\mathrm{Aut}(C)| = 360.$$

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where f and h are scalars, and g is a polynomial of degree n. (Note that $n = 0$ is a special case).

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- So the dimension is $3 + 1 + 1 + n + 1 - 1 = n + 5$.
- Check: $\mathbb{F}_1 = \mathrm{Bl}_p \mathbb{P}^2$, $\dim \operatorname{Aut}(\mathbb{F}_1) = 8 - 2 = 6$.

Birational automorphisms

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- Noether's Theorem: $\mathrm{Bir}(\mathbb{P}^2)$ is generated by $\mathrm{PGL}_2(\mathbb{C})$ and σ.

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- One can use this factorisation to prove Noether's theorem.

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  blows down the cone over $C=(f=0)$.
- So if R is any set of generators of $\mathrm{Bir}(\mathbb{P}^3)$, then

$$\bigcup_{g\in\mathbb{N}} \mathcal{M}_g \subset R,$$

  so that any generating set is infinite dimensional.

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- If Z is a surface, then π is a conic bundle.

Quartic Threefolds

Theorem: If $X \subset \mathbb{P}^4$ is a smooth quartic threefold, then $\mathrm{Bir}(X) = \mathrm{Aut}(X) = \mathrm{Aut}(X, \mathbb{P}^4)$ is finite.

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Question: If X is a general quartic, is X unirational?

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- The case when $q = 0$ is probably the most interesting (e.g X is rationally connected or X is birational to Calabi-Yau).
- Even the case of rational surfaces is unresolved.

Finiteness of minimal models

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- The case when X is of general type is in BCHM.

Curves of genus $g \geq 2$

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Riemann-Hurwitz:

$$K_C = \pi^*(K_B + \Delta),$$

where

$$\Delta = \sum_{b \in B} \frac{r_b - 1}{r_b} b.$$

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Case by case analysis.

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- Note that this question is entirely topological. Can we find a topological cover ramified over 0, 1 and ∞ to order 2, 3 and 7?
- Can we find an appropriate representation on the free group on two letters?
- Question: Is the Wiman sextic the curve with the maximum number of automorphisms, amongst all smooth curves of genus 10?

Higher dimensions

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- If $X = C$ is a smooth curve, then C is of general type if and only if $g \geq 2$ and $\operatorname{vol}(C, K_C) = 2g - 2$.

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- $n = 2$, $c = (42)^2$. Take $S = C \times C$, where C achieves maximum. $K_S = p^* K_C + q^* K_C$ is ample, $\mathrm{vol}(S, K_S) = 2(2g-2)^2$ and $|\operatorname{Aut}(S)| = (42)^2 2(2g-2)^2$.

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- Stupid Question: Is $c = (42)^n$?

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- $|\operatorname{Aut}(X)| = (n+3)^{n+2}(n+2)!$ and $\operatorname{vol}(X, K_X) = (n+3)$, ratio is $(n+3)^{n+1}(n+2)!$ which beats $(42)^n$ ($n = 5$ will do).

Review of finite simple groups

- Let $V = \mathbb{F}_{q^2}^m$. There is a sesquilinear pairing

$$V \times V \longrightarrow \mathbb{F}_{q^2} \qquad \text{given by} \qquad \sum a_i \bar{b}_i,$$

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- Question Are there constants c, d such that

$$|\mathrm{Bir}(X)| \leq c\,\mathrm{vol}(X, K_X)^d.$$

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- Same strategy as before. Change models so that $G = \mathrm{Aut}(Y) = \mathrm{Bir}(Y)$. G is finite. If $\pi\colon Y \longrightarrow X = Y/G$ is the quotient map, then $K_Y = \pi^*(K_X + \Delta)$.

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$\mathrm{vol}(X, r(K_X + \Delta)) \geq \mathrm{vol}(W, H) = 1$, so that

$\mathrm{vol}(X, K_X + \Delta) \geq 1/r^n$.