# ACC FOR LOG CANONICAL THRESHOLDS

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*To Vyacheslav Shokurov on the occasion of his sixtieth birthday*

ABSTRACT. We show that log canonical thresholds satisfy the ACC.

## CONTENTS

## 1. Introduction

We work over an algebraically closed field of characteristic zero. ACC stands for the ascending chain condition whilst DCC stands for the descending chain condition.

Suppose that $(X, \Delta)$ is a log canonical pair and $M \geq 0$ is $\mathbb{R}$-Cartier. The **log canonical threshold** of $M$ with respect to $(X, \Delta)$ is

$$\operatorname{lct}(X, \Delta; M) = \sup\{\, t \in \mathbb{R} \,|\, (X, \Delta + tM) \text{ is log canonical} \,\}.$$

Let $\mathfrak{T} = \mathfrak{T}_n(I)$ denote the set of log canonical pairs $(X, \Delta)$, where $X$ is a variety of dimension $n$ and the coefficients of $\Delta$ belong to a set $I \subset [0, 1]$. Set

$$\operatorname{LCT}_n(I, J) = \{\, \operatorname{lct}(X, \Delta; M) \,|\, (X, \Delta) \in \mathfrak{T}_n(I) \,\},$$

where the coefficients of $M$ belong to a subset $J$ of the positive real numbers.

**Theorem 1.1** (ACC for the log canonical threshold). *Fix a positive integer $n$, $I \subset [0, 1]$ and a subset $J$ of the positive real numbers.*

*If $I$ and $J$ satisfy the DCC then $\operatorname{LCT}_n(I, J)$ satisfies the ACC.*

(1.1) was conjectured by Shokurov [33], see also [22] and [24]. When the dimension is three, [22] proves that 1 is not an accumulation point from below and (1.1) follows from the results of [3]. More recently (1.1) was proved for complete intersections [10] and even when $X$ belongs to a bounded family, [11].

The log canonical threshold is an interesting invariant of the pair $(X, \Delta)$ and the divisor $M$ which is a measure of the complexity of the singularities of the triple $(X, \Delta; M)$. It has made many appearances in many different forms, especially in the case of hypersurfaces, see [24], [25] and [34]. The ACC for the log canonical threshold plays a role in inductive approaches to higher dimensional geometry. For example, after [6], we have the following application of (1.1):

**Corollary 1.2.** *Assume termination of flips for $\mathbb{Q}$-factorial kawamata log terminal pairs in dimension $n - 1$.*

*Let $(X, \Delta)$ be a kawamata log terminal pair where $X$ is a $\mathbb{Q}$-factorial projective variety of dimension $n$. If $K_X + \Delta$ is numerically equivalent to a divisor $D \geq 0$ then any sequence of $(K_X + \Delta)$-flips terminates.*

(1.1) is a consequence of the following theorem, which was conjectured by Alexeev [3] and Kollár [22]:

**Theorem 1.3.** *Fix a positive integer $n$ and a set $I \subset [0,1]$ which satisfies the DCC. Let $\mathfrak{D}$ be the set of log canonical pairs $(X, \Delta)$ such that the dimension of $X$ is $n$ and the coefficients of $\Delta$ belong to $I$.*

*Then there is a constant $\delta > 0$ and a positive integer $m$ with the following properties:*

*(1) the set*
$$\{\, \mathrm{vol}(X, K_X + \Delta) \,|\, (X, \Delta) \in \mathfrak{D} \,\},$$
*also satisfies the DCC.*

*Further, if $(X, \Delta) \in \mathfrak{D}$ and $K_X + \Delta$ is big, then*

*(2) $\mathrm{vol}(X, K_X + \Delta) \geq \delta$, and*
*(3) $\phi_{m(K_X + \Delta)}$ is birational.*

Note that, by convention, $\phi_{m(K_X + \Delta)} = \phi_{\lfloor m(K_X + \Delta) \rfloor}$. (1.3) was proved for surfaces in [3]. (1.3) is a generalisation of [15, 1.3], which deals with the case that $(X, \Delta)$ is the quotient of a smooth projective variety $Y$ of general type by its automorphism group.

One of the original motivations for (1.3) is to prove the boundedness of the moduli functor for canonically polarised varieties, see [26]. We plan to pursue this application of (1.3) in a forthcoming paper.

To state more results it is convenient to give a simple reformulation of (1.1):

**Theorem 1.4.** *Fix a positive integer $n$ and a set $I \subset [0,1]$, which satisfies the DCC.*

*Then there is a finite subset $I_0 \subset I$ with the following properties:*
*If $(X, \Delta)$ is a log pair such that*

*(1) $X$ is a variety of dimension $n$,*
*(2) $(X, \Delta)$ is log canonical,*
*(3) the coefficients of $\Delta$ belong to $I$, and*
*(4) there is a non kawamata log terminal centre $Z \subset X$ which is contained in every component of $\Delta$,*

*then the coefficients of $\Delta$ belong to $I_0$.*

(1.4) follows, cf. [33], [27, §18], almost immediately from the existence of divisorial log terminal modifications and from:

**Theorem 1.5.** *Fix a positive integer $n$ and a set $I \subset [0,1]$, which satisfies the DCC.*

*Then there is a finite subset $I_0 \subset I$ with the following properties:*
*If $(X, \Delta)$ is a log pair such that*

*(1) $X$ is a projective variety of dimension $n$,*
*(2) $(X, \Delta)$ is log canonical,*
*(3) the coefficients of $\Delta$ belong to $I$, and*
*(4) $K_X + \Delta$ is numerically trivial,*

*then the coefficients of $\Delta$ belong to $I_0$.*

We use finiteness of log canonical models to prove a boundedness result for log pairs:

**Theorem 1.6.** *Fix a positive integer $n$ and two real numbers $\delta$ and $\epsilon > 0$.*

*Let $\mathfrak{D}$ be a set of log pairs $(X, \Delta)$ such that*

- *$X$ is a projective variety of dimension $n$,*
- *$K_X + \Delta$ is ample,*
- *the coefficients of $\Delta$ are at least $\delta$, and*
- *the total log discrepancy of $(X, \Delta)$ is greater than $\epsilon$.*

*If $\mathfrak{D}$ is log birationally bounded then $\mathfrak{D}$ is a bounded family.*

Log birationally bounded is defined in (3.5.1). We use (1.5) and (1.6) to prove some boundedness results about Fano varieties.

**Corollary 1.7.** *Fix a positive integer $n$, a real number $\epsilon > 0$ and a set $I \subset [0, 1]$ which satisfies the DCC.*

*Let $\mathfrak{D}$ be the set of all log pairs $(X, \Delta)$, where*

- *$X$ is a projective variety of dimension $n$,*
- *the coefficients of $\Delta$ belong to $I$,*
- *the total log discrepancy of $(X, \Delta)$ is greater than $\epsilon$,*
- *$K_X + \Delta$ is numerically trivial, and*
- *$-K_X$ is ample.*

*Then $\mathfrak{D}$ forms a bounded family.*

As a consequence we are able to prove a result on the boundedness of Fano varieties which was conjectured by Batyrev (cf. [9]):

**Corollary 1.8.** *Fix two positive integers $n$ and $r$.*

*Let $\mathfrak{D}$ be the set of all kawamata log terminal pairs $(X, \Delta)$, where $X$ is a projective variety of dimension $n$ and $-r(K_X + \Delta)$ is an ample Cartier divisor.*

*Then $\mathfrak{D}$ forms a bounded family.*

**Definition 1.9.** *Let $(X, \Delta)$ be a log canonical pair, where $X$ is projective of dimension $n$ and $-(K_X + \Delta)$ is ample. The **Fano index** of $(X, \Delta)$ is the largest real number $r$ such that we can write*

$$-(K_X + \Delta) \sim_{\mathbb{R}} rH,$$

*where $H$ is a Cartier divisor.*

*Fix a set $I \subset [0,1]$ and a positive integer $n$. Let $\mathfrak{D}$ be the set of log canonical pairs $(X, \Delta)$, where $X$ is projective of dimension $n$, $-(K_X + \Delta)$ is ample and the coefficients of $\Delta$ belong to $I$.*

*The set*

$$R = R_n(I) = \{ \, r \in \mathbb{R} \, | \, r \text{ is the Fano index of } (X, \Delta) \in \mathfrak{D} \, \},$$

*is called the **Fano spectrum** of $\mathfrak{D}$.*

**Corollary 1.10.** *Fix a set $I \subset [0,1]$ and a positive integer $n$.*

*If $I$ satisfies the DCC then the Fano spectrum satisfies the ACC.*

(1.10) was proved in dimension 2 in [1] and for $R \cap [n-2, \infty)$ in [2].

Now given any set which satisfies the ACC it is natural to try to identify the accumulation points. (1.1) implies that $\mathrm{LCT}_n(I) = \mathrm{LCT}_n(I, \mathbb{N})$ satisfies the ACC. Kollár, cf. [24], [32], [20], conjectured that the accumulation points in dimension $n$ are log canonical thresholds in dimension $n-1$:

**Theorem 1.11.** *If $1$ is the only accumulation point of $I \subset [0,1]$ and $I = I_+$ then the accumulation points of $\mathrm{LCT}_n(I)$ are $\mathrm{LCT}_{n-1}(I) - \{\, 1 \,\}$. In particular, if $I \subset \mathbb{Q}$ then the accumulation points of $\mathrm{LCT}_n(I)$ are rational numbers.*

See §3.4 for the definition of $I_+$. (1.11) was proved if $X$ is smooth in [20]. Note that in terms of inductive arguments it is quite useful to identify the accumulation points, especially to know that they are rational.

Finally, recall:

**Conjecture 1.12** (Borisov-Alexeev-Borisov)**.** *Fix a positive integer $n$ and a positive real number $\epsilon > 0$.*

*Let $\mathfrak{D}$ be the set of all projective varieties $X$ of dimension $n$ such that there is a divisor $\Delta$ where $(X, \Delta)$ has log discrepancy at least $\epsilon$ and $-(K_X + \Delta)$ is ample.*

*Then $\mathfrak{D}$ forms a bounded family.*

Note that (1.1), (1.4), (1.5), (1.2) and (1.11) are known to follow from (1.12), (cf. [32]). Instead we use birational boundedness of log pairs of general type cf. (1.3) to prove these results.

## 2. Description of the proof

**Theorem A** (ACC for the log canonical threshold)**.** *Fix a positive integer $n$ and a set $I \subset [0,1]$, which satisfies the DCC.*

*Then there is a finite subset $I_0 \subset I$ with the following property:*

*If $(X, \Delta)$ is a log pair such that*

5

*(1) $X$ is a variety of dimension $n$,*
*(2) $(X, \Delta)$ is log canonical,*
*(3) the coefficients of $\Delta$ belong to $I$, and*
*(4) there is a non kawamata log terminal centre $Z \subset X$ which is contained in every component of $\Delta$,*

*then the coefficients of $\Delta$ belong to $I_0$.*

**Theorem B** (Upper bounds for the volume). *Let $n \in \mathbb{N}$ and let $I \subset [0, 1)$ be a set which satisfies the DCC. Let $\mathfrak{D}$ be the set of kawamata log terminal pairs $(X, \Delta)$, where $X$ is projective of dimension $n$, $K_X + \Delta$ is numerically trivial and the coefficients of $\Delta$ belong to $I$.*

*Then the set*

$$\{\, \mathrm{vol}(X, \Delta) \,|\, (X, \Delta) \in \mathfrak{D} \,\},$$

*is bounded from above.*

**Theorem C** (Birational boundedness). *Fix a positive integer $n$ and a set $I \subset [0, 1]$, which satisfies the DCC. Let $\mathfrak{B}$ be the set of log canonical pairs $(X, \Delta)$, where $X$ is projective of dimension $n$, $K_X + \Delta$ is big and the coefficients of $\Delta$ belong to $I$.*

*Then there is a positive integer $m$ such that $\phi_{m(K_X + \Delta)}$ is birational, for every $(X, \Delta) \in \mathfrak{B}$.*

**Theorem D** (ACC for numerically trivial pairs). *Fix a positive integer $n$ and a set $I \subset [0, 1]$, which satisfies the DCC.*

*Then there is a finite subset $I_0 \subset I$ with the following property:*
*If $(X, \Delta)$ is a log pair such that*

*(1) $X$ is projective of dimension $n$,*
*(2) the coefficients of $\Delta$ belong to $I$,*
*(3) $(X, \Delta)$ is log canonical, and*
*(4) $K_X + \Delta$ is numerically trivial,*

*then the coefficients of $\Delta$ belong to $I_0$.*

The proof of Theorem A, Theorem B, Theorem C, and Theorem D proceeds by induction:

- Theorem $D_{n-1}$ implies Theorem $A_n$, cf. (5.3).
- Theorem $D_{n-1}$ and Theorem $A_{n-1}$ imply Theorem $B_n$, cf. (6.2).
- Theorem $C_{n-1}$, Theorem $A_{n-1}$, and Theorem $B_n$ imply Theorem $C_n$, cf. (7.4).
- Theorem $D_{n-1}$ and Theorem $C_n$ implies Theorem $D_n$, cf. (8.1).

2.1. **Sketch of the proof.** The basic idea of the proof of (1.1) goes back to Shokurov and we start by explaining this.

Consider the following simple family of plane curve singularities,
$$C = (y^a + x^b = 0) \subset \mathbb{C}^2,$$
where $a$ and $b$ are two positive integers. A priori, to calculate the log discrepancy $c$, one should take a log resolution of the pair $(X = \mathbb{C}^2, C)$, write down the log discrepancy of every exceptional divisor $E_i$ with respect to the pair $(X, tC)$ as a function of $t$ and then find out the largest value $c$ of $t$ for which all of these log discrepancies are non-negative. However there is an easier way. We know that when $t = c$ there is at least one divisor of log discrepancy zero (and every other divisor has non-negative log discrepancy). Let $\pi \colon Y \longrightarrow X$ extract just this divisor. To construct $\pi$ we simply contract all other divisors on the log resolution.

Almost by definition we can write
$$K_Y + E + cD = \pi^*(K_X + cC),$$
where $E$ is the exceptional divisor and $D$ is the strict transform of $C$. Restrict both sides of this equation to $E$. As the RHS is a pullback, we get a numerically trivial divisor.

To compute the LHS we apply adjunction. $E$ is a copy of $\mathbb{P}^1$. One slightly delicate issue is that $Y$ is singular along $E$ and the adjunction formula has to take account of this. In fact $Y \longrightarrow X$ is precisely the weighted blow up of $X = \mathbb{C}^2$, with weights $(a, b)$, in the given coordinates $x$, $y$. There are two singular points $p$ and $q$ of $Y$ along $C$, of index $a$ and $b$, and $D$ intersects $C$ transversally at another point $r$. If we apply adjunction we get
$$(K_Y + E + cD)|_E = K_E + \left(\frac{a-1}{a}\right)p + \left(\frac{b-1}{b}\right)q + cr.$$
As $(K_Y + E + cD)|_E$ is numerically trivial we have $(K_Y + E + cD) \cdot E = 0$ so that
$$-2 + \frac{a-1}{a} + \frac{b-1}{b} + c = 0,$$
and so
$$c = \frac{1}{a} + \frac{1}{b}.$$
Now let us consider the general case. As with the example above the first step is to extract divisors of log discrepancy zero, $\pi \colon Y \longrightarrow X$. To construct $\pi$ we mimic the argument above; pick a log resolution for the pair $(X, \Delta + C)$ and contract every divisor whose log discrepancy is not zero. The fact that we can do this in all dimensions follows from the MMP (minimal model program), see (3.3.1) and $\pi$ is called a divisorially log terminal modification.

The next step is the same, restrict to the general fibre of some divisor of log discrepancy zero, see (5.1). There are similar formulae for the coefficients of the restricted divisor, see (4.1). In this way, we reduce the problem from a local one in dimension $n$ to a global problem in dimension $n-1$, see §5. This explains how to go from Theorem $D_{n-1}$ to Theorem $A_n$, see the proof of (5.3).

The global problem involves log canonical pairs $(X, \Delta)$, where $X$ is projective and $K_X + \Delta$ is numerically trivial. One reason that the dimension one case is easy is that there is only one possibility for $X$, $X$ must be isomorphic to $\mathbb{P}^1$. In higher dimensions it is not hard, running the MMP again, to reduce to the case where $X$ has Picard number one, so that at least $X$ is a Fano variety and $\Delta$ is ample. In this case we perturb $\Delta$ by increasing one of its coefficients to get a kawamata log terminal pair $(X, \Lambda)$ such that $K_X + \Lambda$ is ample. We then exploit the fact that some fixed multiple $m(K_X + \Lambda)$ of $K_X + \Lambda$ gives a birational map $\phi_{m(K_X+\Lambda)}$. By definition this means that $\phi_{\lfloor m(K_X+\Lambda)\rfloor}$ is a birational map, which in particular means that $K_X + \Lambda_{\lfloor m \rfloor}$ (see (3.1) for the definition of $\Lambda_{\lfloor m \rfloor}$) is big . This forces $\Delta \leq \Lambda_{\lfloor m \rfloor}$ which implies that there are lots of gaps. This explains how to go from Theorem $C_n$ to Theorem $D_n$, see the proof of (8.1).

It is clear then that the main thing to prove is that if $(X, \Delta)$ is a kawamata log terminal pair, $K_X + \Delta$ is big and the coefficients of $\Delta$ belong to a DCC set then some fixed multiple of $K_X + \Delta$, gives a birational map $\phi_{m(K_X+\Delta)}$. Following some ideas of Tsuji, we developed a fairly general method to prove such a result in [15], see (3.5.2) and (3.5.5). We use the technique of cutting non kawamata log terminal centres as developed in [5], see [24]. The main issue is to find a boundary on the non kawamata log terminal centre so that we can run an induction.

There are two key hypotheses to apply (3.5.5). One of them requires that the volume of $K_X + \Delta$ restricted to appropriate non kawamata log terminal centres is bounded from below. The other places a requirement on the coefficients of $\Delta$ which is stronger than the DCC.

The first condition follows by induction on the dimension and a strong version of Kawamata's subadjunction formula, (4.2), which we now explain. If $(X, \Lambda)$ is a log pair and $V$ is a non kawamata log terminal centre such that $(X, \Lambda)$ is log canonical at the generic point of $V$, then one can write

$$(K_X + \Lambda)|_W = K_W + \Theta_b + J,$$

where $W$ is the normalisation of $V$, $\Theta_b$ is the discriminant divisor and $J$ is the moduli part. Not much is known about the moduli part $J$

beyond the fact that it is pseudo-effective. On the other hand $\Theta_b \geq 0$ behaves very well. If $(X, \Lambda)$ is log canonical at the generic point of a prime divisor $B$ on $W$ then the coefficient of $B$ in $\Theta_b$ is at most one. In fact there is a simple way to compute the coefficient of $B$ involving the log canonical threshold. By assumption there is a log canonical place, that is, a valuation with centre $V$ of log discrepancy zero. Then we can find a divisorially log terminal modification $g\colon Y \longrightarrow X$ such that the centre of this log canonical place is a divisor $S$ on $Y$. Note that there is a commutative diagram

$$
\begin{array}{ccc}
S & \longrightarrow & Y \\
f\downarrow & & \downarrow g \\
W & \longrightarrow & X.
\end{array}
$$

If we pullback $K_X + \Delta$ to $Y$ and restrict to $S$ we get a divisor $\Phi'$ on $S$. Let

$$\lambda = \sup\{\, t \in \mathbb{R} \mid (S, \Phi' + tf^*B) \text{ is log canonical over a}$$
$$\text{neighbourhood of the generic point of } B \,\},$$

be the log canonical threshold. Then the coefficient of $B$ in $\Theta_b$ is $1 - \lambda$.

In practice we start with a divisor $\Delta$ whose coefficients belong to $I$ such that $(X, \Delta)$ is kawamata log terminal. We then find a divisor $\Delta_0$, whose coefficients we have no control on, and $V$ is a non kawamata log terminal centre of $(X, \Lambda = \Delta + \Delta_0)$. It follows that the coefficients of $\Phi'$ do not behave well and we have no control on the coefficients of $\Theta_b$.

To circumvent this we simply mimic the same construction for $(X, \Delta)$ rather than $(X, \Lambda)$. First we construct a divisor $\Phi$ on $S$ whose coefficients of $\Phi$ belong to $D(I)$, see (4.1). Then we construct a divisor $\Theta$ whose coefficients automatically belong to the set

$$\{\, a \mid 1 - a \in \mathrm{LCT}_{n-1}(D(I)) \,\} \cup \{\, 1 \,\}.$$

It is clear from the construction that $\Theta_b \geq \Theta$, so that if we bound the volume of $K_W + \Theta$ from below we bound the volume of $(K_X + \Delta + \Delta_0)|_W$ from below.

On the other hand, as part of the induction we assume that Theorem $A_{n-1}$ holds. Hence $\mathrm{LCT}_{n-1}(D(I))$ satisfies the ACC and the coefficients of $\Theta$ belong to a set which satisfies the DCC. The final step is to observe that if we choose $V$ to pass through a general point then it belongs to a family which covers $X$. If we assume that $V$ is a general member of such a family then we can pullback $K_X + \Delta$ to this family and restrict to $V$. It is straightforward to check that the difference between $K_W + \Theta$ and $(K_X + \Delta)|_W$ on a log resolution of the

family is pseudo-effective (for example, if $X$ and $V$ are smooth then this follows from the fact that the first chern class of the normal bundle is pseudo-effective), so that if $K_X + \Delta$ is big then so is $K_W + \Theta$. In this case we know the volume is bounded from below by induction.

We now explain the condition on the coefficients. To apply (3.5.5) we require that either $I$ is a finite set or

$$I = \{ \frac{r-1}{r} \,|\, r \in \mathbb{N} \}.$$

The first lemma, (7.2), simply assumes this condition on $I$ and we deduce the result in this case.

The key is then to reduce to the case when $I$ is finite. Given any positive integer $p$ and a log pair $(X, \Delta)$, let $\Delta_{\lfloor p \rfloor}$ denote the largest divisor less than $\Delta$ such that $p\Delta_{\lfloor p \rfloor}$ is integral. Given $I$ it suffices to find a fixed positive integer $p$ such that if we start with $(X, \Delta)$ such that $K_X + \Delta$ is big and the coefficients belong to $I$ then $K_X + \Delta_{\lfloor p \rfloor}$ is big, since the coefficients of $\Delta_{\lfloor p \rfloor}$ belong to the finite set

$$\{ \frac{i}{p} \,|\, 1 \le i \le p \}.$$

Let

$$\lambda = \inf\{ t \in \mathbb{R} \,|\, K_X + t\Delta \text{ is big} \},$$

be the pseudo-effective threshold. A simple computation, (7.4), shows that it suffices to bound $\lambda$ away from one. Running the MMP we reduce to the case when $X$ has Picard number one. Since $K_X + \lambda\Delta$ is numerically trivial and kawamata log terminal, Theorem B implies that the volume of $\Delta$ is bounded away from one. Passing to a log resolution we may assume that $(X, D)$ has simple normal crossings where $D$ is the sum of the components of $\Delta$. As $K_X + D$ is big then so is $K_X + \frac{r-1}{r}D$ for any positive integer $r$ which is sufficiently large. It follows that some fixed multiple of $K_X + \frac{r-1}{r}D$ gives a birational map, and (3.5.2) implies that $(X, D)$ belongs to log birationally bounded family. In this case, it is easy to bound the pseudo-effective threshold $\lambda$ away from one, see (7.3). This explains how to go from Theorem $B_n$ to Theorem $C_n$, cf. (7.4).

We now explain the last implication. Suppose that $(X, \Delta)$ is kawamata log terminal and $K_X + \Delta$ is numerically trivial. If the volume of $\Delta$ is large then we may find a divisor $\Pi$ numerically equivalent to a small multiple of $\Delta$ with large multiplicity at a general point, so that $(X, \Pi)$ is not kawamata log terminal. In particular we may find $\Phi$ arbitrarily close to $\Delta$ such that $(X, \Phi)$ is not kawamata log terminal. The key lemma is to show that this is impossible, (6.1). By assumption we

may extract a divisor $S$ of log discrepancy zero with respect to $(X, \Phi)$. After we run the MMP we get down a log pair $(Y, S+\Gamma)$ where $\Gamma$ is the strict transform of $\Delta$ and both $K_Y + S + \Gamma$ and $-(K_Y + S + (1-\epsilon)\Gamma)$ are ample. Here $\epsilon > 0$ is arbitrarily close to zero. If we restrict to $S$ and apply adjunction, it is easy to see that this contradicts either ACC for the log canonical threshold or ACC for numerically trivial pairs. This explains how to go from Theorem $D_{n-1}$ and Theorem $A_{n-1}$ to Theorem $B_n$, cf. (6.2).

It is interesting to note that if $(X, \Delta)$ is log canonical then there is no bound on the volume of $\Delta$:

**Example 2.1.1.** *Let $X$ be the weighted projective surface $\mathbb{P}(p, q, r)$, where $p$, $q$ and $r$ are three positive integers and let $\Delta$ be the sum of the three coordinate lines. Then $K_X + \Delta \sim_{\mathbb{Q}} 0$ and*

$$\mathrm{vol}(X, \Delta) = \frac{(p+q+r)^2}{pqr}.$$

*But the set*

$$\{ \, \frac{(p+q+r)^2}{pqr} \, | \, (p, q, r) \in \mathbb{N}^3 \, \},$$

*is dense in the positive real numbers, cf. [19, 22.5].*

We now explain the proof of (1.11) which mirrors the proof of (1.1). We are given a sequence of log pairs $(X, \Delta) = (X_i, \Delta_i)$ and we want to identify the limit points of the log canonical thresholds. The first step is to show that the set of log canonical thresholds is essentially the same as the set of pseudo-effective thresholds. In §5 we showed that every log canonical threshold in dimension $n+1$ is a numerically trivial threshold in dimension $n$. To show the reverse inclusion, one takes the cone $(Y, \Gamma)$ over a log canonical pair $(X, \Delta)$ where $K_X + \Delta$ is numerically trivial, (11.5).

In this way we are reduced to looking at log canonical pairs $(X, \Delta)$ such that $K_X + \Delta$ is numerically trivial. The basic idea is to generate a component of coefficient one and apply adjunction. To this end, we need to deal with the case where some coefficients of $\Delta$ don't necessarily belong to $I$ but instead they are increasing towards one, (11.7).

Running the MMP we reduce to the case of Picard number one, Case A, Step 1 and Case B, Steps 3 and 5. We may also assume that the non kawamata log terminal locus is a divisor. In particular $-K_X$ is ample, any two components of $\Delta$ intersect and we may assume that the number of components of $\Delta$ is constant, (11.6). If $(X, \Delta)$ is not kawamata log terminal then there is a component of coefficient one and we are done, Case B, Step 2.

The argument now splits into two cases. Case A deals with the case that the coefficients of $\Delta$ are bounded away from one. In this case if the volume of $\Delta$ is arbitrarily large then we can create a component of coefficient one and we reduce to the other case, Case B. Otherwise (1.6) implies that $(X, \Delta)$ belongs to a bounded family, which contradicts the fact that the coefficients of $\Delta$ are not constant.

So we may assume we are in Case B, namely that some of the coefficients of the components of $\Delta$ are approaching one. We decompose $\Delta$ as $A + B + C$ where the coefficients of $A$ are approaching one, the coefficients of $B$ are fixed, and we are trying to identify the limit of the coefficients of $C$. Using the fact that the Picard number of $X$ is one, we may increase the coefficients of $A$ to one and decrease the coefficients of $C$, without changing the limit of the coefficients of $C$. At this point we apply adjunction and induction, Case B, Step 6.

## 3. Preliminaries

3.1. **Notation and Conventions.** If $D = \sum d_i D_i$ is an $\mathbb{R}$-divisor on a normal variety $X$, then the *round down* of $D$ is $\lfloor D \rfloor = \sum \lfloor d_i \rfloor D_i$, where $\lfloor d \rfloor$ denotes the largest integer which is at most $d$, the *fractional part* of $D$ is $\{D\} = D - \lfloor D \rfloor$, and the *round up* of $D$ is $\lceil D \rceil = -\lfloor -D \rfloor$. If $m$ is a positive integer, then let

$$D_{\lfloor m \rfloor} = \frac{\lfloor mD \rfloor}{m}.$$

Note that $D_{\lfloor m \rfloor}$ is the largest divisor less than or equal to $D$ such that $mD_{\lfloor m \rfloor}$ is integral.

The sheaf $\mathcal{O}_X(D)$ is defined by

$$\mathcal{O}_X(D)(U) = \{\, f \in K(X) \,|\, (f)|_U + D|_U \geq 0 \,\},$$

so that $\mathcal{O}_X(D) = \mathcal{O}_X(\lfloor D \rfloor)$. Similarly we define $|D| = |\lfloor D \rfloor|$. If $X$ is normal, and $D$ is an $\mathbb{R}$-divisor on $X$, the *rational map $\phi_D$ associated to $D$* is the rational map determined by the restriction of $\lfloor D \rfloor$ to the smooth locus of $X$.

We say that $D$ is $\mathbb{R}$-*Cartier* if it is a real linear combination of Cartier divisors. If $f\colon Y \longrightarrow X$ is a morphism then $D|_Y$ denotes the pullback of $D$ to $Y$, $f^*D$. In general $D|_Y$ is only well-defined up to $\mathbb{R}$-linear equivalence. However if $f(Y)$ is not contained in the support of $D$ then $D|_Y$ is a well-defined $\mathbb{R}$-Cartier divisor. An $\mathbb{R}$-Cartier divisor $D$ on a normal variety $X$ is *nef* if $D \cdot C \geq 0$ for any curve $C \subset X$. We say that two $\mathbb{R}$-divisors $D_1$ and $D_2$ are $\mathbb{R}$-*linearly equivalent*, denoted $D_1 \sim_{\mathbb{R}} D_2$, if the difference is an $\mathbb{R}$-linear combination of principal divisors.

A *log pair* $(X, \Delta)$ consists of a normal variety $X$ and a $\mathbb{R}$-Weil divisor $\Delta \geq 0$ such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. The *support* of $\Delta = \sum_{i \in I} d_i D_i$ (where $d_i \neq 0$) is the sum $D = \sum_{i \in I} D_i$. If $(X, \Delta)$ has simple normal crossings, a *stratum* of $(X, \Delta)$ is an irreducible component of the intersection $\cap_{j \in J} D_j$, where $J$ is a non-empty subset of $I$ (in particular, a stratum of $(X, \Delta)$ is always a proper closed subset of $X$). If we are given a morphism $X \longrightarrow T$, then we say that $(X, \Delta)$ has *simple normal crossings over $T$* if $(X, \Delta)$ has simple normal crossings and both $X$ and every stratum of $(X, D)$ is smooth over $T$. We say that the birational morphism $f \colon Y \longrightarrow X$ only *blows up strata* of $(X, \Delta)$, if $f$ is the composition of birational morphisms $f_i \colon X_{i+1} \longrightarrow X_i$, $1 \leq i \leq k$, with $X = X_0$, $Y = X_{k+1}$, and $f_i$ is the blow up of a stratum of $(X_i, \Delta_i)$, where $\Delta_i$ is the sum of the strict transform of $\Delta$ and the exceptional locus.

A *log resolution* of the pair $(X, \Delta)$ is a projective birational morphism $\mu \colon Y \longrightarrow X$ such that the exceptional locus is the support of a $\mu$-ample divisor and $(Y, G)$ has simple normal crossings, where $G$ is the support of the strict transform of $\Delta$ and the exceptional divisors. If we write

$$K_Y + \Gamma + \sum b_i E_i = \mu^*(K_X + \Delta)$$

where $\Gamma$ is the strict transform of $\Delta$, then $b_i$ is called the *coefficient of $E_i$ with respect to $(X, \Delta)$*. The *log discrepancy* of $E_i$ is $a(E_i, X, \Delta) = 1 - b_i$. The *log discrepancy* of $(X, \Delta)$ is the infimum over all log resolutions of the log discrepancy of any exceptional divisor. The *total log discrepancy* of $(X, \Delta)$ is the minimum of the log discrepancy of $(X, \Delta)$ and $1 - a$ where $a$ ranges over the coefficients of the components of $\Delta$. The pair $(X, \Delta)$ is *kawamata log terminal* (respectively *log canonical*; *purely log terminal*; *divisorially log terminal*) if $b_i < 1$ for all $i$ and $\lfloor \Delta \rfloor = 0$ (respectively $b_i \leq 1$ for all $i$ and for all log resolutions; $b_i < 1$ for all $i$ and for all log resolutions; the coefficients of $\Delta$ belong to $[0, 1]$ and there exists a log resolution such that $b_i < 1$ for all $i$). If we drop the condition that $\Delta \geq 0$ but all of the coefficients of $\Gamma$ are at most one then we say that $(X, \Delta)$ is *sub log canonical*. A *non kawamata log terminal centre* is the centre of any valuation associated to a divisor $E_i$ with $b_i \geq 1$. In this paper, we only consider valuations $\nu$ of $X$ whose centre on some birational model $Y$ of $X$ is a divisor.

We now introduce some results some of which are well known to experts but which are included for the convenience of the reader.

## 3.2. The volume.

**Definition 3.2.1.** *Let $X$ be an irreducible projective variety of dimension $n$ and let $D$ be an $\mathbb{R}$-divisor. The **volume** of $D$ is*

$$\mathrm{vol}(X, D) = \limsup_{m \to \infty} \frac{n! h^0(X, \mathcal{O}_X(mD))}{m^n}.$$

*We say that $D$ is **big** if $\mathrm{vol}(X, D) > 0$.*

For more background, see [31].

**Lemma 3.2.2.** *Let $X$ be a projective variety and let $(X, \Delta)$ be a log pair.*

*If $D$ is an $\mathbb{R}$-divisor and $\mathrm{vol}(X, D) > n^n$ then for every point $x \in X$ we may find $\Pi \sim_{\mathbb{R}} D$ such that $(X, \Delta + \Pi)$ is not kawamata log terminal at $x \in X$.*

*Proof.* Arguing as in the proof of [24, 6.7.1] we may assume that $x \in X$ is a general point, so that in particular $x$ is a smooth point of $X$. As the volume is a continuous function of $D$ we may assume that $D$ is a $\mathbb{Q}$-divisor, [30, 2.2.44]. The result then follows as in the proof of [24, 6.1]. $\qquad\square$

**Lemma 3.2.3.** *Let $X$ be a quasi-projective $\mathbb{Q}$-factorial variety and let $(X, \Delta)$ be a kawamata log terminal pair.*

*If $(X, \Delta + D)$ is not log canonical, where $D \geq 0$ is big, then we may find $0 \leq D' \sim_{\mathbb{R}} tD$, for some $0 < t < 1$ such that $(X, \Delta + D')$ has exactly one log canonical place.*

*Proof.* As $(X, \Delta + D)$ is not log canonical we may find $\delta > 0$ such that $(X, \Delta + (1 - \delta)D)$ is not log canonical. As $D$ is big we may find divisors $A \geq 0$ and $B \geq 0$ such that $D \sim_{\mathbb{R}} A + B$ and $A$ is ample. Replacing $D$ by $(1 - \delta)D + \delta A + \delta B$ we may assume that there is an ample divisor $A \geq 0$ such that $D \geq A$.

Let

$$\pi \colon Y \longrightarrow X$$

be a log resolution. We may write

$$K_Y + \Gamma + \sum a_i E_i = \pi^*(K_X + \Delta + tD),$$

where $\Gamma$ is the strict transform of $\Delta$ and $a_i$ are linear functions of $t$. By assumption $a_i < 1$ when $t = 0$ and there is an index $i$ such that $a_i > 1$ when $t = 1$. It follows that we may find $t \in (0, 1)$ such that $a_i \leq 1$ for all indices with equality for at least one index $i$. Possibly using $A$ to tie-break, see [24], we may assume that there is at most one index $i$ such that $a_i = 1$. $\qquad\square$

### 3.3. Divisorially log terminal modifications.

If $(X, \Delta)$ is not kawamata log terminal then we may find a modification which is divisorially log terminal, so that the non kawamata log terminal locus is a divisor:

**Proposition 3.3.1.** *Let $(X, \Delta)$ be a log pair where $X$ is a variety and the coefficients of $\Delta$ belong to $[0, 1]$.*
*Then there is a projective birational morphism $\pi \colon Y \longrightarrow X$ such that*

*(1) $Y$ is $\mathbb{Q}$-factorial,*
*(2) $\pi$ only extracts divisors of log discrepancy at most zero,*
*(3) if $E = \sum E_i$ is the sum of the $\pi$-exceptional divisors and $\Gamma$ is the strict transform of $\Delta$, then $(Y, \Gamma + E)$ is divisorially log terminal and*

$$K_Y + E + \Gamma = \pi^*(K_X + \Delta) + \sum_{a(E_i, X, B) < 0} a(E_i, X, B) E_i.$$

*(4) Further, if $(X, \Delta)$ is log canonical and $S$ is a component of $\Delta$ then there is a nef divisor of the form $-T - F$, where $T$ is the strict transform of $S$ and $F \geq 0$ is a sum of exceptional divisor whose centres are contained in $S$.*

*Any birational morphism $\pi \colon Y \longrightarrow X$ satisfying (1–3) is called a **divisorially log terminal modification**.*

*Proof.* The proof of (1–3) is due to the first author and can be found in [13], [28, 3.1], and also [4].

Now suppose that $(X, \Delta)$ is log canonical and $S$ is a component of $\Delta$. In this case
$$K_Y + E + \Gamma = \pi^*(K_X + \Delta).$$
Pick $\epsilon > 0$ so that $\Gamma - \epsilon T \geq 0$. Note that $(Y, E + \Gamma - \epsilon T)$ is divisorially log terminal, as $Y$ is $\mathbb{Q}$-factorial and $(Y, E + \Gamma)$ is divisorially log terminal. By Theorem 1.1 of [7] or by Theorem 1.6 of [16], we may replace $Y$ by a log terminal model of $(Y, E + \Gamma - \epsilon T)$ over $X$, gaining the fact that $-T$ is nef over $X$, at the expense of temporarily losing the property that $(Y, \Gamma + E)$ is divisorially log terminal, whilst preserving the condition that $K_Y + E + \Gamma$ is log canonical and numerically trivial over $X$. If $g \colon W \longrightarrow Y$ is a divisorially log terminal modification of $(Y, \Gamma + E)$ and we replace $Y$ by $W$ then $g^*(-T)$ is a nef divisor over $X$ of the correct form. $\square$

### 3.4. DCC sets.

We say that a set $I$ of real numbers satisfies the *descending chain condition* or DCC, if it does not contain any infinite strictly decreasing sequence. For example,

$$I = \{ \frac{r-1}{r} \mid r \in \mathbb{N} \},$$

satisfies the DCC. Let $I \subset [0,1]$. We define

$$I_+ := \{\, 0 \,\} \cup \{\, j \in [0,1] \mid j = \sum_{p=1}^{l} i_p, \text{ for some } i_1, i_2, \ldots, i_l \in I \,\},$$

and

$$D(I) := \{\, a \le 1 \mid a = \frac{m-1+f}{m}, m \in \mathbb{N}, f \in I_+ \,\}.$$

As usual, $\overline{I}$ denotes the closure of $I$. Note that the set $D(I)$ appears when we apply adjunction, (4.1).

**Proposition 3.4.1.** *Let $I \subset [0,1]$.*
*(1) $D(D(I)) = D(I) \cup \{\, 1 \,\}$.*
*(2) $I$ satisfies the DCC if and only if $\overline{I}$ satisfies the DCC.*
*(3) $I$ satisfies the DCC if and only if $D(I)$ satisfies the DCC.*

*Proof.* Straightforward, see for example [32, 4.4]. $\qquad\qquad\square$

3.5. **Bounded pairs.** We recall some results and definitions from [15], stated in a convenient form.

**Definition 3.5.1.** *We say that a set $\mathfrak{X}$ of varieties is **birationally bounded** if there is a projective morphism $Z \longrightarrow T$, where $T$ is of finite type, such that for every $X \in \mathfrak{X}$, there is a closed point $t \in T$ and a birational map $f \colon Z_t \dashrightarrow X$.*

*We say that a set $\mathfrak{D}$ of log pairs is **log birationally bounded** (respectively **bounded**) if there is a log pair $(Z, B)$, where the coefficients of $B$ are all one, and a projective morphism $Z \longrightarrow T$, where $T$ is of finite type, such that for every $(X, \Delta) \in \mathfrak{D}$, there is a closed point $t \in T$ and a birational map $f \colon Z_t \dashrightarrow X$ (respectively isomorphism of varieties) such that the support of $B_t$ is not the whole of $Z_t$ and yet $B_t$ contains the support of the strict transform of $\Delta$ and any $f$-exceptional divisor (respectively $f(B_t) = \Delta$).*

**Theorem 3.5.2.** *Fix a positive integer $n$ and a set $I \subset [0,1] \cap \mathbb{Q}$, which satisfies the DCC. Let $\mathfrak{B}_0$ be a set of log canonical pairs $(X, \Delta)$, where $X$ is projective of dimension $n$, $K_X + \Delta$ is big and the coefficients of $\Delta$ belong to $I$.*

*Suppose that there is a constant $M$ such that for every $(X, \Delta) \in \mathfrak{B}_0$ there is a positive integer $k$ such that $\phi_{k(K_X+\Delta)}$ is birational and*

$$\mathrm{vol}(X, k(K_X + \Delta)) \le M.$$

*Then the set*

$$\{\, \mathrm{vol}(X, K_X + \Delta) \mid (X, \Delta) \in \mathfrak{B}_0 \,\},$$

*satisfies the DCC.*

*Proof.* Follows from (2.3.4), (3.1) and (1.9) of [15]. □

Recall:

**Definition 3.5.3.** *Let $X$ be a normal projective variety and let $D$ be a big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $X$.*

*If $x$ and $y$ are two general points of $X$ then, possibly switching $x$ and $y$, we may find $0 \leq \Delta \sim_{\mathbb{Q}} (1 - \epsilon)D$, for some $0 < \epsilon < 1$, where $(X, \Delta)$ is not kawamata log terminal at $y$, $(X, \Delta)$ is log canonical at $x$ and $\{x\}$ is a non kawamata log terminal centre, then we say that $D$ is* **potentially birational**.

Note that this is a slight variation on the definition which appears in [15], where general is replaced by very general.

**Theorem 3.5.4.** *Let $(X, \Delta)$ be a kawamata log terminal pair, where $X$ is projective of dimension $n$ and let $H$ be an ample $\mathbb{Q}$-divisor. Suppose there is a constant $\gamma \geq 1$ and a family of subvarieties $V \longrightarrow B$ with the following property.*

*If $x$ and $y$ are two general points of $X$ then, possibly switching $x$ and $y$, we can find $b \in B$ and $0 \leq \Delta_b \sim_{\mathbb{Q}} (1 - \delta)H$, for some $\delta > 0$, such that $(X, \Delta + \Delta_b)$ is not kawamata log terminal at $y$ and there is a unique non kawamata log terminal place of $(X, \Delta + \Delta_b)$ whose centre $V_b$ contains $x$. Further there is a divisor $D$ on $W$, the normalisation of $V_b$, such that $\phi_D$ is birational and $\gamma H|_W - D$ is pseudo-effective.*

*Then $mH$ is potentially birational, where $m = 2p^2\gamma + 1$ and $p = \dim V_b$.*

*Proof.* Let $x$ and $y$ be two general points of $X$. Possibly switching $x$ and $y$, we will prove by descending induction on $k$ that there is a $\mathbb{Q}$-divisor $\Delta_0 \geq 0$ such that:

$(\flat)_k$ $\Delta_0 \sim_{\mathbb{Q}} \lambda H$, for some $\lambda < 2(p - k)p\gamma + 1$, where $(X, \Delta + \Delta_0)$ is log canonical at $x$, not kawamata log terminal at $y$ and there is a non kawamata log terminal centre $Z \subset V_b$ of dimension at most $k$ containing $x$.

Suppose $k = p$. $(X, \Delta + \Delta_b)$ is not kawamata log terminal but log canonical at $x$ since there is a unique non kawamata log terminal place whose centre contains $x$. Thus $\Delta_0 = \Delta_b \sim_{\mathbb{Q}} \lambda H$, where $\lambda = 1 - \delta < 1$, satisfies $(\flat)_k$ and so this is the start of the induction.

Now suppose that we may find a $\mathbb{Q}$-divisor $\Delta_0$ satisfying $(\flat)_k$. We may assume that $Z$ is the minimal non kawamata log terminal centre containing $x$ and that $Z$ has dimension $k$. Let $Y \subset W$ be the inverse image of $Z$. As $x$ is a general point of $X$ it is also a general point of $W$, $Y$ and $Z$. In particular the restriction of $\gamma H|_W - D$ to $Y$ is

pseudo-effective, $Y \longrightarrow Z$ is birational, and as $\phi_D$ is birational and $x$ is general, the restriction of $\phi_D$ to $Y$ is birational. Thus

$$\text{vol}(Y, \gamma H|_Y) \geq \text{vol}(Y, D|_Y) \geq 1,$$

where the last inequality is proved, for example, in [14, 2.2]. Note that

$$\text{vol}(Z, \gamma H|_Z) = \text{vol}(Y, \gamma H|_Y),$$

as $H$ is nef, see for example [23, VI.2.15]. Thus

$$\text{vol}(Z, 2p\gamma H|_V) > \text{vol}(Z, 2k\gamma H|_V) \geq 2k^k,$$

so that by [15, 2.3.5], we may find $\Delta_1 \sim_{\mathbb{Q}} \mu H$, where $\mu < 2p\gamma$ and constants $0 < a_i \leq 1$ such that $(X, \Delta + a_0\Delta_0 + a_1\Delta_1)$ is log canonical at $x$, not kawamata log terminal at $y$ and there is a non kawamata log terminal centre $Z'$ containing $x$, whose dimension is less than $k$. As

$$a_0\Delta_0 + a_1\Delta_1 \sim_{\mathbb{Q}} (a_0\lambda + a_1\mu)H,$$

and

$$\lambda' = a_0\lambda + a_1\mu < 2(p-k)p\gamma + 1 + 2p\gamma = 2(p-(k-1))p\gamma + 1,$$

$a_0\Delta_0 + a_1\Delta_1$ satisfies $(\flat)_{k-1}$. This completes the induction and the proof. $\qquad \square$

**Theorem 3.5.5.** *Fix a positive integer $n$. Let $\mathfrak{B}_0$ be a set of kawamata log terminal pairs $(X, \Delta)$, where $X$ is projective of dimension $n$ and $K_X + \Delta$ is ample.*

*Suppose that there are positive integers $p$, $k$ and $l$ such that for every $(X, \Delta) \in \mathfrak{B}_0$ we have:*

(1) *There is a family of subvarieties $V \longrightarrow B$ such that if $x$ and $y$ are two general points of $X$ then, possibly switching $x$ and $y$, we can find $b \in B$ and $0 \leq \Delta_b \sim_{\mathbb{Q}} (1 - \delta)H$, for some $\delta > 0$, such that $(X, \Delta + \Delta_b)$ is not kawamata log terminal at $y$ and there is a unique non kawamata log terminal place of $(X, \Delta + \Delta_b)$ whose centre $V_b$ contains $x$, where $H = k(K_X + \Delta)$. Further there is a divisor $D$ on $W$, the normalisation of $V_b$, such that $\phi_D$ is birational and $lH|_W - D$ is pseudo-effective.*

(2) *Either $p\Delta$ is integral or the coefficients of $\Delta$ belong to*

$$\left\{ \frac{r-1}{r} \,\middle|\, r \in \mathbb{N} \right\}.$$

*Then there is a positive integer $m$ such that $\phi_{mk(K_X + \Delta)}$ is birational, for every $(X, \Delta) \in \mathfrak{B}_0$.*

*Proof.* Let $m_0 = 2(n-1)^2 l + 1$. (3.5.4) implies that $m_0 H$ is potentially birational. But then [15, 2.3.4.1] implies that $\phi_{K_X + \lceil m_0 j H \rceil}$ is birational for all positive integers $j$.

If $p\Delta$ is integral then

$$K_X + \lceil m_0 k p (K_X + \Delta) \rceil = \lfloor (m_0 k p + 1)(K_X + \Delta) \rfloor,$$

and if the coefficients of $\Delta$ belong to

$$\{ \frac{r-1}{r} \mid r \in \mathbb{N} \},$$

then

$$K_X + \lceil m_0 k p (K_X + \Delta) \rceil = \lfloor (m_0 k p + 1)(K_X + \Delta) \rfloor.$$

Let $m = (m_0 + 1)p$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

## 4. Adjunction

We will need the following basic result about adjunction (see for example §16 in [27]).

**Lemma 4.1.** *Let $(X, \Delta = S' + B)$ be a log canonical pair, where $S$ has coefficient one in $\Delta$. If $S$ is the normalisation of $S'$ then there is a divisor $\Theta = \mathrm{Diff}_S(B)$ on $S$ such that*

$$(K_X + \Delta)|_S = K_S + \Theta.$$

*(1) If $(X, \Delta)$ is purely log terminal then $(S, \Theta)$ is kawamata log terminal.*
*(2) If $(X, \Delta)$ is divisorially log terminal then $(S, \Theta)$ is divisorially log terminal.*
*(3) If $B = \sum b_i B_i$ then the coefficients of $\Theta$ belong to the set $D(\{ b_1, b_2, \ldots, b_m \})$.*

*In particular, if $(X, \Delta)$ is divisorially log terminal and the coefficients of $B$ belong to the set $I$ then the coefficients of $\Theta$ belong to the set $D(I)$.*

**Theorem 4.2.** *Let $I$ be a subset of $[0, 1]$ which contains $1$. Let $X$ be a projective variety of dimension $n$ and let $V$ be an irreducible closed subvariety, with normalisation $W$. Suppose we are given a log pair $(X, \Delta)$ and an $\mathbb{R}$-Cartier divisor $\Delta' \geq 0$, with the following properties:*
*(1) the coefficients of $\Delta$ belong to $I$,*
*(2) $(X, \Delta)$ is kawamata log terminal, and*
*(3) there is a unique non kawamata log terminal place $\nu$ for $(X, \Delta + \Delta')$, with centre $V$.*

*Then there is a divisor $\Theta$ on $W$ whose coefficients belong to*

$$\{\, a \mid 1 - a \in \mathrm{LCT}_{n-1}(D(I)) \,\} \cup \{\, 1 \,\},$$

*such that the difference*

$$(K_X + \Delta + \Delta')|_W - (K_W + \Theta),$$

*is pseudo-effective.*

*Now suppose that $V$ is the general member of a covering family of subvarieties of $X$. Let $\psi\colon U \longrightarrow W$ be a log resolution of $W$ and let $\Psi$ be the sum of the strict transform of $\Theta$ and the exceptional divisors. Then*

$$K_U + \Psi \geq (K_X + \Delta)|_U.$$

*Proof.* Since there is a unique non kawamata log terminal place with centre $V$, it follows that $(X, \Delta + \Delta')$ is log canonical but not kawamata log terminal at the generic point of $V$, see (2.31) of [29]. Let $g\colon Y \longrightarrow X$ be a divisorially log terminal modification of $(X, \Delta + \Delta')$, (3.3.1), so that the centre of $\nu$ is a divisor $S$ on $Y$ and this is the only exceptional divisor with centre $V$. As $(X, \Delta + \Delta')$ is divisorially log terminal, $S$ is normal and so there is a commutative diagram

$$
\begin{array}{ccc}
S & \longrightarrow & Y \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\
W & \longrightarrow & X.
\end{array}
$$

We may write

$$K_Y + S + \Gamma = g^*(K_X + \Delta) + E \qquad \text{and} \qquad K_Y + S + \Gamma + \Gamma' = g^*(K_X + \Delta + \Delta'),$$

where $\Gamma$ is the sum of the strict transform of $\Delta$ and the exceptional divisors, apart from $S$. In particular the coefficients of $\Gamma$ belong to $I$. As $(X, \Delta)$ is kawamata log terminal, $E \geq 0$. As $g$ is a divisorially log terminal modification of $(X, \Delta + \Delta')$, $\Gamma' \geq 0$ and $(Y, S + \Gamma)$ is divisorially log terminal. We may write

$$(K_Y + S + \Gamma)|_S = K_S + \Phi \qquad \text{and} \qquad (K_Y + S + \Gamma + \Gamma')|_S = K_S + \Phi'.$$

Note that the coefficients of $\Phi$ belong to $D(I)$. Let $B$ be a prime divisor on $W$. Let

$$\mu = \sup\{\, t \in \mathbb{R} \mid (S, \Phi + t f^* B) \text{ is log canonical over a}$$
$$\text{neighbourhood of the generic point of } B \,\},$$

be the log canonical threshold over a neighbourhood of the generic point of $B$. We define $\Theta$ by

$$\mathrm{mult}_B(\Theta) = 1 - \mu.$$

It is clear that the coefficients of $\Theta$ belong to
$$\{\, a \mid 1 - a \in \mathrm{LCT}_{n-1}(D(I)) \,\} \cup \{\, 1 \,\}.$$

Let
$$\lambda = \sup\{\, t \in \mathbb{R} \mid (S, \Phi' + t f^* B) \text{ is log canonical over a}$$
$$\text{neighbourhood of the generic point of } B \,\},$$

be the log canonical threshold over a neighbourhood of the generic point of $B$. We define a divisor $\Theta_b$ on $W$ by
$$\mathrm{mult}_B(\Theta_b) = 1 - \lambda.$$

As $\Gamma' \geq 0$ we have $\Phi \leq \Phi'$, so that $\lambda \leq \mu$. But then
$$\Theta \leq \Theta_b.$$

Note that $\Theta_b$ is precisely the divisor defined in Kawamata's subadjunction formula, see Theorem 1 and 2 of [18] and also (8.5.1) and (8.6.1) of [25]. It follows that the difference
$$(K_X + \Delta + \Delta')|_W - (K_W + \Theta_b),$$

is pseudo-effective, so that the difference
$$(K_X + \Delta + \Delta')|_W - (K_W + \Theta),$$

is certainly pseudo-effective.

Now suppose that $V$ is the general member of a covering family of subvarieties of $X$. We first relate the definition of $\Theta$, which uses the log canonical threshold on $S$, to a log canonical threshold on $X$. Let $B$ be a prime divisor on $W$ and let $A$ be its image on $V$. Pick any $\mathbb{Q}$-divisor $H \geq 0$ on $X$ which is $\mathbb{Q}$-Cartier in a neighbourhood of the generic point of $A$ and which does not contain $V$ such that
$$\mathrm{mult}_B(H|_W) = 1.$$

We have
$$K_Y + S + \Gamma + t g^* H = g^*(K_X + \Delta + tH) + E,$$

and so
$$(K_Y + S + \Gamma + t g^* H)|_S = K_S + \Phi + t f^* B,$$

over a neighbourhood of the generic point of $B$. Now if $(X, \Delta + tH)$ is not log canonical in a neighbourhood of the generic point of $A$ then $K_Y + S + \Gamma + t g^* H$ is not log canonical over a neighbourhood of the generic point of $B$. Inversion of adjunction on $Y$, cf [17], implies that $K_Y + S + \Gamma + t g^* H$ is log canonical over a neighbourhood of the generic

point of $B$ if and only if $K_S + \Phi + tf^*B$ is log canonical over a neighbourhood of the generic point of $B$. It follows that if

$$\mu = \sup\{\, t \in \mathbb{R} \mid (S, \Phi + tf^*B) \text{ is log canonical over a}$$
$$\text{neighbourhood of the generic point of } B \,\},$$

the log canonical threshold of $f^*B$ over a neighbourhood of the generic point of $B$, and

$$\xi = \sup\{\, t \in \mathbb{R} \mid (X, \Delta + tH) \text{ is log canonical at the generic point of } A \,\},$$

the log canonical threshold of $H$ at the generic point of $A$, then $\mu \leq \xi$.

By assumption we may pick a component $R_0$ of the Hilbert scheme $\mathcal{H}$ whose universal family dominates $X$, such that $V$ is the general member of $R_0$. Let $\pi_0 \colon Z_0 \longrightarrow R_0$ be the restriction of the normalisation of the universal family. Cutting by hyperplanes in $R_0$ we may find $R \subset R_0$ with $V \in R$ such that if $\pi \colon Z \longrightarrow R$ is the restriction of $\pi_0$ then the natural morphism $h \colon Z \longrightarrow X$ is generically finite (note that if we take the hyperplanes successively from a fixed sequence of general pencils of hyperplanes then we won't lose the fact that $V$ is a general element). We may write

$$K_Z + \Xi = h^*(K_X + \Delta).$$

Possibly blowing up, we may assume that $(Z, \Xi)$ has simple normal crossings over a dense open subset $R_1$ of $R$. Let $U$ be the fibre of $\pi$ corresponding to $W$. As $V$ is a general member of $R_0$, we may assume that $r = \pi(U) \in R_1$ and so $(U, \Xi|_U)$ has simple normal crossings. As the coefficients of $\Xi|_U$ are at most one it follows that $(U, \Xi|_U)$ is sub log canonical. Therefore it is enough to check that

$$K_U + \Psi \geq (K_X + \Delta)|_U = K_U + \Xi|_U,$$

on the given model and in fact we just have to check that $\Psi \geq \Xi|_U$.

Let $C$ be a prime divisor on $U$. If $\operatorname{mult}_C \Xi|_U \leq 0$ there is nothing to prove as $\Psi \geq 0$. If $C$ is an exceptional divisor of $U \longrightarrow V$ then $\operatorname{mult}_C \Psi = 1$ and there is again nothing to prove as $\operatorname{mult}_C \Xi|_U \leq 1$.

Otherwise pick a prime component $G$ of $\Xi$ such that $\operatorname{mult}_C(G|_U) = 1$. If $h(G)$ is a divisor then let $H = h(G)/e$ where $e$ is the ramification index at $G$. Note that the pullback of $H$ to $W$ is $\mathbb{Q}$-Cartier in a neighbourhood of the generic point of $B = \psi(C)$. Otherwise, pick a $\mathbb{Q}$-Cartier divisor $H \geq 0$, which does not contain $V$, such that $\operatorname{mult}_G(h^*H) = 1$. Either way, as $r \in R$ is general it follows that $\operatorname{mult}_C(h^*H|_U) = 1$. But then

$$\operatorname{mult}_B(H|_W) = \operatorname{mult}_C(h^*H|_U) = 1.$$

We may write

$$K_Z + \Xi + \xi h^* H = h^*(K_X + \Delta + \xi H).$$

As $(X, \Delta + \xi H)$ is log canonical in a neighbourhood of the generic point of $B$, $K_Z + \Xi + \xi h^* H$ is sub log canonical in a neighbourhood of the generic point of $C$. Note that in a neighbourhood of the generic point of $C$,

$$(K_Z + \Xi + \xi h^* H)|_U = K_U + \Xi|_U + \xi C + J,$$

where $J \geq 0$. As $r$ is a general point of $R$, $(U, \Xi|_U + \xi C + J)$ is sub log canonical in a neighbourhood of the generic point of $C$. It follows that

$$\mathrm{mult}_C \Xi|_U + \xi \leq 1,$$

so that

$$\mathrm{mult}_C \Psi = \mathrm{mult}_B \Theta = 1 - \mu \geq 1 - \xi \geq \mathrm{mult}_C \Xi|_U.$$

Thus $\Psi \geq \Xi|_U$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

## 5. Global to local

**Lemma 5.1.** *Fix a positive integer $n$ and a set $1 \in I \subset [0,1]$.*

*Suppose $(X, \Delta)$ is a log canonical pair where $X$ is a variety of dimension $n + 1$, the coefficients of $\Delta$ belong to $I$ and there is a non kawamata log terminal centre $V \subset X$. Suppose that $c \in I$ is the coefficient of some component $M$ of $\Delta$ which contains $V$.*

*Then we may find a log canonical pair $(S, \Theta)$ where $S$ is a projective variety of dimension at most $n$, the coefficients of $\Theta$ belong to $D(I)$, $K_S + \Theta$ is numerically trivial and some component of $\Theta$ has coefficient*

$$\frac{m - 1 + f + kc}{m},$$

*where $m$, $k \in \mathbb{N}$ and $f \in D(I)$.*

*Proof.* Possibly passing to an open subset of $X$ and replacing $V$ by a maximal (with respect to inclusion) non kawamata log terminal centre, we may assume that $X$ is quasi-projective. If $V$ is a divisor then $M = V$ is a component of $\Delta$ with coefficient one so that $c = 1$. As $1 \in I$ we may take $(S, \Theta) = (\mathbb{P}^1, p + q)$, where $p$ and $q$ are two points of $\mathbb{P}^1$.

Otherwise, let $\pi \colon Y \longrightarrow X$ be a divisorially log terminal modification of $(X, \Delta)$. Then $Y$ is $\mathbb{Q}$-factorial and we may write

$$K_Y + E + \Gamma = \pi^*(K_X + \Delta),$$

where $\Gamma$ is the strict transform of $\Delta$, $E$ is the sum of the exceptional divisors and the pair $(Y, E + \Gamma)$ is divisorially log terminal. By (4) of (3.3.1) we may choose $\pi$ so that there is a nef divisor of the form

$-N - F$, where $N$ is the strict transform of $M$ and $F \geq 0$ is a sum of exceptional divisors whose centres are contained in $M$.

By assumption $\pi$ is not an isomorphism over the generic point of $V$. It follows that $N$ must intersect an exceptional divisor $S$ of $\pi$ whose centre is $V$. We may write

$$(K_Y + E + \Gamma)|_S = K_S + \Theta,$$

by adjunction, where $(S, \Theta)$ is divisorially log terminal, the coefficients of $\Theta$ belong to $D(I)$ and some component of $\Theta$ has a coefficient of the form

$$\frac{m - 1 + f + kc}{m},$$

where $m$, $k \in \mathbb{N}$ and $f \in D(I)$. Note that $N \cap S$ dominates $V$. If $v \in V$ is a general point then $(S_v, \Theta_v)$ is divisorially log terminal, $S_v$ is projective of dimension at most $n$, the coefficients of $\Theta_v$ belong to $D(I)$, some component of $\Theta_v$ has a coefficient of the form

$$\frac{m - 1 + f + kc}{m},$$

and $K_{S_v} + \Theta_v$ is numerically trivial. $\qquad\square$

**Lemma 5.2.** *Let $I \subset [0, 1]$ be a set which satisfies the DCC.*
*If $J_0 \subset [0, 1]$ is a finite set then*

$$I_0 = \left\{\, c \in I \,\middle|\, \frac{m - 1 + f + kc}{m} \in J_0, \text{ for some } k,\, m \in \mathbb{N} \text{ and } f \in D(I) \,\right\}$$

*is a finite set.*

*Proof.* We may assume that $c \neq 0$. Suppose that

$$l = \frac{m - 1 + f + kc}{m} \in J_0.$$

Then $kc \leq 1$. As $I$ satisfies the DCC, we may find $\delta > 0$ such that $c > \delta$. It follows that $k < 1/\delta$ so that $k$ can take on only finitely many values. As $J_0$ is finite, we may find $\epsilon > 0$ such that if $l < 1$ then $l < 1 - \epsilon$. But then $m < \frac{1}{\epsilon}$. If $l = 1$ then $f + kc = 1$, in which case we may take $m = 1$. Either way, we may assume that $m$ takes on only finitely many values.

Fix $k$, $m$ and $l$. Then

$$c = \frac{(ml - m + 1) - f}{k}.$$

The LHS belongs to $I$, a set which satisfies the DCC. The RHS belongs to a set which satisfies the ACC. But the only set which satisfies both the DCC and the ACC is a finite set. $\qquad\square$

24

**Lemma 5.3.** *Theorem $D_{n-1}$ implies Theorem $A_n$.*

*Proof.* As $I$ satisfies the DCC so does $J = D(I)$. As we are assuming Theorem $D_{n-1}$, there is a finite set $J_0 \subset J$ such that if $(S, \Theta)$ is a log canonical pair where $S$ is projective of dimension at most $n-1$, the coefficients of $\Theta$ belong to $J$ and $K_S + \Theta$ is numerically trivial, then the coefficients of $\Theta$ belong to $J_0$. Let

$$I_0 = \{ c \in I \mid \frac{m - 1 + f + kc}{m} \in J_0 \text{ for some } k \text{ and } m \in \mathbb{N} \text{ and } f \in I_+ \}.$$

As $J_0$ is a finite set, (5.2) implies that $I_0$ is also a finite set.

Suppose that $(X, \Delta)$ is a log canonical pair where $X$ is a quasi-projective variety of dimension $n$, the coefficients of $\Delta$ belong to $I$, and there is a non kawamata log terminal centre $Z \subset X$ which is contained in every component of $\Delta$. (5.1) implies that the coefficients of $\Delta$ belong to $I_0$. $\qquad \square$

## 6. Upper bounds on the volume

**Lemma 6.1.** *Using the notation of Theorem $B_n$, Theorem $D_{n-1}$ and Theorem $A_{n-1}$ imply that there is a constant $\epsilon > 0$ with the following property:*

*If $(X, \Delta) \in \mathfrak{D}$, where $X$ has dimension $n$, $\Delta$ is big and $K_X + \Phi$ is numerically trivial, where*

$$\Phi \geq (1 - \delta)\Delta,$$

*for some $\delta < \epsilon$, then $(X, \Phi)$ is kawamata log terminal.*

*Proof.* Theorem $D_{n-1}$ and Theorem $A_{n-1}$ imply that we may find $\epsilon > 0$ with the following property: if $S$ is a projective variety of dimension $n - 1$, $(S, \Theta)$ and $(S, \Theta')$ are two log pairs, the coefficients of $\Theta$ belong to $D(I)$, and

$$(1 - \epsilon)\Theta \leq \Theta' \leq \Theta,$$

then $(S, \Theta)$ is log canonical if $(S, \Theta')$ is log canonical, and moreover $\Theta = \Theta'$ if in addition $K_S + \Theta'$ is numerically trivial.

Suppose that $(X, \Phi)$ is not kawamata log terminal, where

$$\Phi \geq (1 - \delta)\Delta,$$

for some $\delta < \epsilon$ and $K_X + \Phi$ is numerically trivial. As $\delta < \epsilon$ and $\Phi$ is big we may assume that $K_X + \Phi$ is not log canonical. Pick $\lambda \in (0, 1]$ such that $(X, (1 - \lambda)\Delta + \lambda\Phi)$ is log canonical but not kawamata log terminal. As $\Phi$ is big, $\delta < \epsilon$ and $(X, \Delta)$ is kawamata log terminal, (3.2.3) implies that, perturbing $\Phi$, we may assume $(X, (1 - \lambda)\Delta + \lambda\Phi)$ has only one non kawamata log terminal place.

Replacing $\Phi$ by $(1 - \lambda)\Delta + \lambda\Phi$ we may assume that $(X, \Phi)$ is purely log terminal and the non kawamata log terminal locus is irreducible. Let $\phi\colon Y \longrightarrow X$ be a divisorially log terminal modification of $(X, \Phi)$. We may write

$$K_Y + \Psi = \phi^*(K_X + \Phi) \qquad \text{and} \qquad K_Y + \Gamma + aS = \phi^*(K_X + \Delta),$$

where $S = \lfloor \Psi \rfloor$ is a prime divisor, $\Gamma$ is the strict transform of $\Delta$ and $a < 1$, as $(X, \Delta)$ is kawamata log terminal.

As $K_Y + \Psi$ is numerically trivial, $K_Y + \Psi - S$ is not pseudo-effective. By [8, 1.3.3], we may run $f\colon Y \dashrightarrow W$ the $(K_Y + \Psi - S)$-MMP until we end with a Mori fibre space $\pi\colon W \longrightarrow Z$. As $K_Y + \Psi$ is numerically trivial, every step of this MMP is $S$-positive, so that the strict transform $T$ of $S$ dominates $Z$. Let $F$ be the general fibre of $\pi$. Replacing $Y$, $\Gamma$ and $\Psi$ by $F$ and the restriction of $\pi_*\Gamma$ and $\pi_*\Psi$ to $F$, we may assume that $S$, $\Psi$ and $\Gamma$ are $\mathbb{Q}$-linearly equivalent to multiples of the same ample divisor.

In particular $K_Y + \Gamma + S$ is ample. As $\Psi \geq (1 - \epsilon)\Gamma + S$, it follows that $K_Y + (1 - \eta)\Gamma + S$ is numerically trivial, for some $0 < \eta < \epsilon$, and $K_Y + (1 - \epsilon)\Gamma + S$ is log canonical. We may write

$$\begin{aligned}
(K_Y + (1 - \epsilon)\Gamma + S)|_S &= K_S + \Theta_1, \\
(K_Y + (1 - \eta)\Gamma + S)|_S &= K_S + \Theta_2, \quad \text{and} \\
(K_Y + \Gamma + S)|_S &= K_S + \Theta,
\end{aligned}$$

where the coefficients of $\Theta$ belong to $D(I)$. Note that

$$(1 - \epsilon)\Theta \leq \Theta_1 \leq \Theta_2 \leq \Theta,$$

where by (4.1) the first inequality follows from the inequality

$$t\left(\frac{m - 1 + f}{m}\right) \leq \frac{m - 1 + tf}{m} \qquad \text{for any} \qquad t \leq 1.$$

As $(S, \Theta_1)$ is log canonical, it follows that $(S, \Theta)$ is log canonical. In particular $(S, \Theta_2)$ is also log canonical. As $K_S + \Theta_2$ is numerically trivial, $\Theta = \Theta_2$, a contradiction. $\qquad \square$

**Lemma 6.2.** *Theorem $D_{n-1}$ and Theorem $A_{n-1}$ imply Theorem $B_n$.*

*Proof.* Let $\epsilon > 0$ be the constant given by (6.1). If $(X, \Delta) \in \mathfrak{D}$, $\Delta$ is big, $\Pi \sim_{\mathbb{R}} \eta\Delta$ and $(X, \Pi + (1 - \eta)\Delta)$ is not kawamata log terminal, then (6.1) implies that $\eta \geq \epsilon$. But then (3.2.2) implies that
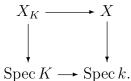
$$\mathrm{vol}(X, \Delta) \leq \left(\frac{n}{\epsilon}\right)^n. \qquad\qquad \square$$

## 7. Birational boundedness

**Lemma 7.1.** *Let $(X, \Delta)$ be a log pair, where $X$ is a projective variety of dimension $n$ and let $D$ be a big $\mathbb{R}$-divisor.*

*If $\mathrm{vol}(X, D) > (2n)^n$ then there is a family $V \longrightarrow B$ of subvarieties of $X$ such that if $x$ and $y$ are two general points of $X$ then we may find $b \in B$ and $0 \le \Delta_b \sim_{\mathbb{R}} D$ such that $(X, \Delta + \Delta_b)$ is not kawamata log terminal at $y$ and there is a unique non kawamata log terminal place of $(X, \Delta + \Delta_b)$ whose centre $V_b$ contains $x$. Further, if $B_1, B_2, \ldots, B_k$ are the irreducible components of $B$ and $V_i \longrightarrow B_i$ is the corresponding family then the natural map $V_i \longrightarrow X$ is dominant.*

*Proof.* Let $K$ be the algebraic closure of the function field of $X$. There is a fibre square

$$
\begin{array}{ccc}
X_K & \longrightarrow & X \\
\downarrow & & \downarrow \\
\mathrm{Spec}\,K & \longrightarrow & \mathrm{Spec}\,k.
\end{array}
$$

Let $\xi$ be the closed point of $X_K$ corresponding to the generic point of $X$, and let $\Delta_K$ and $D_K$ be the pullbacks of $\Delta$ and $D$ to $X_K$. (3.2.2) implies that we may find $0 \le D_\xi \sim_{\mathbb{R}} D_K/2$ such that $(X_K, \Delta_K + D_\xi)$ is not log canonical at $\xi$. By standard arguments we may spread out $D_\xi$ to a family of divisors $D_t$, $t \in T$, where there is dominant morphism $g \colon T \longrightarrow X$ such that $(X, \Delta + D_t)$ is not log canonical at $x = g(t)$ and where $D_t \sim_{\mathbb{R}} D/2$.

Let $y$ be a general point of $X$. Pick $s$ such that $(X, \Delta + D_s)$ is not log canonical at $y = g(s)$, where $D_s \sim_{\mathbb{R}} D/2$. Let

$$
\beta = \beta_{s,t} = \sup\{\, \lambda \in \mathbb{R} \mid (X, \Delta + \lambda(D_t + D_s)) \text{ is log canonical at } x \,\},
$$

be the log canonical threshold. Thus $(X, \Delta + \beta(D_s + D_t))$ is log canonical but not kawamata log terminal at $x$. Possibly switching $s$ and $t$, we may assume that $(X, \Delta + \beta(D_s + D_t))$ is not kawamata log terminal at $y$. Perturbing, by (3.2.3) we may assume that there is a unique non kawamata log terminal place of $(X, \Delta + \beta(D_t + D_s))$ whose centre $V_{(s,t)}$ contains $x$ (as $y$ is general, we will not lose the property that $(X, \Delta + \beta(D_t + D_s))$ is not kawamata log terminal at $y$). Decomposing $B = T \times T$ into finitely many locally closed subsets, we may assume that the log canonical threshold is constant on each irreducible component of $B$, and moreover that $V_{s,t}$ forms a family $V \longrightarrow B$. Possibly discarding components of $B$, we may assume that every component of $V$ dominates $X$. Then the image of $B$ in $X \times X$ contains an open subset of the form $U \times U$. $\qquad\square$

27

**Lemma 7.2.** *Assume Theorem $C_{n-1}$ and Theorem $A_{n-1}$. Fix a positive integer $p$.*

*Let $\mathfrak{B}_1$ be the set of kawamata log terminal pairs $(X, \Delta)$, where $X$ is projective of dimension n, $K_X + \Delta$ is big and either $p\Delta$ is integral or the coefficients of $\Delta$ belong to*

$$\{ \frac{r-1}{r} \mid r \in \mathbb{N} \}.$$

*Then there is a positive integer $m$ such that $\phi_{m(K_X+\Delta)}$ is birational, for every $(X, \Delta) \in \mathfrak{B}_1$.*

*Proof.* Passing to a log canonical model of $(X, \Delta)$ we may assume that $K_X + \Delta$ is ample.

Pick a positive integer $k$ such that $\mathrm{vol}(X, k(K_X + \Delta)) > (2n)^n$. We will apply (3.5.5) to $k(K_X + \Delta)$. (2) holds by hypothesis.

Let

$$J = \{ 1 - a \mid a \in \mathrm{LCT}_{n-1}(D(I)) \} \cup \{ 1 \}.$$

Theorem $A_{n-1}$ implies that $J$ satisfies the DCC.

Theorem $C_{n-1}$ implies that there is a positive integer $l$ such that if $(U, \Psi)$ is a log canonical pair, where $U$ is projective of dimension at most $n - 1$, the coefficients of $\Psi$ belong to $J$ and $K_U + \Psi$ is big, then $\phi_{l(K_U+\Psi)}$ is birational.

Apply (7.1) to $k(K_X + \Delta)$ to get a family $V \longrightarrow B$. Let $b \in B$ be a general point. Let $\nu\colon W \longrightarrow V_b$ be the normalisation of $V_b$ and let $0 \leq \Delta_b \sim_{\mathbb{R}} k(K_X + \Delta)$ be the divisor given by (7.1), so that $V_b$ is the unique non kawamata log terminal place of $(X, \Delta + \Delta_b)$ containing $x$. $(4.2)_n$ implies that we may find $\Theta$ on $W$ such that

$$(K_X + \Delta + \Delta_b)|_W - (K_W + \Theta),$$

is pseudo-effective, where the coefficients of $\Theta$ belong to $J$.

Let $\psi\colon U \longrightarrow W$ be a log resolution of $(W, \Theta)$ and let $\Psi$ be the sum of the strict transform of $\Theta$ and the exceptional divisors. $(4.2)_n$ implies that

$$(K_U + \Psi) \geq (K_X + \Delta)|_U,$$

so that $K_U + \Psi$ is big. As the coefficients of $\Theta$ belong to $J$, it follows that the coefficients of $\Psi$ belong to $J$. But then $\phi_{l(K_U+\Psi)}$ is birational. It is easy to see (1) of (3.5.5) holds.

As the hypotheses of (3.5.5) hold, there is a positive integer $m_0$ such that $\phi_{m_0k(K_X+\Delta)}$ is birational. If $\mathrm{vol}(X, K_X+\Delta) \geq 1$ then $\mathrm{vol}(X, 2(n+1)(K_X + \Delta)) > (2n)^n$ and $\phi_{2m_0(n+1)(K_X+\Delta)}$ is birational.

Otherwise, if $\mathrm{vol}(X, K_X + \Delta) < 1$, then we may find $k$ such that

$$(2n)^n < \mathrm{vol}(X, k(K_X + \Delta)) \leq (4n)^n.$$

It follows that

$$\operatorname{vol}(X, m_0 k(K_X + \Delta)) \leq (4m_0 n)^n.$$

(3.5.2) implies that there is a constant $0 < \delta < 1$ such that if $(X, \Delta) \in \mathfrak{B}$, then

$$\operatorname{vol}(X, K_X + \Delta) > \delta.$$

In this case,

$$\operatorname{vol}(X, \alpha(K_X + \Delta)) > (2n)^n,$$

where

$$\alpha = \frac{2n}{\delta},$$

and we may take $m = \max(m_0 \lceil \alpha \rceil, 2m_0(n+1))$. $\qquad \square$

**Lemma 7.3.** *Using the notation of Theorem $C_n$, assume Theorem $C_{n-1}$, Theorem $A_{n-1}$, and Theorem $B_n$.*

*Then there is a constant $\beta < 1$ such that if $(X, \Delta) \in \mathfrak{B}$ then the pseudo-effective threshold*

$$\lambda = \inf\{\, t \in \mathbb{R} \mid K_X + t\Delta \text{ is big}\,\},$$

*is at most $\beta$.*

*Proof.* We may assume that $1 \in I$. Suppose that $(X, \Delta) \in \mathfrak{B}$. Let $\pi \colon W \longrightarrow X$ be a log resolution of $(X, \Delta)$. We may write

$$K_W + \Xi = \pi^*(K_X + \Delta) + F,$$

where $\Xi$ is the strict transform of $\Delta$ plus the sum of the exceptional divisors and $F \geq 0$ is exceptional as $(X, \Delta)$ is log canonical. Let

$$\mu = \inf\{\, t \in \mathbb{R} \mid K_W + t\Xi \text{ is big}\,\}.$$

be the pseudo-effective threshold. As $\pi_*(K_W + \mu\Xi) = K_X + \mu\Delta$ is pseudo-effective it follows that $\lambda \leq \mu$ and so it suffices to bound $\mu$ away from one. Replacing $(X, \Delta)$ by $(W, \Xi)$ we may assume that $(X, \Delta)$ has simple normal crossings.

We may assume that $\lambda > 1/2$, so that $K_X$ is not pseudo-effective. As $K_X + \Delta$ is big we may find $0 \leq D \sim_{\mathbb{R}} (K_X + \Delta)$. If $\epsilon > 0$ then

$$(1 + \epsilon)(K_X + \lambda\Delta) \sim_{\mathbb{R}} K_X + \mu\Delta + \epsilon D,$$

where $\mu = (1 + \epsilon)\lambda - \epsilon < \lambda$. It follows that if $\epsilon$ is sufficiently small then $K_X + \mu\Delta + \epsilon D$ is kawamata log terminal. By [8, 1.4.2], we may run $f \colon X \dashrightarrow Y$ the $(K_X + \lambda\Delta)$-MMP with scaling until $K_Y + \Gamma$ is kawamata log terminal and nef, where $\Gamma = f_*(\lambda\Delta)$. Now we may run the $(K_Y + \mu f_*\Delta)$-MMP with scaling of $f_*D$ until we get to a Mori fibre space $\pi \colon Y \longrightarrow Z$; all steps of this MMP are $(K_Y + \Gamma)$-trivial, as all steps of this MMP are $(K_Y + \mu f_*\Delta + \epsilon f_*D)$-trivial, so that $(Y, \Gamma)$

remains kawamata log terminal and nef. Replacing $(X, \Delta)$ by a log resolution, we may assume that $f$ is a morphism. Replacing $X$ by the general fibre of the composition of $f$ and $\pi$, we may assume that $Z$ is a point, so that $K_Y + \Gamma$ is numerically trivial.

Suppose that we have a sequence of such log pairs $(X_l, \Delta_l) \in \mathfrak{B}$. We may assume that the pseudo-effective threshold is an increasing sequence,

$$\lambda_1 < \lambda_2 < \lambda_3 < \dots,$$

and it suffices to bound this sequence away from one. Let

$$J = \{ \lambda_l i \, | \, i \in I, l \in \mathbb{N} \}.$$

Then $J$ satisfies the DCC, as $\lambda_l$ are an increasing sequence.

Theorem $B_n$ implies that there is a constant $C$ such that $\mathrm{vol}(Y, \Gamma) < C$ for any $\Gamma$ whose coefficients belong to $J$. Let $\alpha$ be the smallest non-zero element of $J$ and let $G = G_l$ be the sum of the components of $\Gamma = \Gamma_l$. Let $Y = Y_l$. Then

$$\begin{aligned}
\mathrm{vol}(Y, K_Y + G) &= \mathrm{vol}(Y, G - \Gamma) \\
&\leq \mathrm{vol}(Y, G) \\
&\leq \mathrm{vol}(Y, \frac{1}{\alpha}\Gamma) \\
&\leq \frac{C}{\alpha^n}.
\end{aligned}$$

Let $D$ be the sum of the components of $\Delta$. Certainly $K_X + D$ is big. We may write

$$K_X + D = f^*(K_Y + G) + F,$$

where $F$ is supported on the exceptional locus. It follows that

$$\mathrm{vol}(X, K_X + D) \leq \mathrm{vol}(Y, K_Y + G) \leq \frac{C}{\alpha^n}.$$

Given $(X_l, D_l)$ we may pick $r \in \mathbb{N}$ such that

$$K_{X_l} + \Theta_l = K_{X_l} + \frac{r-1}{r} D_l$$

is big. As the coefficients of $\Theta_l$ belong to

$$\{ \frac{r-1}{r} \, | \, r \in \mathbb{N} \},$$

(7.2) implies that

$$\{ (X_l, \Theta_l) \, | \, l \in \mathbb{N} \},$$

is log birationally bounded. But then

$$\{ (X_l, \Delta_l) \, | \, l \in \mathbb{N} \},$$

30

is log birationally bounded. In particular, [15, 1.9] implies that there is a constant $\delta > 0$ such that

$$\mathrm{vol}(X_l, K_{X_l} + \Delta_l) \geq \delta,$$

for every $l \in \mathbb{N}$. In this case

$$\delta \leq \mathrm{vol}(X, K_X + \Delta) \leq \mathrm{vol}(Y, K_Y + \frac{1}{\lambda}\Gamma) = (\frac{1}{\lambda} - 1)^n \,\mathrm{vol}(Y, \Gamma) \leq (\frac{1}{\lambda} - 1)^n C,$$

so that we may take

$$\beta = \frac{1}{1 + \left(\frac{\delta}{C}\right)^{1/n}}. \qquad \square$$

**Lemma 7.4.** *Theorem $C_{n-1}$, Theorem $A_{n-1}$, and Theorem $B_n$ imply Theorem $C_n$.*

*Proof.* Replacing $I$ by

$$I \cup \{ \frac{r-1}{r} \mid r \in \mathbb{N} \} \cup \{1\},$$

we may assume that 1 is both an accumulation point of $I$ and an element of $I$. Let $\alpha$ be the smallest non-zero element of $I$. By (7.3) there is a constant $\beta < 1$ such that if $(X, \Delta) \in \mathfrak{B}$ then the pseudo-effective threshold

$$\lambda = \inf\{ t \in \mathbb{R} \mid K_X + t\Delta \text{ is big} \},$$

is at most $\beta$.

Pick $(X, \Delta) \in \mathfrak{B}$. Let $\pi \colon Y \longrightarrow X$ be a log resolution of $(X, \Delta)$. Then we may write

$$K_Y + \Gamma = \pi^*(K_X + \Delta) + E,$$

where $\Gamma$ is the strict transform of $\Delta$ plus the sum of the exceptional divisors. Replacing $(X, \Delta)$ by $(Y, \Gamma)$ we may assume that $(X, \Delta)$ is log smooth. If $S = \lfloor \Delta \rfloor$, then we may pick $r \in \mathbb{N}$ such that

$$K_X + \Delta' = K_X + \frac{r-1}{r}S + \{\Delta\},$$

is big. Replacing $(X, \Delta)$ by $(X, \Delta')$, we may assume that $(X, \Delta)$ is kawamata log terminal.

Pick $p$ such that

$$p > \frac{2}{\alpha(1 - \beta)}.$$

31

If $a$ is the coefficient of a component of $\Delta$ then

$$\frac{\lfloor pa \rfloor}{p} > a - \frac{1}{p}$$
$$> a - \frac{\alpha(1-\beta)}{2}$$
$$\geq a - \frac{a(1-\beta)}{2}$$
$$= \frac{a(1+\beta)}{2}.$$

It follows that

$$\frac{\beta+1}{2}\Delta \leq \Delta_{\lfloor p \rfloor} \leq \Delta,$$

so that $K_X + \Delta_{\lfloor p \rfloor}$ is big. Since the coefficients of $\Delta_{\lfloor p \rfloor}$ belong to

$$I_0 = \{\,\frac{i}{p} \mid 1 \leq i \leq p-1\,\},$$

(7.2) implies that there is a positive integer $m$ such that $\phi_{m(K_X+\Delta_{\lfloor p \rfloor})}$ is birational. But then $\phi_{m(K_X+\Delta)}$ is birational as well. $\qquad\square$

## 8. Numerically trivial log pairs

**Lemma 8.1.** *Theorem $D_{n-1}$ and Theorem $C_n$ imply Theorem $D_n$.*

*Proof.* We may assume that $1 \in I$ and $n > 1$.

As we are assuming Theorem $D_{n-1}$ there is a finite set $J_0 \subset J = D(I)$ with the following property. If $(S, \Theta)$ is a log pair such that $S$ is projective of dimension $n-1$, the coefficients of $\Theta$ belong to $J$, $(S, \Theta)$ is log canonical, and $K_S + \Theta$ is numerically trivial, then the coefficients of $\Theta$ belong to $J_0$. Let $I_1$ be the largest subset of $I$ such that $D(I_1) \subset J_0$. (5.2) implies that $I_1$ is finite.

Theorem $C_n$ implies that there is a constant $m$ with the following property: if $(Y, \Gamma)$ is log canonical, $Y$ is a projective variety of dimension $n$, $K_Y + \Gamma$ is big and the coefficients of $\Gamma$ belong to $I$, then $\phi_{m(K_Y+\Gamma)}$ is birational.

For every $1 \leq l \leq m$, let

$$A_l = [(l-1)/m, l/m),$$

and $A_{m+1} = \{\,1\,\}$ so that

$$[0,1] = \bigcup_{l=1}^{m+1} A_l.$$

32

Let $I_2$ be the union of the largest elements of $A_l \cap I$ (if $A_l \cap I$ does not have a largest element, either because it is empty or because it has infinitely many elements, then we ignore the elements of $A_l \cap I$). Then $I_2$ has at most $m + 1$ elements, so that $I_2$ is certainly finite. Let $I_0$ be the union of $I_1$ and $I_2$.

Suppose that $(X, \Delta)$ satisfies (1–4) of Theorem $D_n$. Let $\pi\colon Y \longrightarrow X$ be a divisorially log terminal modification, so that $Y$ is $\mathbb{Q}$-factorial. As $(X, \Delta)$ is log canonical if we write

$$K_Y + \Gamma = \pi^*(K_X + \Delta),$$

then $\Gamma$ is the strict transform of $\Delta$ plus the exceptional divisors, so that $(Y, \Gamma)$ is numerically trivial and divisorially log terminal. Replacing $(X, \Delta)$ by $(Y, \Gamma)$, we may assume that $X$ is $\mathbb{Q}$-factorial. Further $(X, \Delta)$ is kawamata log terminal if and only if $\lfloor \Delta \rfloor = 0$. Suppose that $B$ is a prime component of $\Delta$ with coefficient $i$. It suffices to prove that $i \in I_0$. We may assume that $i \neq 1$. Suppose that $B$ intersects a component of $\lfloor \Delta \rfloor$. If $S$ is the normalisation of this component then by adjunction we may write

$$(K_X + \Delta)|_S = K_S + \Theta,$$

where the coefficients of $\Theta$ belong to $J = D(I)$ by (4.1). As $S$ is projective of dimension $n - 1$, $(S, \Theta)$ is log canonical, and $K_S + \Theta$ is numerically trivial, the coefficients of $\Theta$ belong to $J_0$. But then $i \in I_1$.

As $K_X + \Delta$ is numerically trivial, $K_X + \Delta - iB$ is not pseudo-effective. By [8, 1.3.3] we may run $f\colon X \dashrightarrow Y$ the $(K_X + \Delta - iB)$-MMP until we reach a Mori fibre space. As $K_X + \Delta$ is numerically trivial, it follows that every step of this MMP is $B$-positive. If at some step of this MMP we contract a component $S$ of $\lfloor \Delta \rfloor$ then this component intersects $B$ and $i \in I_1$ by the argument above. Otherwise, it follows that $(Y, f_*\Delta)$ is kawamata log terminal if and only if $\lfloor f_*\Delta \rfloor = 0$. Further $B$ is not contracted and so replacing $(X, \Delta)$ by $(Y, f_*\Delta)$, we may assume that $X$ is a Mori fibre space $\pi\colon X \longrightarrow Z$, where $B$ dominates $Z$.

If $Z$ is not a point, then replacing $X$ by the general fibre of $\pi$ we are done by induction. So we may assume that $X$ has Picard number one. If $\lfloor \Delta \rfloor \neq 0$ then any component $S$ of $\lfloor \Delta \rfloor$ intersects $B$ and so $i \in I_1$. Otherwise $\lfloor \Delta \rfloor = 0$ and we may assume that $(X, \Delta)$ is kawamata log terminal.

Suppose that $j \in I$ and $j > i$. Let $\pi\colon Y \longrightarrow X$ be a log resolution of $(X, \Delta)$. Let $\Gamma_0$ be the strict transform of $\Delta$, let $E$ by the sum of the exceptional divisors, and let $C$ be the strict transform of $B$. Set

$$\Gamma = \Gamma_0 + E + (j - i)C.$$

Then $(Y, \Gamma)$ is log canonical and the coefficients of $\Gamma$ belong to $I$. We may write
$$K_Y + \Gamma_0 + E = \pi^*(K_X + \Delta) + F,$$
where $F \geq 0$ contains the full exceptional locus. Pick $\epsilon > 0$ such that $F \geq \epsilon E$. Note that $(j - i)C + \epsilon E > \delta \pi^* B$ for any $\delta > 0$ sufficiently small, so that
$$K_Y + \Gamma = (K_Y + \Gamma_0 + (1 - \epsilon)E) + (j - i)C + \epsilon E,$$
is big. Hence $\phi_{m(K_Y + \Gamma)}$ is birational, so that $K_Y + \Gamma_{\lfloor m \rfloor}$ is big. But then $K_X + \Lambda_{\lfloor m \rfloor}$ is big, where
$$\Lambda = \pi_* \Gamma = \Delta + (j - i)B.$$
It follows that if $i \in A_l$, then $j \geq l/m$, so that $i$ is the largest element of the interval $A_l$ which also belongs to $I$. Hence $i \in I_2$. $\qquad \square$

## 9. PROOFS OF THEOREMS

*Proof of* (1.5) *and* (1.4). This is Theorem A and Theorem D. $\qquad \square$

*Proof of* (1.1). Suppose that $c_1, c_2, \ldots \in \mathrm{LCT}_n(I, J)$, where $c_i \leq c_{i+1}$. It suffices to show that $c_i = c_{i+1}$ for $i$ sufficiently large. By assumption we may find log canonical pairs $(X_i, \Delta_i)$ and $\mathbb{R}$-Cartier divisors $M_i$, where $X_i$ is a variety of dimension $n$, the coefficients of $\Delta_i$ belong to $I$, the coefficients of $M_i$ belong to $J$ and $c_i$ is the log canonical threshold,
$$c_i = \sup\{\, t \in \mathbb{R} \mid (X_i, \Delta_i + c_i M_i) \text{ is log canonical}\,\}.$$
Let $\Theta_i = \Delta_i + c_i M_i$ and
$$K = I \cup \{\, c_i j \mid i \in \mathbb{N}, j \in J \,\}.$$
Then $(X_i, \Theta_i)$ is log canonical, $X_i$ is a variety of dimension $n$, the coefficients of $\Theta_i$ belong to $K$ and there is a non kawamata log terminal centre $V$ contained in the support of $M_i$. Possibly throwing away components of $\Theta_i$ which don't contain $V$ and passing to an open subset which contains the generic point of $V$, we may assume that every component of $\Theta_i$ contains $V$.

As $K$ satisfies the DCC, (1.5) implies that the coefficients of $\Theta_i$ belong to a finite subset $K_0$ of $K$. It follows $c_i = c_{i+1}$ for $i$ sufficiently large. $\qquad \square$

*Proof of* (1.3). (3) is Theorem C.

Fix a constant $V > 0$ and let
$$\mathfrak{D}_V = \{\, (X, \Delta) \in \mathfrak{D} \mid 0 < \mathrm{vol}(X, K_X + \Delta) \leq V \,\}.$$

(3) implies that $\phi_{m(K_X+\Delta)}$ is birational. (3.5.2) implies that the set

$$\{\,\mathrm{vol}(X, K_X + \Delta)\,|\,(X, \Delta) \in \mathfrak{D}_V\,\},$$

satisfies the DCC, which implies that (1) and (2) of (1.3) hold in dimension $n$. $\qquad\square$

**Lemma 9.1.** *Let $Z \longrightarrow T$ be a projective morphism to a variety and suppose that $(Z, \Phi)$ has simple normal crossings over $T$. Suppose that the restriction of any irreducible component of $\Phi$ to any fibre is irreducible. Suppose that $(Z, \Phi)$ is kawamata log terminal and there is a closed point $0 \in T$ such that $K_{Z_0} + \Phi_0$ is big. Let $0 \le \Theta \le \Phi$ be any divisor with the same support as $\Phi$.*

*Then we may find finitely many birational contractions $f_i\colon Z \dashrightarrow X_i$ over $T$ such that if $f\colon Z_t \dashrightarrow Y$ is the log canonical model of $(Z_t, \Psi)$ for some $t \in T$ and $\Theta_t \le \Psi \le \Phi_t$ then $f = f_{it}$ for some index $i$.*

*Proof.* [15, 1.7] implies that $K_Z + \Phi$ is big over $T$. Pick

$$0 \le D \sim_{\mathbb{R},T} (K_Z + \Phi).$$

Let

$$B = \frac{\epsilon}{1 - \epsilon} D.$$

If we pick $\epsilon > 0$ sufficiently small then $K_Z + B + \Phi$ is kawamata log terminal and we may find a divisor $0 \le \Theta' \le \Theta$ with

$$K_Z + \Theta = \epsilon(K_Z + \Phi) + (1 - \epsilon)(K_Z + \Theta').$$

If $\Theta \le \Xi \le \Phi$ then

$$K_Z + \Xi \sim_{\mathbb{R},T} (1 - \epsilon)(K_Z + B + \Xi'),$$

where $\Theta' \le \Xi' \le \Xi$. It is proved in [8, 1.1.5] that there are finitely many $f_1, f_2, \ldots, f_k$ birational contractions $f_i\colon Z \dashrightarrow X_i$ over $T$ such that if $g\colon Z \dashrightarrow X$ is the log canonical model of $K_Z + \Xi$ over $T$ then $g = f_i$ for some index $1 \le i \le k$.

It suffices to show that if $\Xi|_{Z_t} = \Psi$ and $g$ is the log canonical model of $K_Z + \Xi$ then $f = g_t$. For this we may assume that $T$ is affine.

In this case the (relative) log canonical model is given by taking Proj

$$X_i = \mathrm{Proj}(Z, R(Z, k(K_Z + \Xi))),$$

of the (truncation of the) canonical ring

$$R(Z, k(K_Z + \Xi)) = \bigoplus_{m \in \mathbb{N}} H^0(Z, \mathcal{O}_Z(mk(K_Z + \Xi))).$$

On the other hand [15, 1.7] implies that if $k$ is sufficiently divisible then

$$R(Z, k(K_Z + \Xi)) \longrightarrow R(Z_t, k(K_{Z_t} + \Psi)),$$

35

is surjective and so $f = g_t$. $\hfill\square$

*Proof of* (1.6). By definition there is a log pair $(Z, B)$ and a projective morphism $Z \longrightarrow T$, where $T$ is of finite type with the following property. If $(X, \Delta) \in \mathfrak{D}$ then there is a closed point $t \in T$ and a birational map $f\colon X \dashrightarrow Z_t$ such that the support of $B_t$ is a divisor on $Z_t$ which contains the support of the strict transform of $\Delta$ and any $f^{-1}$-exceptional divisor.

We may assume that $T$ is reduced. Decomposing $T$ into a finite union of locally closed subsets and throwing away some components, we may assume that every fibre $Z_t$ is a variety and that $B$ does not contain $Z_t$; blowing up and decomposing $T$ into a finite union of locally closed subsets, we may assume that $(Z, B)$ has simple normal crossings; passing to an open subset of $T$, we may assume that the fibres of $Z \longrightarrow T$ are log pairs, so that $(Z, B)$ has simple normal crossings over $T$; passing to a finite cover of $T$, we may assume that every stratum of $(Z, B)$ has irreducible fibres over $T$; decomposing $T$ into a finite union of locally closed subsets, we may assume that $T$ is smooth; finally passing to a connected component of $T$, we may assume that $T$ is integral.

Let $a = 1 - \epsilon < 1$. By assumption $\delta \leq a \leq 1$. Let $\Phi = aB$ and $\Theta = \delta B$, so that $\Phi$, $\Theta$ and $B$ have the same support but the coefficients of $\Phi$ are all $a$, the coefficients of $\Theta$ are all $\delta$ and the coefficients of $B$ are all one. As $(Z, \Phi)$ is kawamata log terminal it follows that there are only finitely many valuations of log discrepancy at most one with respect to $(Z, \Phi)$. As $(Z, \Phi)$ has simple normal crossings there is a sequence of blow ups $Y \longrightarrow Z$ of strata, which extracts every divisor of log discrepancy at most one. Note that as $(Z, \Phi)$ has simple normal crossings over $T$, it follows that if $t \in T$ is a closed point then every valuation of log discrepancy at most one with respect to $(Z_t, \Phi_t)$ has centre a divisor on $Y_t$.

Suppose that $(X, \Delta) \in \mathfrak{D}$. Then there is a closed point $t \in T$ and a birational map $f\colon X \dashrightarrow Z_t$ such that the support of $B_t$ contains the support of the strict transform of $\Delta_t$ and any $f^{-1}$-exceptional divisor. Let $p\colon W \longrightarrow X$ and $q\colon W \longrightarrow Z_t$ resolve $f$. Let $S$ be the sum of the $p$-exceptional divisors and let $\Xi$ be the sum of the strict transform of $\Delta$ and $aS$, so that $S$ and $\Xi$ are divisors on $W$. We may write

$$K_W + \Xi = p^*(K_X + \Delta) + E,$$

where $E$ is a sum of $p$-exceptional divisors and $E \geq 0$ as the log discrepancy of $(X, \Delta)$ is greater than $\epsilon$.

Let $\Psi = q_* \Xi$. We may write

$$p^*(K_X + \Delta) + E + F = q^*(K_{Z_t} + \Psi),$$

where $F$ is $q$-exceptional. As $p^*(K_X + \Delta)$ is nef, it is $q$-nef so that $E + F \geq 0$ by negativity of contraction. If $\nu$ is any valuation whose centre is a divisor on $X$ then

$$
\begin{aligned}
a(Z_t, \Phi_t, \nu) &\leq a(Z_t, \Psi, \nu) && \text{as } \Phi_t \geq \Psi \\
&\leq a(X, \Delta, \nu) && \text{as } E + F \geq 0 \\
&\leq 1 && \text{as the centre of } \nu \text{ is a divisor on } X.
\end{aligned}
$$

Therefore the induced birational map $Y_t \dashrightarrow X$ is a birational contraction. Thus replacing $Z$ by $Y$ and $B$ by its strict transform union the exceptional divisor, we may assume that $g = f^{-1} \colon Z_t \dashrightarrow X$ is a birational contraction. In this case $F$ is $p$-exceptional and so $g$ is the log canonical model of $(Z_t, \Theta_t)$.

Since there are only finitely integral divisors $0 \leq B' \leq B$, replacing $B$ we may assume that $\Psi$ has the same support as $B_t$. $K_{Z_t} + \Phi_t$ is big as $K_{Z_t} + \Psi$ is big and $\Phi_t \geq \Psi$. Finally $\Theta_t \leq \Psi \leq \Phi_t$ and so we are done by (9.1). $\qquad \square$

## 10. Proofs of Corollaries

*Proof of* (1.2). This follows from (1.1) and the main result of [6]. $\quad \square$

*Proof of* (1.7). (1.5) implies that there is a finite subset $I_0 \subset I$ such that the coefficients of $\Delta$ belong to $I_0$. Thus there is a positive integer $r$ such that $r\Delta$ is integral.

On the other hand, Theorem B implies that there is a constant $C$ such that $\mathrm{vol}(X, \Delta) < C$. Let $D$ be the sum of the components of $\Delta$. Then $K_X + D$ is big and

$$
\begin{aligned}
\mathrm{vol}(X, K_X + D) &= \mathrm{vol}(X, D - \Delta) \\
&\leq \mathrm{vol}(X, D) \\
&\leq \mathrm{vol}(X, r\Delta) \\
&\leq Cr^n.
\end{aligned}
$$

Let $\pi \colon Y \longrightarrow X$ be a log resolution of $(X, \Delta)$. Let $G$ be the sum of the strict transform of the components of $\Delta$ and the exceptional divisors. Then $(Y, G)$ has simple normal crossings. Pick $\eta > 0$ such that $(X, (1+\eta)\Delta)$ is kawamata log terminal and the log discrepancy is greater than $\epsilon$. Then $K_X + (1+\eta)\Delta$ is ample and we may write

$$
K_Y + \Gamma = \pi^*(K_X + (1+\eta)\Delta),
$$

where $\Gamma \leq G$. As $K_Y + \Gamma$ is big it follows that $K_Y + G$ is big. (1.3) implies that there is a positive integer $m$ such that $\phi_{m(K_Y+G)}$ is birational, for every $(X, \Delta) \in \mathfrak{D}$. But then $\mathfrak{D}$ is log birationally bounded by [15, 2.4.2.3-4]. Now apply (1.6). $\qquad\square$

*Proof of* (1.8). Let $D = -r(K_X + \Delta)$. Then $D$ is an ample Cartier divisor and $D - (K_X + \Delta)$ is ample. By Kollár's effective base point free theorem (cf. [21]), there is a fixed positive integer $m$ such that the linear system $|mD|$ is base point free. Pick a general divisor $H \in |mD|$. Then $(X, \Lambda = \Delta + \frac{1}{mr}H)$ is kawamata log terminal and

$$K_X + \Lambda \sim_{\mathbb{Q}} 0.$$

Note the coefficients of $\Lambda$ belong to the finite set

$$I = \{\, \frac{i}{r} \mid 1 \leq i \leq r - 1 \,\} \cup \{\, \frac{1}{mr} \,\}.$$

There are two ways to proceed. On the one hand we may apply (1.7).
    Here is a more direct approach. Theorem B implies that

$$\mathrm{vol}(X, \Lambda),$$

is bounded from above. But then

$$\mathrm{vol}(X, mD) \leq (mr)^n \, \mathrm{vol}(X, \Lambda),$$

is bounded from above. $\qquad\square$

*Proof of* (1.10). Suppose that $r_1 \leq r_2 \leq \ldots$ is a non-decreasing sequence in $R$. For each $i$ we may find $(X, \Delta) = (X_i, \Delta_i) \in \mathfrak{D}$ and a Cartier divisor $H$ such that $-(K_X + \Delta) \sim_{\mathbb{R}} rH$. By the cone theorem we may find a curve $C$ such that $-(K_X + \Delta) \cdot C \leq 2n$, cf. Theorem 18.2 of [13]. In particular $r \leq 2n$, as $H \cdot C \geq 1$. By Fujino's extension, [12], of Kollár's effective base point free theorem, [21], to the case of log canonical pairs, there is a fixed positive integer $m$ such that the linear system $|mH|$ is base point free. Possibly replacing $m$ by a multiple we may assume that $m > 2n$. Pick a general divisor $D \in |mH|$.
    Then $(X, \Lambda = \Delta + \frac{r}{m}D)$ is log canonical and

$$K_X + \Lambda \sim_{\mathbb{R}} 0.$$

Then the coefficients of $\Lambda_i = \Lambda$ belong to the set

$$I \cup \{\, \frac{r_i}{m} \mid i \in \mathbb{N} \,\},$$

which satisfies the DCC. (1.4) implies that the coefficients of $\Lambda$ belong to a finite subset. But then $r_i = r_{i+1}$ is eventually constant and so $R$ satisfies the ACC. $\qquad\square$

# 11. Accumulation points

**Definition 11.1.** *Given $I \subset [0,1]$ and $c \in [0,1]$ let*

$$D_c(I) = \left\{ a \leq 1 \,\middle|\, a = \frac{m - 1 + f + kc}{m}, k, m \in \mathbb{N}, f \in I_+ \right\} \subset D(I \cup \{c\}).$$

*Let $\mathfrak{N}_n(I, c)$ be the set of log canonical pairs $(X, \Delta)$ such that $X$ is a projective variety of dimension $n$, $K_X + \Delta$ is numerically trivial and we may write $\Delta = B + C$, where the coefficients of $B$ belong to $D(I)$ and the coefficients of $C \neq 0$ belong to $D_c(I)$.*

*Let*

$$N_n(I) = \{ c \in [0,1] \,|\, \mathfrak{N}_n(I, c) \text{ is non-empty} \}.$$

**Lemma 11.2.** *Let $n \in \mathbb{N}$ and $I \subset [0,1]$.*

*(1) $\mathrm{LCT}_n(I) \subset \mathrm{LCT}_{n+1}(I)$.*
*(2) $N_n(I) \subset N_{n+1}(I)$.*
*(3) If $f \in I_+$ and $k \in \mathbb{N}$ then*

$$c = \frac{1 - f}{k} \in N_n(I).$$

*Proof.* Let $E$ be an elliptic curve. If $(X, \Delta = \sum d_i \Delta_i)$ is a log pair then $(Y, \Gamma)$ is a log pair, where $Y = X \times E$ and $\Gamma = \sum d_i(\Delta_i \times E)$. By construction $\Gamma$ has the same coefficients as $\Delta$.

Note that $(X, \Delta)$ is log canonical if and only if $(Y, \Gamma)$ is log canonical. This gives (1). Further if $c \in [0,1]$ and $(X, \Delta) \in \mathfrak{N}_n(I, c)$ then $(Y, \Gamma) \in \mathfrak{N}_{n+1}(I, c)$. This is (2).

Using (2), it suffices to prove (3) when $n = 1$. Let $X = \mathbb{P}^1$ and $\Delta = B + C$, where $B = fp + fq$, $C = 2kcr$, and $p$, $q$ and $r$ are three points of $\mathbb{P}^1$. Then $(X, \Delta) \in \mathfrak{N}_1(I, c)$ (take $m = 1$) so that $c \in N_1(I)$. This is (3). $\qquad\square$

For technical reasons, it is convenient to introduce a smaller set than $\mathfrak{N}_n(I, c)$:

**Definition 11.3.** *Given $I \subset [0,1]$ and $c \in [0,1]$ let $\mathfrak{K}_n(I, c) \subset \mathfrak{N}_n(I, c)$ be the subset consisting of kawamata log terminal pairs $(X, \Delta)$, where $X$ is $\mathbb{Q}$-factorial of Picard number one.*

*Let*

$$K_n(I) = \{ c \in [0,1] \,|\, \mathfrak{K}_m(I, c) \text{ is non-empty, for some } m \leq n \}.$$

**Lemma 11.4.** *If $n \in \mathbb{N}$ and $I \subset [0,1]$ then*

$$N_n(I \cup \{1\}) = K_n(I).$$

*In particular, $N_n(I \cup \{1\}) = N_n(I)$.*

*Proof.* By (2) of (11.2), it suffices to show that

$$N_n(I \cup \{1\}) \subset K_n(I).$$

Suppose that $c \in N_n(I \cup \{1\})$. Then we may find $(X, \Delta) \in \mathfrak{N}_n(I \cup \{1\}, c)$. By assumption we may write $\Delta = A + B + C$, where the coefficients of $A$ are one, the coefficients of $B$ belong to $D(I)$ and the coefficients of $C \neq 0$ belong to $D_c(I)$.

Let $\pi \colon X' \longrightarrow X$ be a divisorially log terminal modification of $(X, \Delta)$. If we write

$$K_{X'} + \Delta' = \pi^*(K_X + \Delta),$$

then $X'$ is projective of dimension $n$, $X'$ is $\mathbb{Q}$-factorial, $(X', \Delta')$ is divisorially log terminal and $K_{X'} + \Delta'$ is numerically trivial. Let $B'$ and $C'$ be the strict transforms of $B$ and $C$ and let $A' = \Delta' - B' - C'$. Then the coefficients of $A'$ are one, the coefficients of $B'$ belong to $D(I)$ and the coefficients of $C' \neq 0$ belong to $D_c(I)$. Thus $(X', \Delta') \in \mathfrak{N}_n(I \cup \{1\}, c)$. Replacing $(X, \Delta)$ by $(X', \Delta')$ we may assume that $X$ is $\mathbb{Q}$-factorial and $(X, A + B)$ is divisorially log terminal. Note that $(X, \Delta)$ is kawamata log terminal if and only if $A = 0$.

Suppose that $A$ and $C$ intersect. Let $S$ be an irreducible component of $A$ which intersects $C$. Then we may write

$$(K_X + \Delta)|_S = K_S + \Theta,$$

by adjunction, where $(S, \Theta)$ is divisorially log terminal and moreover we may write $\Theta = A' + B' + C'$, where the coefficients of $A'$ are one, the coefficients of $B'$ belong to $D(I)$ and the coefficients of $C' \neq 0$ belong to $D_c(I)$. Thus $(S, \Theta) \in \mathfrak{N}_{n-1}(I \cup \{1\}, c)$. Hence $c \in N_{n-1}(I \cup \{1\})$ and so $c \in K_{n-1}(I) \subset K_n(I)$, by induction on $n$.

Let $f \colon X \dashrightarrow X'$ be a step of the $(K_X + A + B)$-MMP. As $K_X + \Delta$ is numerically trivial, $f$ is automatically $C$-positive. Suppose that $f$ is birational. Let $A' = f_*A$, $B' = f_*B$ and $C' = f_*C$, so that $\Delta' = f_*\Delta = A' + B' + C'$. $C' \neq 0$, as $f$ is $C$-positive. $X'$ is a projective variety of dimension $n$, $(X', \Delta')$ is log canonical, $K_{X'} + \Delta'$ is numerically trivial, the coefficients of $A'$ are all one, the coefficients of $B'$ belong to $D(I)$ and the coefficients of $C' \neq 0$ belong to $D_c(I)$. Thus $(X', \Delta') \in \mathfrak{N}_n(I \cup \{1\}, c)$. Further $X'$ is $\mathbb{Q}$-factorial and $(X', A' + B')$ is divisorially log terminal. If a component of $A$ is contracted then $A$ and $C$ intersect and we are done. Otherwise $(X', \Delta')$ is kawamata log terminal if and only if $A' = 0$.

If we run the $(K_X + A + B)$-MMP with scaling of an ample divisor then we end with a Mori fibre space. Therefore, replacing $(X, \Delta)$ by $(X', \Delta')$ finitely many times, we may assume that $f \colon X \dashrightarrow Z = X'$ is

a Mori fibre space and $C$ dominates $Z$. If $\dim Z > 0$ then let $z \in Z$ be a general point. Then $(X_z, \Delta_z) \in \mathfrak{N}_{n-k}(I \cup \{\, 1 \,\}, c)$, where $k = \dim Z$, and we are done by induction on the dimension.

So we may assume that $Z$ is a point in which case $X$ has Picard number one. If $A \neq 0$ then $A$ and $C$ intersect and we are done. If $A = 0$ then $(X, \Delta)$ is kawamata log terminal and so $(X, \Delta) \in \mathfrak{K}_n(I, c)$. But then $c \in K_n(I)$. $\hspace{1cm}\square$

**Proposition 11.5.** *If $I \subset [0,1]$, $I = I_+$ and $n \in \mathbb{N}$ then $\mathrm{LCT}_{n+1}(I) = N_n(I)$.*

*Proof.* We first show that $\mathrm{LCT}_{n+1}(I) \subset N_n(I)$. Pick $0 \neq c \in \mathrm{LCT}_{n+1}(I)$. By definition we may find a log canonical pair $(X, \Delta + cM)$ where $X$ has dimension $n+1$, the coefficients of $\Delta$ belong to $I$, $M$ is an integral $\mathbb{Q}$-Cartier divisor and there is a non kawamata log terminal centre $V$ contained in the support of $M$. Possibly passing to an open subset of $X$ and replacing $V$ by a maximal non kawamata log terminal centre, we may assume that $V$ is the only non kawamata log terminal centre of $(X, \Delta + cM)$. In particular, $(X, \Delta)$ is kawamata log terminal.

If $V$ is a component of $M$ then $V$ has coefficient one in $\Delta + cM$ and $c = \frac{1-f}{k} \in N_n(I)$ by (3) of (11.2). Otherwise let $f \colon Y \longrightarrow X$ be a divisorially log terminal modification of $(X, \Delta + cM)$. Then $Y$ is $\mathbb{Q}$-factorial and we may write

$$K_Y + T + \Delta' + cM' = f^*(K_X + \Delta + cM)$$

where $\Delta'$ and $M'$ are the strict transforms of $\Delta$ and $M$, $T$ is the sum of the exceptional divisors and the pair $(Y, T + \Delta' + cM')$ is divisorially log terminal. By (4) of (3.3.1) we may choose $f$ so that $T$ contains the inverse image of $V$. Let $S$ be an irreducible component of $T$ which intersects $M'$. Then we may write

$$(K_Y + T + \Delta' + cM')|_S = K_S + \Theta,$$

by adjunction, where $(S, \Theta)$ is divisorially log terminal and moreover we may write $\Theta = A + B + C$, where the coefficients of $A$ are one, the coefficients of $B$ belong to $D(I)$ and the coefficients of $C \neq 0$ belong to $D_c(I)$. As $S$ is a non kawamata log terminal centre, the centre of $S$ on $X$ is $V$ so that there is a morphism $S \longrightarrow V$. If $v \in V$ is a general point then $(S_v, \Theta_v) \in \mathfrak{N}_k(I \cup \{\, 1 \,\}, c)$, for some $k \leq n$. Thus $c \in N_k(I \cup \{\, 1 \,\}) \subset N_n(I)$.

We now show that $\mathrm{LCT}_{n+1}(I) \supset N_n(I)$. Pick $0 \neq c \in N_n(I)$. Then we may find a pair $(X, \Delta) \in \mathfrak{K}_m(I, c)$, some $m \leq n$. If $m < n$ then we are done by induction on the dimension. Otherwise $X$ has dimension

$n$. As $-K_X$ is ample, we may pick $d$ such that $-dK_X$ is very ample and embed $X$ into projective space by the linear system $|-dK_X|$.

Let $Y$ be the cone over $X$ and let $\Gamma_j$ be the cone over $\Delta_j$. Then $Y$ is a quasi-projective variety of dimension $n+1$. $Y$ is $\mathbb{Q}$-factorial as $X$ has Picard number one. $(Y, \Gamma = \sum d_i \Gamma_i)$ is log canonical but not kawamata log terminal at the vertex $p$ of the cone. By assumption we may write

$$d_i = \frac{m_i - 1 + f_i + k_i c}{m_i},$$

for each $i$, where $m_i$ is a positive integer, $k_i$ is a non-negative integer ($k_i = 0$ if $\Gamma_i$ is a component of $B_i$ and $k_i > 0$ if $\Gamma_i$ is a component of $C_i$) and $f_i \in I_+$. Since we are working locally around $p$, the vertex of $Y$, we may find a cover of $\pi \colon \tilde{Y} \longrightarrow Y$ which ramifies over $\Gamma_i$ to index $m_i$ for every $i$ and is otherwise unramified at the generic point of any divisor. We may write

$$K_{\tilde{Y}} + \tilde{\Gamma} = \pi^*(K_Y + \Gamma),$$

where the coefficients of $\tilde{\Gamma}$ belong to the set

$$\{ f_i + k_i c \,|\, i \}.$$

$\tilde{Y}$ is a $\mathbb{Q}$-factorial quasi-projective variety of dimension $n+1$ and $(\tilde{Y}, \tilde{\Gamma})$ is log canonical but not kawamata log terminal over any point $q$ lying over $p$. Let

$$\Theta = \sum f_i \Gamma_i \qquad \text{and} \qquad M_i = \sum k_i \Gamma_i.$$

Then the coefficients of $\Theta$ belong to $I_+ = I$, $M_i$ is an integral $\mathbb{Q}$-Cartier divisor and

$$c = \sup\{ t \in \mathbb{R} \,|\, (X, \Theta + tM) \text{ is log canonical} \},$$

is the log canonical threshold. But then $c \in \mathrm{LCT}_{n+1}(I)$. $\qquad \square$

**Lemma 11.6.** *Let $(X, \Delta)$ be a log canonical pair, where $X$ is $\mathbb{Q}$-factorial of dimension $n$ and Picard number one and $K_X + \Delta$ is numerically trivial.*

*If the coefficients of $\Delta$ are at least $\delta$ then $\Delta$ has at most $\frac{n+1}{\delta}$ components.*

*Proof.* [27, 18.24] implies that the sum of the coefficients of $\Delta$ is at most $n + 1$. $\qquad \square$

**Proposition 11.7.** *Fix a positive integer $n$ and a set $I \subset [0,1]$ whose only accumulation point is one such that $I = I_+$.*

*Let $c_1, c_2, \ldots \in [0,1]$ be a strictly decreasing sequence with limit $c \neq 0$ with the following property. There is a sequence of log canonical pairs*

$(X_i, \Delta_i)$ such that $X_i$ is a projective variety of dimension $n$, $K_{X_i} + \Delta_i$ is numerically trivial and we may write $\Delta_i = A_i + B_i + C_i$, where the coefficients of $A_i$ are approaching one, the coefficients of $B_i$ belong to $D(I)$ and the coefficients of $C_i \neq 0$ belong to $D_{c_i}(I)$.

Then $c \in N_{n-1}(I)$.

*Proof.* We may assume that $A_i$ and $B_i + C_i$ have no common components. Replacing $B_i$ by $B_i - \lfloor B_i \rfloor$ and $A_i$ by $A_i + \lfloor B_i \rfloor$ we may assume that $\lfloor \Delta_i \rfloor = \lfloor A_i \rfloor$. As the coefficients of $A_i + B_i$ belong to a set which satisfies the DCC, (1.5) implies that not all of the coefficients of $C_i$ are increasing. In particular at least one coefficient of $C_i$ is bounded away from one.

Let $a_i$ be the total log discrepancy of $(X_i, \Delta_i)$.

**Case A:** $\lim a_i > 0$.

In this case, we assume that $a_i$ is bounded away from zero.

**Case A, Step 1:** We reduce to the case $X_i$ is $\mathbb{Q}$-factorial and the Picard number of $X_i$ is one.

As we are assuming that $a_i$ is bounded away from zero, $A_i = 0$ and so $(X_i, \Delta_i) \in \mathfrak{N}_n(I, c_i)$, so that $c_i \in N_n(I) = K_n(I)$, by (11.4). Thus we may assume that $(X_i, \Delta_i) \in \mathfrak{K}_m(I, c_i)$, for some $m \leq n$. If $m < n$ then we are done by induction. Otherwise we may assume that $X_i$ is $\mathbb{Q}$-factorial and the Picard number of $X_i$ is one.

Possibly passing to a subsequence, (11.6) implies that we may assume that the number of components of $B_i$ and $C_i$ is fixed. As the only accumulation point of $D(I)$ is one and the coefficients of $B_i$ are bounded away from one, possibly passing to a subsequence we may assume that the coefficients of $B_i$ are fixed and that the coefficients of $C_i$ have the form

$$\frac{r-1}{r} + \frac{f}{r} + \frac{kc_i}{r},$$

where $k$, $r$ and $f$ depend on the component but not on $i$.

Given $t \in [0, 1]$, let $C_i(t)$ be the divisor with the same components as $C_i$ but now with coefficients

$$\frac{r-1}{r} + \frac{f}{r} + \frac{kt}{r},$$

so that $C_i = C_i(c_i)$. Let

$$h_i = \sup\{ t \mid (X_i, B_i + C_i(t)) \text{ is log canonical} \},$$

be the log canonical threshold. Set $h = \lim h_i$.

**Case A, Step 2:** We reduce to the case $h > c$.

Suppose that $h \leq c$. As $c_i \leq h_i$, it follows that $h = c$. Now

$$h_i \in \mathrm{LCT}_n(D(I)) = N_{n-1}(I),$$

so that we are done by induction in this case.

**Case A, Step 3:** We reduce to the case $\mathrm{vol}(X_i, C_i)$ is unbounded.
Suppose not, suppose that $\mathrm{vol}(X_i, C_i)$ is bounded from above. Let

$$d_i = \frac{c_i + h_i}{2} \qquad \text{and} \qquad d = \frac{c + h}{2}.$$

Then the coefficients of $(X_i, B_i + C_i(d))$ are fixed. The log discrepancy of $(X_i, B_i + C_i(d_i))$ is at least $a_i/2$ so that the log discrepancy of $(X_i, B_i + C_i(d))$ is bounded away from zero. As $h > c$, possibly passing to a tail of the sequence, we may assume that $d > c_i$ so that $K_{X_i} + B_i + C_i(d)$ is ample. Note that

$$\mathrm{vol}(X_i, K_{X_i} + B_i + C_i(d)) = \mathrm{vol}(X_i, C_i(d) - C_i)$$

is bounded from above by assumption. (1.3) implies that there is a positive integer $m$ such that $\phi_{m(K_{X_i} + B_i + C_i(d))}$ is birational. But then $\{(X_i, \Delta_i) \,|\, i \in \mathbb{N}\}$ is log birationally bounded by [15, 2.4.2.4]. (1.6) implies that $(X_i, \Delta_i)$ belongs to a bounded family. Thus we may find an ample Cartier divisor $H_i$ such that the intersection numbers $T_i \cdot H_i^{n-1}$ and $-K_{X_i} \cdot H_i^{n-1}$ are bounded, where $T_i$ is any component of $\Delta_i$. Possibly passing to a subsequence, we may assume that these intersection numbers are constant. But then

$$(K_{X_i} + \Delta_i) \cdot H_i^{n-1} = 0, \qquad A_i \cdot H_i^{n-1} = 0 \qquad \text{and} \qquad B_i \cdot H_i^{n-1}$$

is independent of $i$, whilst $C_i \cdot H_i^{n-1}$ is not constant, a contradiction.

**Case A, Step 4:** We finish case A.

As $\mathrm{vol}(X_i, C_i)$ is unbounded (3.2.2) implies that we may find $\epsilon_i > 0$ and divisors $0 \leq C_i' \sim_{\mathbb{R}} \epsilon_i C_i$ such that $(X_i, \Delta_i + C_i')$ is not log canonical. Passing to a subsequence, and using (3.2.3), we may find $g_i < c_i$ and a divisor

$$0 \leq \Theta_i \sim_{\mathbb{R}} C_i - C_i(g_i) \qquad \text{with} \qquad \lim g_i = c,$$

such that $(X_i, \Phi_i = B_i + C_i(g_i) + \Theta_i)$ has a unique non kawamata log terminal place. If $\phi \colon Y_i \longrightarrow X_i$ is a divisorially log terminal modification then $\phi$ extracts a unique prime divisor $S_i$ of log discrepancy zero with respect to $(X_i, \Phi_i)$. We may write

$$K_{Y_i} + \Psi_i = \phi^*(K_{X_i} + \Phi_i) \qquad \text{and} \qquad K_{Y_i} + B_i' + C_i' + s_i S_i = \phi^*(K_{X_i} + \Delta_i),$$

where $S_i = \lfloor \Psi_i \rfloor$, $B_i'$ and $C_i'$ are the strict transform of $B_i$ and $C_i$, and $s_i < 1$, as $(X_i, \Delta_i)$ is kawamata log terminal.

As $K_{Y_i} + \Psi_i$ is numerically trivial, $K_{Y_i} + \Psi_i - S_i$ is not pseudo-effective. By [8, 1.3.3], we may run $f \colon Y_i \dashrightarrow W_i$ the $(K_{Y_i} + \Psi_i - S_i)$-MMP until we end with a Mori fibre space $\pi_i \colon W_i \longrightarrow Z_i$. As $K_{Y_i} + \Psi_i$ is numerically trivial, every step of this MMP is $S_i$-positive, so that the strict transform $T_i$ of $S_i$ dominates $Z_i$. Let $F_i$ be the general fibre of

44

$\pi_i$. Replacing $Y_i$, $B_i'$, $C_i'$ and $\Psi_i$ by $F_i$ and the restriction of $f_* B_i'$, $f_* C_i'$ and $f_* \Psi_i$ to $F_i$, we may assume that $S_i$, $\Psi_i$, $B_i'$ and $C_i'$ are multiples of the same ample divisor. In particular $K_{Y_i} + B_i' + C_i' + S_i$ is ample.

We let $C_i'(t)$ denote the strict transform of $C_i(t)$. We may write

$$(K_{Y_i} + S_i + B_i' + C_i'(t))|_{S_i} = K_{S_i} + B_i'' + C_i''(t),$$

where the coefficients of $B_i''$ belong to $D(I)$ and the coefficients of $C_i''(t) \neq 0$ belong to $D_t(I)$. We let $C_i'' = C_i''(c_i)$.

There are two cases. Suppose that $(S_i, B_i'' + C_i'')$ is not log canonical. Let

$$k_i = \sup\{\, t \mid (S_i, B_i'' + C_i''(t)) \text{ is log canonical}\,\},$$

be the log canonical threshold. Then $k_i \in \mathrm{LCT}_{n-1}(D(I)) = N_{n-2}(I)$. Then $k = \lim k_i \in N_{n-2}(I) \subset N_{n-1}(I)$ by induction on $n$. As $(S_i, B_i'' + C_i''(g_i))$ is kawamata log terminal, $k_i \in (g_i, c_i)$. Thus

$$c = \lim c_i = \lim k_i = k \in N_{n-1}(I).$$

Otherwise we may suppose that $(S_i, B_i'' + C_i'')$ is log canonical. Let

$$l_i = \sup\{\, t \mid (S_i, B_i'' + C_i''(t)) \text{ is pseudo-effective}\,\},$$

be the pseudo-effective threshold. Then $l_i \in N_{n-1}(I)$ and $l = \lim l_i \in N_{n-1}(I)$ by induction on $n$. On the other hand $l_i \in (g_i, c_i)$. Thus

$$c = \lim c_i = \lim l_i = l \in N_{n-1}(I).$$

**Case B:** $\lim a_i = 0$.

In this case, we assume that $a_i$ approaches $0$.

**Case B, Step 1:** We reduce to the case $A_i \neq 0$, $X_i$ is $\mathbb{Q}$-factorial and $(X_i, \Delta_i)$ is kawamata log terminal if and only if $\lfloor A_i \rfloor = 0$.

Possibly passing to a subsequence we may assume that $a_i \geq a_{i+1}$ and $a_i \leq 1$. If $(X_i, \Delta_i)$ is not divisorially log terminal or $A_i \neq 0$ but $X_i$ is not $\mathbb{Q}$-factorial then let $\pi_i \colon X_i' \longrightarrow X_i$ be a divisorially log terminal modification. If $A_i = 0$ then let $\pi_i \colon X_i' \longrightarrow X_i$ extract a divisor of log discrepancy $a_i$, where $X_i'$ is $\mathbb{Q}$-factorial. Either way, we may write

$$K_{X_i'} + \Delta_i' = \pi_i^*(K_{X_i} + \Delta_i),$$

where $\Delta_i'$ is a sum of the strict transform of $\Delta_i$ and a divisor which is exceptional. Let $B_i'$ and $C_i'$ be the strict transforms of $B_i$ and $C_i$ and let $A_i' = \Delta_i' - B_i' - C_i' \neq 0$. Then $X_i'$ is a $\mathbb{Q}$-factorial projective variety of dimension $n$, $(X_i', \Delta_i')$ is a divisorially log terminal pair, $K_{X_i'} + \Delta_i'$ is numerically trivial, the coefficients of $A_i' \neq 0$ are approaching one, the coefficients of $B_i'$ belong to $D(I)$ and the coefficients of $C_i' \neq 0$ belong to $D_{c_i}(I)$. Replacing $(X_i, \Delta_i)$ by $(X_i', \Delta_i')$, we may assume that $A_i \neq 0$ and $X_i$ is $\mathbb{Q}$-factorial. Moreover $(X_i, \Delta_i)$ is kawamata log terminal if and only if $\lfloor A_i \rfloor = 0$.

**Case B, Step 2:** We are done if the support of $C_i$ and $\lfloor A_i \rfloor$ intersect.

Suppose that a component of $C_i$ intersects the normalisation of a component $S_i$ of $\lfloor A_i \rfloor$. Then we may write

$$(K_{X_i} + \Delta_i)|_{S_i} = K_{S_i} + \Theta_i,$$

by adjunction. $S_i$ is projective of dimension $n-1$, $(S_i, \Theta_i)$ is log canonical, $K_{S_i} + \Theta_i$ is numerically trivial, and we may write $\Theta_i = A_i' + B_i' + C_i'$, where the coefficients of $A_i'$ approach one, the coefficients of $B_i'$ belong to $D(I)$ and the coefficients of $C_i' \neq 0$ belong to $D_{c_i}(I)$. In this case, the limit $c$ belongs to $N_{n-2}(I) \subset N_{n-1}(I)$ by induction.

**Case B, Step 3:** We are done if $f_i \colon X_i \longrightarrow Z_i$ is a Mori fibre space, $A_i$ dominates $Z_i$ and $\dim Z_i > 0$.

Let $F_i$ be the general fibre of $f_i$. We may write

$$(K_{X_i} + \Delta_i)|_{F_i} = K_{F_i} + \Theta_i,$$

by adjunction. $F_i$ is projective of dimension at most $n - 1$, $(F_i, \Theta_i)$ is log canonical, $K_{F_i} + \Theta_i$ is numerically trivial, and we may write $\Theta_i = A_i' + B_i' + C_i'$, where the coefficients of $A_i'$ approach one, the coefficients of $B_i'$ belong to $D(I)$ and the coefficients of $C_i'$ belong to $D_{c_i}(I)$.

There are two cases. Suppose that $C_i' = 0$. Then (1.5) implies that the coefficients of $A_i'$ are fixed, so that $\lfloor A_i' \rfloor = A_i'$. But then $\lfloor A_i \rfloor \neq 0$ dominates $Z_i$. On the other hand, as $C_i' = 0$, $C_i$ does not intersect $F_i$, that is, $C_i$ does not dominate $Z_i$. But then $C_i$ must contain a fibre so that $A_i$ and $C_i$ intersect and we are done by Case B, Step 2. Otherwise $C_i' \neq 0$. In this case $c_i \in N_{n-1}(I)$ so that

$$c = \lim c_i \in N_{n-2}(I) \subset N_{n-1}(I),$$

by induction.

**Case B, Step 4:** We reduce to the case $(X_i, \Delta_i)$ is kawamata log terminal.

Suppose not, suppose that $(X_i, \Delta_i)$ is not kawamata log terminal. By Case B, Step 1, this implies that $S_i = \lfloor A_i \rfloor$ is not the zero divisor. Let $\Theta_i = \Delta_i - S_i$. We run the $(K_{X_i} + \Theta_i)$-MMP with scaling of some ample divisor. Let $f_i \colon X_i \dashrightarrow X_i'$ be a step of the $(K_{X_i} + \Theta_i)$-MMP. As $K_{X_i} + \Delta_i$ is numerically trivial, $f_i$ is automatically $S_i$-positive. Let $A_i' = f_{i*}A_i$, $B_i' = f_{i*}B_i$ and $C_i' = f_{i*}C_i$. First suppose that $f_i$ is birational. If $C_i' = 0$ then (1.5) implies that the coefficients of $A_i'$ are all one. As $f_i$ contracts $C_i$ it does not contract a component of $A_i$ and so it follows that the coefficients of $A_i$ are all one, that is, $S_i = A_i$. As $f_i$ contracts $C_i$ and $f_i$ is $S_i$-positive, $C_i$ intersects $S_i$ and we are done by Case B, Step 2. Therefore we may assume that $C_i' \neq 0$ and we

46

may replace $(X_i, \Delta_i)$ by $(X'_i, \Delta'_i)$. As the MMP must terminate with a Mori fibre space, replacing $(X_i, \Delta_i)$ with $(X'_i, \Delta'_i)$ finitely many times, we may assume that $f_i \colon X_i \longrightarrow Z_i = X'_i$ is a Mori fibre space and $S_i$ dominates $Z_i$. By Case B, Step 3, we may assume that $Z_i$ is a point. But then the support of $S_i$ and $C_i$ intersect and we are done by Case B, Step 2.

**Case B, Step 5:** We reduce to the case $X_i$ has Picard number one.

We run the $(K_{X_i} + B_i + C_i)$-MMP with scaling of some ample divisor. Let $f_i \colon X_i \dashrightarrow X'_i$ be a step of the $(K_{X_i} + B_i + C_i)$-MMP. As $K_{X_i} + \Delta_i$ is numerically trivial $f_i$ is automatically $A_i$-positive. Let $A'_i = f_{i*}A_i$, $B'_i = f_{i*}B_i$ and $C'_i = f_{i*}C_i$. First suppose that $f_i$ is birational. Suppose $C'_i = 0$. As $f_i$ contracts only one divisor and $A_i$ and $C_i$ are non-zero by assumption, it follows that $A'_i \neq 0$. (1.5) implies that the coefficients of $A'_i$ are all one, which contradicts the fact that $(X_i, \Delta_i)$ is kawamata log terminal. Therefore we may assume that $C'_i \neq 0$ and we may replace $(X_i, \Delta_i)$ by $(X'_i, \Delta'_i)$. As the MMP must terminate with a Mori fibre space, replacing $(X_i, \Delta_i)$ with $(X'_i, \Delta'_i)$ finitely many times, we may assume that $f_i \colon X_i \longrightarrow Z_i = X'_i$ is a Mori fibre space and $A_i$ dominates $Z_i$.

By Case B, Step 3 we may assume that $Z_i$ is a point, so that $X_i$ has Picard number one.

**Case B, Step 6:** We finish case B and the proof.

Possibly passing to a subsequence, (11.6) implies that we may assume that the number of components of $B_i$ and $C_i$ is fixed. As the only accumulation point of $D(I)$ is one and the coefficients of $B_i$ are bounded away from one, possibly passing to a subsequence we may assume that the coefficients of $B_i$ are fixed and that the coefficients of $C_i$ have the form

$$\frac{r-1}{r} + \frac{f}{r} + \frac{kc_i}{r},$$

where $k$, $r$ and $f$ depend on the component but not on $i$.

Given $t \in [0, 1]$, let $C_i(t)$ be the divisor with the same components as $C_i$ but now with coefficients

$$\frac{r-1}{r} + \frac{f}{r} + \frac{kt}{r},$$

so that $C_i = C_i(c_i)$.

Let $T_i$ be the sum of the components of $A_i$, so that $T_i$ has the same components as $A_i$ but now every component has coefficient one. Then $A_i \leq T_i$ and $C_i(c) \leq C_i$. Note that $(X_i, A_i + B_i + C_i(c))$ is kawamata log terminal as $(X_i, A_i + B_i + C_i)$ is kawamata log terminal. Let

$$s_i = \sup\{\, s \in [0, 1] \mid (X_i, A_i + B_i + C_i(c) + s(T_i - A_i)) \text{ is log canonical} \,\}$$

be the log canonical threshold. Then

$$A_i + B_i + C_i(c) \leq A_i + B_i + C_i(c) + s_i(T_i - A_i) \leq T_i + B_i + C_i(c),$$

As the coefficients of $A_i + B_i + C_i(c)$ belong to a set which satisfies the DCC and the coefficients of $T_i - A_i$ approach zero, the coefficients of $A_i + B_i + C_i(c) + s_i(T_i - A_i)$ belong to a set which satisfies the DCC. Therefore, possibly passing to a tail of the sequence, (1.4) implies that $s_i = 1$, so that $(X_i, T_i + B_i + C_i(c))$ is log canonical.

Suppose that $(X_i, T_i + B_i + C_i)$ is not log canonical. Let

$$d_i = \sup\{\, t \in [c, c_i) \,|\, (X_i, T_i + B_i + C_i(t)) \text{ is log canonical}\,\},$$

be the log canonical threshold. Then $d_i \in \mathrm{LCT}_n(D(I)) = N_{n-1}(I)$ and $c = \lim d_i$ and so we are done by induction on the dimension.

Thus we may assume that $(X_i, T_i + B_i + C_i)$ is log canonical. Let

$$e_i = \sup\{\, t \in \mathbb{R} \,|\, K_{X_i} + T_i + B_i + C_i(t)) \text{ is pseudo-effective}\,\},$$

be the pseudo-effective threshold. Suppose that $e_i < c$. Let

$$f_i = \sup\{\, t \in \mathbb{R} \,|\, K_{X_i} + t T_i + B_i + C_i(c)) \text{ is pseudo-effective}\,\},$$

be the pseudo-effective threshold. As $e_i < c$, $f_i < 1$ and $\lim f_i = 1$, so that the coefficients of $f_i T_i + B_i + C_i(c)$ belong to a set which satisfies the DCC, which contradicts (1.5). Thus $e_i \geq c$. On the other hand $e_i < c_i$ as $K_{X_i} + T_i + B_i + C_i$ is strictly bigger than $K_{X_i} + A_i + B_i + C_i$, which is numerically trivial. Thus $\lim e_i = c$. Possibly passing to a subsequence we may assume that either $e_i > e_{i+1}$ for all $i$ or $e_i = c$. In the former case we might as well replace $C_i = C_i(c_i)$ by $C_i(e_i)$. In this case some component of $C_i$ intersects a component $S_i$ of $T_i$ and we are done by Case B, Step 2. In the latter case we restrict to a component $S_i$ of $T_i$ and apply adjunction to conclude that $c = e_i \in N_{n-1}(I)$.  $\square$

*Proof of* (1.11). By (11.5) it suffices to prove that the accumulation points of $N_n(I)$ belong to $N_{n-1}(I)$. Suppose that $c_1, c_2, \ldots \in [0, 1]$ is a strictly decreasing sequence of real numbers such that $\mathfrak{N}(I, c_i)$ is non-empty. Pick $(X_i, \Delta_i) \in \mathfrak{N}(I, c_i)$. By assumption we may write $\Delta_i = B_i + C_i$ where the coefficients of $B_i$ belong to $D(I)$ and the coefficients of $C_i \neq 0$ belong to $D_{c_i}(I)$, and so (11.7) implies that the limit $c$ belongs to $N_{n-1}(I)$.  $\square$

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