THE SARKISOV PROGRAM

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ABSTRACT. Any two birational Mori fibre spaces are connected by a sequence of Sarkisov links.

CONTENTS

1. INTRODUCTION

We prove that any two birational Mori fibre spaces are connected by a sequence of elementary transformations, known as Sarkisov links:

Theorem 1.1. *Suppose that $\phi\colon X \longrightarrow S$ and $\psi\colon Y \longrightarrow T$ are two Mori fibre spaces with $\mathbb{Q}$-factorial terminal singularities.*

Then X and Y are birational if and only if they are related by a sequence of Sarkisov links.

Recall the following:

Conjecture 1.2. *Let (Z, Φ) be a kawamata log terminal pair.*

Then we may run $f\colon Z \dashrightarrow X$ the $(K_Z + \Phi)$-MMP such that either

(1) *(X, Δ) is a log terminal model, that is $K_X + \Delta$ is nef, or*
(2) *there is a Mori fibre space $\phi\colon X \longrightarrow S$, that is $\rho(X/S) = 1$ and $-(K_X + \Delta)$ is ϕ-ample,*

where $\Delta = f_*\Phi$.

We will refer to the log terminal model X and the Mori fibre space ϕ as the output of the $(K_Z + \Phi)$-MMP. If $h\colon Z \dashrightarrow X$ is any sequence of divisorial contractions and flips for the $(K_Z + \Phi)$-MMP then we say that h is the result of running the $(K_Z + \Phi)$-MMP. In other words if h is the result of running the $(K_Z + \Phi)$-MMP then X does not have to be either a log terminal model or a Mori fibre space.

By [1] the only unknown case of (1.2) is when $K_Z + \Phi$ is pseudo-effective but neither Φ nor $K_Z + \Phi$ is big. Unfortunately the output is not unique in either case. We will call two Mori fibre spaces $\phi\colon X \longrightarrow S$ and $\psi\colon Y \longrightarrow T$ *Sarkisov related* if X and Y are outcomes of running the $(K_Z + \Phi)$-MMP, for the same $\mathbb{Q}$-factorial kawamata log terminal pair (Z, Φ). This defines a category, which we call the Sarkisov category, whose objects are Mori fibre spaces and whose morphisms are the induced birational maps $X \dashrightarrow Y$ between two Sarkisov related Mori fibre spaces. Our goal is to show that every morphism in this category is a product of Sarkisov links. In particular a Sarkisov link should connect two Sarkisov related Mori fibre spaces.

Theorem 1.3. *If $\phi\colon X \longrightarrow S$ and $\psi\colon Y \longrightarrow T$ are two Sarkisov related Mori fibres spaces then the induced birational map $\sigma\colon X \dashrightarrow Y$ is a composition of Sarkisov links.*

Note that if X and Y are birational and have $\mathbb{Q}$-factorial terminal singularities, then ϕ and ψ are automatically the outcome of running the K_Z-MMP for some projective variety Z, so that (1.1) is an easy consequence of (1.3).

It is proved in [1] that the number of log terminal models is finite if either Φ or $K_Z + \Phi$ is big, and it is conjectured that in general the number of log terminal models is finite up to birational automorphisms. Moreover Kawamata, see [5], has proved:

Theorem 1.4. *Suppose that $\sigma\colon X \dashrightarrow Y$ is a birational map between two $\mathbb{Q}$-factorial varieties which is an isomorphism in codimension one.*

If $K_X + \Delta$ and $K_Y + \Gamma$ are kawamata log terminal and nef and Γ is the strict transform of Δ then σ is the composition of $(K_X + \Delta)$-flops.

Note that if the pairs (X, Δ) and (Y, Γ) both have $\mathbb{Q}$-factorial terminal singularities then the birational map σ is automatically an isomorphism in codimension one.

We recall the definition of a Sarkisov link. Suppose that $\phi\colon X \longrightarrow S$ and $\psi\colon Y \longrightarrow T$ are two Mori fibre spaces. A Sarkisov link $\sigma\colon X \dashrightarrow Y$

between ϕ and ψ is one of four types:

$$
\begin{array}{cccc}
\text{I} & \text{II} & \text{III} & \text{IV}
\end{array}
$$
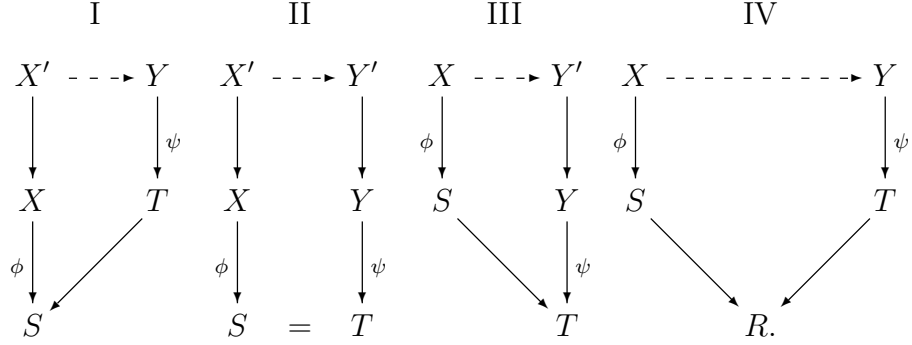
There is a divisor Ξ on the space L on the top left (be it $L = X$ or $L = X'$) such that $K_L + \Xi$ is kawamata log terminal and numerically trivial over the base (be it S, T, or R). Every arrow which is not horizontal is an extremal contraction. If the target is X or Y it is a divisorial contraction. The horizontal dotted arrows are compositions of $(K_L + \Xi)$-flops. Links of type IV break into two types, IV_m and IV_s. For a link of type IV_m both s and t are Mori fibre spaces. For a link of type IV_s both s and t are small birational contractions. In this case R is not $\mathbb{Q}$-factorial; for every other type of link all varieties are $\mathbb{Q}$-factorial. Note that there is an induced birational map $\sigma\colon X \dashrightarrow Y$ but not necessarily a rational map between S and T.

The Sarkisov program has its origin in the birational classification of ruled surfaces. A link of type I corresponds to the diagram
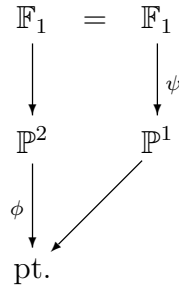
Note that there are no flops for surfaces so the top horizontal map is always the identity. The top vertical arrow on the left is the blow up of a point in $\mathbb{P}^2$ and ψ is the natural map given by the pencil of lines.

A link of type III is the same diagram, reflected in a vertical line,

$$
\begin{array}{ccc}
\mathbb{F}_1 & = & \mathbb{F}_1 \\
{\scriptstyle\phi}\downarrow & & \downarrow \\
\mathbb{P}^1 & & \mathbb{P}^2 \\
& \searrow & \downarrow{\scriptstyle\psi} \\
& & \text{pt.}
\end{array}
$$

A link of type II corresponds to the classical elementary transformation between ruled surfaces,

$$
\begin{array}{ccc}
X' & = & Y' \\
\downarrow & & \downarrow \\
X & & Y \\
{\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\psi} \\
S & = & T.
\end{array}
$$

The birational map $X' \longrightarrow X$ blows up a point in one fibre and the birational map $Y' \longrightarrow Y$ blows down the old fibre. Finally a link of type IV corresponds to switching between the two ways to project $\mathbb{P}^1 \times \mathbb{P}^1$ down to $\mathbb{P}^1$,

$$
\begin{array}{ccc}
\mathbb{P}^1 \times \mathbb{P}^1 & = & \mathbb{P}^1 \times \mathbb{P}^1 \\
{\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\psi} \\
\mathbb{P}^1 & & \mathbb{P}^1 \\
& \searrow \quad \swarrow & \\
& \text{pt.} &
\end{array}
$$

It is a fun exercise to factor the classical Cremona transformation $\sigma\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, $[X : Y : Z] \longrightarrow [X^{-1} : Y^{-1} : Z^{-1}]$ into a product of Sarkisov links. Indeed one can use the Sarkisov program to give a very clean proof that the birational automorphism of $\mathbb{P}^2$ is generated by this birational map σ and PGL(3). More generally the Sarkisov program can sometimes be used to calculate the birational automorphism group of Mori fibre spaces, especially Fano varieties. With this said, note that the following problem seems quite hard:

Question 1.5. *What are generators of the birational automorphism group of $\mathbb{P}^3$?*

Note that a link of type IV_s only occurs in dimension four or more. For an example of a link of type IV_s simply take $S \dashrightarrow T$ to be a flop between threefolds, let $S \longrightarrow R$ be the base of the flop and let $X = S \times \mathbb{P}^1$ and $Y = T \times \mathbb{P}^1$ with the obvious maps down to S and T. It is conceivable that one can factor a link of type IV_s into links of type I and III. However given any positive integer k it is easy to write down examples of links of type IV which cannot be factored into fewer than k links of type I, II or III.

Let us now turn to a description of the proof of (1.3). The proof is based on the original ideas of the Sarkisov program (as explained by Corti and Reid [3]; see also [2]). We are given a birational map $\sigma \colon X \dashrightarrow Y$ and the objective is to factor σ into a product of Sarkisov links. In the original proof one keeps track of some subtle invariants and the idea is to prove:

- the first Sarkisov link σ_1 exists,
- if one chooses σ_1 appropriately then the invariants improve, and
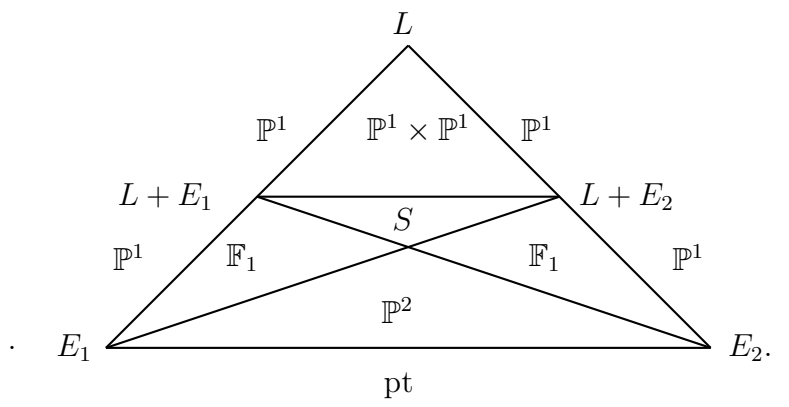- the invariants cannot increase infinitely often.

Sarkisov links arise naturally if one plays the 2-ray game. If the relative Picard number is two then there are only two rays to contract and this gives a natural way to order the steps of the minimal model program. One interesting feature of the original proof is that it is a little tricky to prove the existence of the first Sarkisov link, even if we assume existence and termination of flips. In the original proof one picks a linear system on Y and pulls it back to X. There are then three invariants to keep track of; the singularities of the linear system on X, as measured by the canonical threshold, the number of divisors of log discrepancy one (after rescaling to the canonical threshold) and the pseudo-effective threshold. Even for threefolds it is very hard to establish that these invariants satisfy the ascending chain condition.

Our approach is quite different. We don't consider any linear systems nor do we try to keep track of any invariants. Instead we use one of the main results of [1], namely finiteness of ample models for kawamata log terminal pairs $(Z, A + B)$. Here A is a fixed ample $\mathbb{Q}$-divisor and B ranges over a finite dimensional affine space of Weil divisors. The closure of the set of divisors B with the same ample model is a disjoint union of finitely many polytopes and the union of all of these polytopes corresponds to divisors in the effective cone.

Now if the space of Weil divisors spans the Néron-Severi group then one can read off which ample model admits a contraction to another ample model from the combinatorics of the polytopes, (3.3). Further this property is preserved on taking a general two dimensional slice,

(3.4). Sarkisov links then correspond to points on the boundary of the effective cone which are contained in more than two polytopes, (3.7). To obtain the required factorisation it suffices to simply traverse the boundary. In other words instead of considering the closed cone of curves and playing the 2-ray game we look at the dual picture of Weil divisors and we work inside a carefully chosen two dimensional affine space. The details of the correct choice of underlying affine space are contained in §4.

To illustrate some of these ideas, let us consider an easy case. Let S be the blow up of $\mathbb{P}^2$ at two points. Then S is a toric surface and there are five invariant divisors. The two exceptional divisors, E_1 and E_2, the strict transform L of the line which meets E_1 and E_2, and finally the strict transform L_1 and L_2 of two lines, one of which meets E_1 and one of which meets E_2. Then the cone of effective divisors is spanned by the invariant divisors and according to [4] the polytopes we are looking for are obtained by considering the chamber decomposition given by the invariant divisors. Since $L_1 = L + E_1$ and $L_2 = L + E_2$ the cone of effective divisors is spanned by L, E_1 and E_2. Since $-K_S$ is ample, we can pick an ample $\mathbb{Q}$-divisor A such that $K_S + A \sim_{\mathbb{Q}} 0$ and $K_S + A + E_1 + E_2 + L$ is divisorially log terminal. Let V be the real vector space of Weil divisors spanned by E_1, E_2 and L. In this case projecting $\mathcal{L}_A(V)$ from the origin we get

$$\begin{array}{c}
L \\
\mathbb{P}^1 \quad \mathbb{P}^1 \times \mathbb{P}^1 \quad \mathbb{P}^1 \\
L + E_1 \quad\quad\quad L + E_2 \\
S \\
\mathbb{P}^1 \quad \mathbb{F}_1 \quad\quad \mathbb{F}_1 \quad \mathbb{P}^1 \\
\mathbb{P}^2 \\
E_1 \quad\quad\quad\quad\quad E_2. \\
\mathrm{pt}
\end{array}$$

We have labelled each polytope by the corresponding model. Imagine going around the boundary clockwise, starting just before the point corresponding to L. The point L corresponds to a Sarkisov link of type IV_m, the point $L + E_2$ a link of type II, the point E_2 a link of type III, the point E_1 a link of type I and the point $L + E_1$ another link of type II.

2. Notation and conventions

We work over the field of complex numbers $\mathbb{C}$. An $\mathbb{R}$-Cartier divisor D on a variety X is *nef* if $D \cdot C \geq 0$ for any curve $C \subset X$. We say that two $\mathbb{R}$-divisors D_1, D_2 are $\mathbb{R}$-linearly equivalent ($D_1 \sim_{\mathbb{R}} D_2$) if $D_1 - D_2 = \sum r_i(f_i)$ where $r_i \in \mathbb{R}$ and f_i are rational functions on X. We say that an $\mathbb{R}$-Weil divisor D is *big* if we may find an ample $\mathbb{R}$-divisor A and an $\mathbb{R}$-divisor $B \geq 0$, such that $D \sim_{\mathbb{R}} A + B$. A divisor D is *pseudo-effective*, if for any ample divisor A and any rational number $\epsilon > 0$, the divisor $D + \epsilon A$ is big. If A is a $\mathbb{Q}$-divisor, we say that A is a *general ample $\mathbb{Q}$-divisor* if A is ample and there is a sufficiently divisible integer $m > 0$ such that mA is very ample and $mA \in |mA|$ is very general.

A *log pair* (X, Δ) is a normal variety X and an $\mathbb{R}$-Weil divisor $\Delta \geq 0$ such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. We say that a log pair (X, Δ) is *log smooth*, if X is smooth and the support of Δ is a divisor with global normal crossings. A projective birational morphism $g \colon Y \longrightarrow X$ is a *log resolution* of the pair (X, Δ) if Y is smooth and the strict transform Γ of Δ union the exceptional set E of g is a divisor with normal crossings support. If we write

$$K_Y + \Gamma + E = g^*(K_X + \Delta) + \sum a_i E_i,$$

where $E = \sum E_i$ is the sum of the exceptional divisors then the log discrepancy $a(E_i, X, \Delta)$ of E_i is a_i. By convention the log discrepancy of any divisor B which is not exceptional is $1 - b$, where b is the coefficient of B in Δ. The log discrepancy a is the infimum of the log discrepancy of any divisor.

A pair (X, Δ) is *kawamata log terminal* if $a > 0$. We say that the pair (X, Δ) is *log canonical* if $a \geq 0$. We say that the pair (X, Δ) is *terminal* if the log discrepancy of any exceptional divisor is greater than one.

We say that a rational map $\phi \colon X \dashrightarrow Y$ is a *rational contraction* if there is a resolution $p \colon W \longrightarrow X$ and $q \colon W \longrightarrow Y$ of ϕ such that p and q are contraction morphisms and p is birational. We say that ϕ is a *birational contraction* if q is in addition birational and every

p-exceptional divisor is q-exceptional. If in addition ϕ^{-1} is also a birational contraction, we say that ϕ is a *small birational map*. We refer the reader to [1] for the definitions of negative and non-positive rational contractions and of log terminal models.

If $\mathcal{C}$ is a closed convex in a finite dimensional real vector space then $\mathcal{C}^*$ denotes the dual convex set in the dual real vector space.

3. The combinatorics of ample models

We fix some notation. Z is a smooth projective variety, V is a finite dimensional affine subspace of the real vector space $\mathrm{WDiv}_{\mathbb{R}}(Z)$ of Weil divisors on Z, which is defined over the rationals, and $A \geq 0$ is an ample $\mathbb{Q}$-divisor on Z. We suppose that there is an element Θ_0 of $\mathcal{L}_A(V)$ such that $K_Z + \Theta_0$ is big and kawamata log terminal.

We recall some definitions and notation from [1]:

Definition 3.1. *Let D be an $\mathbb{R}$-divisor on Z.*

*We say that $f\colon Z \dashrightarrow X$ is the **ample model** of D, if f is a rational contraction, X is a normal projective variety and there is an ample divisor H on X such that if $p\colon W \longrightarrow Z$ and $q\colon W \longrightarrow X$ resolve f and we write $p^*D \sim_{\mathbb{R}} q^*H + E$, then $E \geq 0$ and for every $B \sim_{\mathbb{R}} p^*D$ if $B \geq 0$ then $B \geq E$.*

Note that if f is birational then $q_*E = 0$.

Definition 3.2. *Let*

$$V_A = \{\, \Theta \,|\, \Theta = A + B, B \in V \,\},$$
$$\mathcal{L}_A(V) = \{\, \Theta = A + B \in V_A \,|\, K_Z + \Theta \text{ is log canonical and } B \geq 0 \,\},$$
$$\mathcal{E}_A(V) = \{\, \Theta \in \mathcal{L}_A(V) \,|\, K_Z + \Theta \text{ is pseudo-effective} \,\}.$$

Given a rational contraction $f\colon Z \dashrightarrow X$, define

$$\mathcal{A}_{A,f}(V) = \{\, \Theta \in \mathcal{E}_A(V) \,|\, f \text{ is the ample model of } (Z, \Theta) \,\}.$$

In addition, let $\mathcal{C}_{A,f}(V)$ denote the closure of $\mathcal{A}_{A,f}(V)$.

Theorem 3.3. *There are finitely many $1 \leq i \leq m$ rational contractions $f_i\colon Z \dashrightarrow X_i$ with the following properties:*

(1) *$\{\, \mathcal{A}_i = \mathcal{A}_{A,f_i} \,|\, 1 \leq i \leq m \,\}$ is a partition of $\mathcal{E}_A(V)$. $\mathcal{A}_i$ is a finite union of interiors of rational polytopes. If f_i is birational then $\mathcal{C}_i = \mathcal{C}_{A,f_i}$ is a rational polytope.*

(2) *If $1 \leq i \leq m$ and $1 \leq j \leq m$ are two indices such that $\mathcal{A}_j \cap \mathcal{C}_i \neq \varnothing$ then there is a contraction morphism $f_{i,j}\colon X_i \longrightarrow X_j$ and a factorisation $f_j = f_{i,j} \circ f_i$.*

Now suppose in addition that V spans the Néron-Severi group of Z.

(3) *Pick $1 \leq i \leq m$ such that a connected component $\mathcal{C}$ of $\mathcal{C}_i$ inter-sects the interior of $\mathcal{L}_A(V)$. The following are equivalent*
  - *$\mathcal{C}$ spans V.*
  - *If $\Theta \in \mathcal{A}_i \cap \mathcal{C}$ then f_i is a log terminal model of $K_Z + \Theta$.*
  - *f_i is birational and X_i is $\mathbb{Q}$-factorial.*
(4) *If $1 \leq i \leq m$ and $1 \leq j \leq m$ are two indices such that $\mathcal{C}_i$ spans V and Θ is a general point of $\mathcal{A}_j \cap \mathcal{C}_i$ which is also a point of the interior of $\mathcal{L}_A(V)$ then $\mathcal{C}_i$ and $\overline{\mathrm{NE}}(X_i/X_j)^* \times \mathbb{R}^k$ are locally isomorphic in a neighbourhood of Θ, for some $k \geq 0$. Further the relative Picard number of $f_{i,j} \colon X_i \longrightarrow X_j$ is equal to the difference in the dimensions of $\mathcal{C}_i$ and $\mathcal{C}_j \cap \mathcal{C}_i$.*

Proof. (1) is proved in [1].

Pick $\Theta \in \mathcal{A}_j \cap \mathcal{C}_i$ and $\Theta' \in \mathcal{A}_i$ so that

$$\Theta_t = \Theta + t(\Theta' - \Theta) \in \mathcal{A}_i \qquad \text{if} \qquad t \in (0, 1].$$

By finiteness of log terminal models, cf. [1], we may find a positive constant $\delta > 0$ and a birational contraction $f \colon Z \dashrightarrow X$ which is a log terminal model of $K_Z + \Theta_t$ for $t \in (0, \delta]$. Replacing $\Theta' = \Theta_1$ by Θ_δ we may assume that $\delta = 1$. If we set

$$\Delta_t = f_* \Theta_t,$$

then $K_X + \Delta_t$ is kawamata log terminal and nef, and f is $K_Z + \Theta_t$ non-positive for $t \in [0, 1]$. As Δ_t is big the base point free theorem implies that $K_X + \Delta_t$ is semiample and so there is an induced contraction morphism $g_i \colon X \longrightarrow X_i$ together with ample divisors $H_{1/2}$ and H_1 such that

$$K_X + \Delta_{1/2} = g_i^* H_{1/2} \qquad \text{and} \qquad K_X + \Delta_1 = g_i^* H_1.$$

If we set

$$H_t = (2t - 1)H_1 + 2(1 - t)H_{1/2},$$

then

$$\begin{aligned}
K_X + \Delta_t &= (2t - 1)(K_X + \Delta_1) + 2(1 - t)(K_X + \Delta_{1/2}) \\
&= (2t - 1)g_i^* H_1 + 2(1 - t)g_i^* H_{1/2} \\
&= g_i^* H_t,
\end{aligned}$$

for all $t \in [0, 1]$. As $K_X + \Delta_0$ is semiample, it follows that H_0 is semi-ample and the associated contraction $f_{i,j} \colon X_i \longrightarrow X_j$ is the required morphism. This is (2).

Now suppose that V spans the Néron-Severi group of Z. Suppose that $\mathcal{C}$ spans V. Pick Θ in the interior of $\mathcal{C} \cap \mathcal{A}_i$. Let $f \colon Z \dashrightarrow X$ be a

log terminal model of $K_Z + \Theta$. It is proved in [1] that $f = f_j$ for some index $1 \le j \le m$ and that $\Theta \in \mathcal{C}_j$. But then $\mathcal{A}_i \cap \mathcal{A}_j \ne \varnothing$ so that $i = j$.

If f_i is a log terminal model of $K_Z + \Theta$ then f_i is birational and X_i is $\mathbb{Q}$-factorial.

Finally suppose that f_i is birational and X_i is $\mathbb{Q}$-factorial. Fix $\Theta \in \mathcal{A}_i$. Pick any divisor $B \in V$ such that $-B$ is ample $K_{X_i} + f_{i*}(\Theta + B)$ is ample and $\Theta + B \in \mathcal{L}_A(V)$. Then f_i is $(K_Z + \Theta + B)$-negative and so $\Theta + B \in \mathcal{A}_i$. But then $\mathcal{C}_i$ spans V. This is (3).

We now prove (4). Let $f = f_i$ and $X = X_i$. As $\mathcal{C}_i$ spans V, (3) implies that f is birational and X is $\mathbb{Q}$-factorial so that f is a $\mathbb{Q}$-factorial weak log canonical model of $K_Z + \Theta$. Suppose that $E_1, E_2, \dots, E_k$ are the divisors contracted by f. Pick $B_i \in V$ numerically equivalent to E_i. If we let $E_0 = \sum E_i$ and $B_0 = \sum B_i$ then E_0 and B_0 are numerically equivalent. As Θ belongs to the interior of $\mathcal{L}_A(V)$ we may find $\delta > 0$ such that $K_Z + \Theta + \delta E_0$ and $K_Z + \Theta + \delta B_0$ are both kawamata log terminal. Then f is $(K_Z + \Theta + \delta E_0)$-negative and so f is a log terminal model of $K_Z + \Theta + \delta E_0$ and f_j is the ample model of $K_Z + \Theta + \delta E_0$. But then f is also a log terminal model of $K_Z + \Theta + \delta B_0$ and f_j is also the ample model of $K_Z + \Theta + \delta B_0$. In particular $\Theta + \delta B_0 \in \mathcal{A}_j \cap \mathcal{C}_i$. As we are supposing that Θ is general in $\mathcal{A}_j \cap \mathcal{C}_i$, in fact f must be a log terminal model of $K_Z + \Theta$. In particular f is $(K_Z + \Theta)$-negative.

Pick $\epsilon > 0$ such that if $\Xi \in V$ and $\|\Xi - \Theta\| < \epsilon$ then Ξ belongs to the interior of $\mathcal{L}_A(V)$ and f is $(K_Z + \Xi)$-negative. Then the condition that $\Xi \in \mathcal{C}_i$ is simply the condition that $K_X + \Delta = f_*(K_Z + \Xi)$ is nef. Let W be the affine suspace of $\mathrm{WDiv}_{\mathbb{R}}(X)$ given by pushing forward the elements of V and let

$$\mathcal{N} = \{\, \Delta \in W \mid K_X + \Delta \text{ is nef}\,\}.$$

Given $(a_1, a_2, \dots, a_k) \in \mathbb{R}^k$ let $B = \sum a_i B_i$ and $E = \sum a_i E_i$. If $\|B\| < \epsilon$ then, as $\Xi + B$ is numerically equivalent to $\Xi + E$, $K_X + \Delta \in \mathcal{N}$ if and only if $K_X + \Delta + f_* B \in \mathcal{N}$. In particular $\mathcal{C}_i$ is locally isomorphic to $\mathcal{N} \times \mathbb{R}^k$.

But since f_j is the ample model of $K_Z + \Theta$, in fact we can choose ϵ sufficiently small so that $K_X + \Delta$ is nef if and only if $K_X + \Delta$ is nef over X_j, see §3 of [1]. There is a surjective affine linear map from W to the space of Weil divisors on X modulo numerical equivalence over X_j and this induces an isomorphism

$$\mathcal{N} \simeq \overline{\mathrm{NE}}(X/X_j)^* \times \mathbb{R}^l,$$

in a neighbourhood of $f_*\Theta$.

Note that $K_X + f_*\Theta$ is numerically trivial over X_j. As $f_*\Theta$ is big and $K_X + f_*\Theta$ is kawamata log terminal we may find an ample $\mathbb{Q}$-divisor

A' and a divisor $B' \geq 0$ such that

$$K_X + A' + B' \sim_{\mathbb{R}} K_X + f_* \Theta,$$

is kawamata log terminal. But then

$$-(K_X + B') \sim_{\mathbb{R}} -(K_X + \Delta') + A',$$

is ample over X_j. Hence $f_{ij} \colon X \longrightarrow X_j$ is a Fano fibration and so by the cone theorem

$$\rho(X_i/X_j) = \dim \mathcal{N}.$$

This is (4). $\qquad\square$

Corollary 3.4. *If V spans the Néron-Severi group of Z then there is a Zariski dense open subset U of the Grassmannian $G(\alpha, V)$ of real affine subspaces of dimension α such that if $[W] \in U$ and it is defined over the rationals then W satisfies (1-4) of (3.3).*

Proof. Let $U \subset G(\alpha, V)$ be the set of real affine subspaces W of V of dimension α, which contain no face of any $\mathcal{C}_i$ or $\mathcal{L}_A(V)$. In particular the interior of $\mathcal{L}_A(W)$ is contained in the interior of $\mathcal{L}_A(V)$. (3.3) implies that (1-2) always hold for W and (1-4) hold for V and so (3) and (4) clearly hold for $W \in U$. $\qquad\square$

From now on in this section we assume that V has dimension two and satisfies (1-4) of (3.3).

Lemma 3.5. *Let $f \colon Z \dashrightarrow X$ and $g \colon Z \dashrightarrow Y$ be two rational contractions such that $\mathcal{C}_{A,f}$ is two dimensional and $\mathcal{O} = \mathcal{C}_{A,f} \cap \mathcal{C}_{A,g}$ is one dimensional. Assume that $\rho(X) \geq \rho(Y)$ and that $\mathcal{O}$ is not contained in the boundary of $\mathcal{L}_A(V)$. Let Θ be an interior point of $\mathcal{O}$ and let $\Delta = f_* \Theta$.*

Then there is a rational contraction $\pi \colon X \dashrightarrow Y$ which factors $g = \pi \circ f$ and either

(1) *$\rho(X) = \rho(Y) + 1$ and π is a $(K_X + \Delta)$-trivial morphism, in which case, either*
    (a) *π is birational and $\mathcal{O}$ is not contained in the boundary of $\mathcal{E}_A(V)$, in which case, either*
        (i) *π is a divisorial contraction and $\mathcal{O} \neq \mathcal{C}_{A,g}$, or*
        (ii) *π is a small contraction and $\mathcal{O} = \mathcal{C}_{A,g}$, or*
    (b) *π is a Mori fibre space and $\mathcal{O} = \mathcal{C}_{A,g}$ is contained in the boundary of $\mathcal{E}_A(V)$, or*
(2) *$\rho(X) = \rho(Y)$, in which case, π is a $(K_X + \Delta)$-flop and $\mathcal{O} \neq \mathcal{C}_{A,g}$ is not contained in the boundary of $\mathcal{E}_A(V)$.*

Proof. By assumption f is birational and X is $\mathbb{Q}$-factorial. Let $h\colon Z \dashrightarrow W$ be the ample model corresponding to $K_Z+\Theta$. Since Θ is not a point of the boundary of $\mathcal{L}_A(V)$ if Θ belongs to the boundary of $\mathcal{E}_A(V)$ then $K_Z + \Theta$ is not big and so h is not birational. As $\mathcal{O}$ is a subset of both $\mathcal{C}_{A,f}$ and $\mathcal{C}_{A,g}$ there are morphisms $p\colon X \longrightarrow W$ and $q\colon Y \longrightarrow W$ of relative Picard number at most one. There are therefore only two possibilities:

(1) $\rho(X) = \rho(Y) + 1$, or
(2) $\rho(X) = \rho(Y)$.

Suppose we are in case (1). Then q is the identity and $\pi = p\colon X \longrightarrow Y$ is a contraction morphism such that $g = \pi \circ f$. Suppose that π is birational. Then h is birational and $\mathcal{O}$ is not contained in the boundary of $\mathcal{E}_A(V)$. If π is divisorial then Y is $\mathbb{Q}$-factorial and so $\mathcal{O} \neq \mathcal{C}_{A,g}$. If π is a small contraction then Y is not $\mathbb{Q}$-factorial and so $\mathcal{C}_{A,g} = \mathcal{O}$ is one dimensional. If π is a Mori fibre space then $\mathcal{O}$ is contained in the boundary of $\mathcal{E}_A(V)$ and $\mathcal{O} = \mathcal{C}_{A,g}$.

Now suppose we are in case (2). By what we have already proved $\rho(X/W) = \rho(Y/W) = 1$. p and q are not divisorial contractions as $\mathcal{O}$ is one dimensional. p and q are not Mori fibre spaces as $\mathcal{O}$ cannot be contained in the boundary of $\mathcal{E}_A(V)$. Hence p and q are small and the rest is clear. □

Lemma 3.6. *Let $f\colon W \dashrightarrow X$ be a birational contraction between projective $\mathbb{Q}$-factorial varieties. Suppose that (W, Θ) and (W, Φ) are both kawamata log terminal.*

If f is the ample model of $K_W + \Theta$ and $\Theta - \Phi$ is ample then f is the result of running the $(K_W + \Phi)$-MMP.

Proof. By assumption we may find an ample divisor H on W such that $K_W + \Phi + H$ is kawamata log terminal and ample and a positive real number $t < 1$ such that $tH \sim_{\mathbb{R}} \Theta - \Phi$. Note that f is the ample model of $K_W + \Phi + tH$. Pick any $s < t$ sufficiently close to t so that f is $(K_W + \Phi + sH)$-negative and yet f is still the ample model of $K_W + \Phi + sH$. Then f is the unique log terminal model of $K_W + \Phi + sH$. In particular if we run the $(K_W + \Phi)$-MMP with scaling of H then, when the value of the scalar is s, the induced rational map is f. □

We now adopt some more notation for the rest of this section. Let $\Theta = A + B$ be a point of the boundary of $\mathcal{E}_A(V)$ in the interior of $\mathcal{L}_A(V)$. Enumerate $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_k$ the polytopes $\mathcal{C}_i$ of dimension two which contain Θ. Possibly re-ordering we may assume that the intersections $\mathcal{O}_0$ and $\mathcal{O}_k$ of $\mathcal{T}_1$ and $\mathcal{T}_k$ with the boundary of $\mathcal{E}_A(V)$ and $\mathcal{O}_i = \mathcal{T}_i \cap \mathcal{T}_{i+1}$ are

all one dimensional. Let $f_i\colon Z \dashrightarrow X_i$ be the rational contractions associated to $\mathcal{T}_i$ and $g_i\colon Z \dashrightarrow S_i$ be the rational contractions associated to $\mathcal{O}_i$. Set $f = f_1\colon Z \dashrightarrow X$, $g = f_k\colon Z \dashrightarrow Y$, $X' = X_2$, $Y' = X_{k-1}$. Let $\phi\colon X \longrightarrow S = S_0$, $\psi\colon Y \longrightarrow T = S_k$ be the induced morphisms and let $Z \dashrightarrow R$ be the ample model of $K_Z + \Theta$.

$\mathcal{O}_k$ $\mathcal{O}_{k-1}$ $\mathcal{O}_1$

$\mathcal{T}_k$ $\mathcal{T}_{k-1}$ $\cdots$ $\mathcal{T}_2$ $\mathcal{T}_1$

Θ
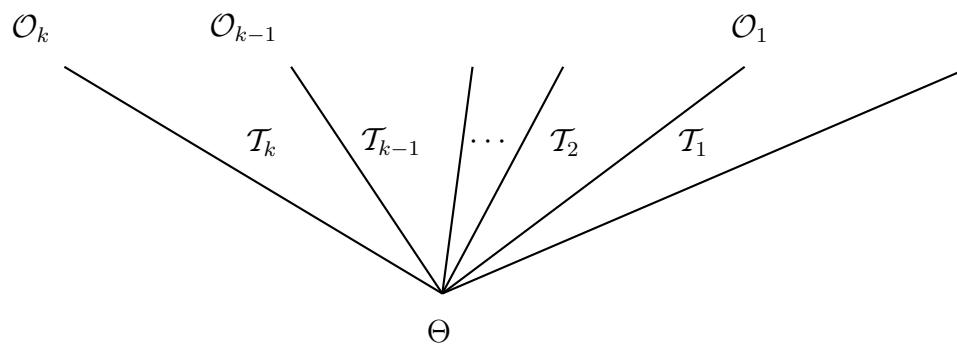
Theorem 3.7. *Suppose Φ is any divisor such that $K_Z + \Phi$ is kawamata log terminal and $\Theta - \Phi$ is ample.*

Then ϕ and ψ are two Mori fibre spaces which are outputs of the $(K_Z + \Phi)$-MMP which are connected by a Sarkisov link if Θ is contained in more than two polytopes.

Proof. We assume for simplicity of notation that $k \geq 3$. The case $k \leq 2$ is similar and we omit it. The incidence relations between the corresponding polytopes yield a commutative heptagon,

$$
\begin{array}{ccc}
X' & \dashrightarrow & Y' \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\
X & & Y \\
\downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} \\
S & & T \\
\searrow{\scriptstyle s} & & \swarrow{\scriptstyle t} \\
& R &
\end{array}
$$

where p and q are birational maps. ϕ and ψ are Mori fibre spaces by (3.5). Pick Θ_1 and Θ_k in the interior of $\mathcal{T}_1$ and $\mathcal{T}_k$ sufficiently close to Θ so that $\Theta_1 - \Phi$ and $\Theta_k - \Phi$ are ample. As X and Y are $\mathbb{Q}$-factorial, (3.6) implies that ϕ and ψ are possible outcomes of the $(K_Z + \Phi)$-MMP. Let $\Delta = f_* \Theta$. Then $K_X + \Delta$ is numerically trivial over R.

Note that there are contraction morphisms $X_i \longrightarrow R$ and that $\rho(X_i/R) \leq 2$. If $\rho(X_i/R) = 1$ then $X_i \longrightarrow R$ is a Mori fibre space. By (3.3) there is facet of $\mathcal{T}_i$ which is contained in the boundary of $\mathcal{E}_A(V)$ and so $i = 1$ or k. Thus $X_i \dashrightarrow X_{i+1}$ is a flop, $1 < i < k - 1$. Since $\rho(X'/R) = 2$ it follows that either p is a divisorial contraction and s is the identity or p is a flop and s is not the identity. We have a similar dichotomy for $q \colon Y' \dashrightarrow Y$ and $t \colon T \longrightarrow R$.

There are then four cases. If s and t are the identity then p and q are divisorial extractions and we have a link of type II.

If s is the identity and t is not then p is a divisorial extraction and q is a flop and we have a link of type I. Similarly if t is the identity and s is not then q is a divisorial extraction and p is a flop and we have a link of type III.

Finally suppose neither s nor t is the identity. Then both p and q are flops. Suppose that s is a divisorial contraction. Let F be the divisor contracted by s and let E be its inverse image in X. Since ϕ has relative Picard number one $\phi^*(F) = mE$, for some positive integer m. Then

$K_X + \Delta + \delta E$ is kawamata log terminal for any $\delta > 0$ sufficiently small and $E = \mathbf{B}(K_X + \Delta + \delta E / R)$. If we run the $(K_X + \Delta + \delta E)$-MMP over R then we end with a birational contraction $X \dashrightarrow W$, which is a Mori fibre space over R. Since $\rho(X/R) = 2$, $W = Y$ and we have a link of type III, a contradiction. Similarly t is never a divisorial contraction. If s is a Mori fibre space then R is $\mathbb{Q}$-factorial and so t must be a Mori fibre space as well. This is a link of type IV_m. If s is small then R is not $\mathbb{Q}$-factorial and so t is small as well. Thus we have a link of type IV_s. $\qquad\square$

4. Proof of (1.3)

Lemma 4.1. *Let $\phi\colon X \longrightarrow S$ and $\psi\colon Y \longrightarrow T$ be two Sarkisov related Mori fibre spaces corresponding to two $\mathbb{Q}$-factorial kawamata log terminal projective varieties (X, Δ) and (Y, Γ).*

Then we may find a smooth projective variety Z, two birational contractions $f\colon Z \dashrightarrow X$ and $g\colon Z \dashrightarrow Y$, a kawamata log terminal pair (Z, Φ), an ample $\mathbb{Q}$-divisor A on Z and a two dimensional rational affine subspace V of $\mathrm{WDiv}_{\mathbb{R}}(Z)$ such that

  (1) *if $\Theta \in \mathcal{L}_A(V)$ then $\Theta - \Phi$ is ample,*
  (2) *$\mathcal{A}_{A,\phi \circ f}$ and $\mathcal{A}_{A,\psi \circ g}$ are not contained in the boundary of $\mathcal{L}_A(V)$,*
  (3) *V satisfies (1-4) of (3.3),*
  (4) *$\mathcal{C}_{A,f}$ and $\mathcal{C}_{A,g}$ are two dimensional, and*
  (5) *$\mathcal{C}_{A,\phi \circ f}$ and $\mathcal{C}_{A,\psi \circ g}$ are one dimensional.*

Proof. By assumption we may find a $\mathbb{Q}$-factorial kawamata log terminal pair (Z, Φ) such that $f\colon Z \dashrightarrow X$ and $g\colon Z \dashrightarrow Y$ are both outcomes of the $(K_Z + \Phi)$-MMP.

Let $p\colon W \longrightarrow Z$ be any log resolution of (Z, Φ) which resolves the indeterminancy of f and g. We may write

$$K_W + \Psi = p^*(K_Z + \Phi) + E',$$

where $E' \geq 0$ and $\Psi \geq 0$ have no common components, E' is exceptional and $p_*\Psi = \Phi$. Pick $-E$ ample over Z with support equal to the full exceptional locus such that $K_W + \Psi + E$ is kawamata log terminal. As p is $(K_W + \Psi + E)$-negative, $K_Z + \Phi$ is kawamata log terminal and Z is $\mathbb{Q}$-factorial, the $(K_W + \Psi + E)$-MMP over Z terminates with the pair (Z, Φ) by (3.6). Replacing (Z, Φ) with $(W, \Psi + E)$, we may assume that (Z, Φ) is log smooth and f and g are morphisms.

Pick general ample $\mathbb{Q}$-divisors $A, H_1, H_2, \ldots, H_k$ on Z such that $H_1, H_2, \ldots, H_k$ generate the Néron-Severi group of Z. Let

$$H = A + H_1 + H_2 + \cdots + H_k.$$

Pick sufficiently ample divisors C on S and D on T such that

$$-(K_X + \Delta) + \phi^* C \qquad \text{and} \qquad -(K_Y + \Gamma) + \psi^* D,$$

are both ample. Pick a rational number $0 < \delta < 1$ such that

$$-(K_X + \Delta + \delta f_* H) + \phi^* C \qquad \text{and} \qquad -(K_Y + \Gamma + \delta g_* H) + \psi^* D,$$

are both ample and $K_Z + \Phi + \delta H$ is both f and g-negative. Replacing H by δH we may assume that $\delta = 1$. Now pick a $\mathbb{Q}$-divisor $\Phi_0 \leq \Phi$ such that $A + (\Phi_0 - \Phi)$,

$$-(K_X + f_* \Phi_0 + f_* H) + \phi^* C \quad \text{and} \quad -(K_Y + g_* \Phi_0 + g_* H) + \psi^* D,$$

are all ample and $K_Z + \Phi_0 + H$ is both f and g-negative.

Pick general ample $\mathbb{Q}$-divisors $F_1 \geq 0$ and $G_1 \geq 0$

$$F_1 \sim_{\mathbb{Q}} -(K_X + f_* \Phi_0 + f_* H) + \phi^* C \quad \text{and} \quad G_1 \sim_{\mathbb{Q}} -(K_Y + g_* \Phi_0 + g_* H) + \psi^* D.$$

Then

$$K_Z + \Phi_0 + H + F + G,$$

is kawamata log terminal, where $F = f^* F_1$ and $G = g^* G_1$.

Let V_0 be the affine subspace of $\mathrm{WDiv}_{\mathbb{R}}(Z)$ which is the translate by Φ_0 of the vector subspace spanned by $H_1, H_2, \ldots, H_k, F, G$. Suppose that $\Theta = A + B \in \mathcal{L}_A(V_0)$. Then

$$\Theta - \Phi = (A + \Phi_0 - \Phi) + (B - \Phi_0),$$

is ample, as $B - \Phi_0$ is nef by definition of V_0. Note that $\Phi_0 + F + H \in \mathcal{A}_{A,\phi \circ f}(V_0)$, $\Phi_0 + G + H \in \mathcal{A}_{A,\psi \circ g}(V_0)$, and f, respectively g, is a weak log canonical model of $K_Z + \Phi_0 + F + H$, respectively $K_Z + \Phi_0 + G + H$. (3.3) implies that V_0 satisfies (1-4) of (3.3).

Since $H_1, H_2, \ldots, H_k$ generate the Néron-Severi group of Z we may find constants $h_1, h_2, \ldots, h_k$ such that G is numerically equivalent to $\sum h_i H_i$. Then $\Phi_0 + F + \delta G + H - \delta(\sum h_i H_i)$ is numerically equivalent to $\Phi_0 + F + H$ and if $\delta > 0$ is small enough $\Phi_0 + F + \delta G + H - \sum \delta h_i H_i \in \mathcal{L}_A(V_0)$. Thus $\mathcal{A}_{A,\phi \circ f}(V_0)$ is not contained in the boundary of $\mathcal{L}_A(V_0)$. Similarly $\mathcal{A}_{A,\psi \circ g}(V_0)$ is not contained in the boundary of $\mathcal{L}_A(V_0)$. In particular $\mathcal{A}_{A,f}(V_0)$ and $\mathcal{A}_{A,g}(V_0)$ span V_0 and $\mathcal{A}_{A,\phi \circ f}(V_0)$ and $\mathcal{A}_{A,\psi \circ g}(V_0)$ span affine hyperplanes of V_0, since $\rho(X/S) = \rho(Y/T) = 1$.

Let V_1 be the translate by Φ_0 of the two dimensional vector space spanned by $F + H - A$ and $F + G - A$. Let V be a small general perturbation of V_1, which is defined over the rationals. Then (2) holds. (1) holds, as it holds for any two dimensional subspace of V_0, (3) holds by (3.4) and this implies that (4) and (5) hold. $\qquad \square$

Proof of (1.3). Pick (Z, Φ), A and V given by (4.1). Pick points $\Theta_0 \in \mathcal{A}_{A, \phi \circ f}(V)$ and $\Theta_1 \in \mathcal{A}_{A, \psi \circ g}(V)$ belonging to the interior of $\mathcal{L}_A(V)$. As V is two dimensional, removing Θ_0 and Θ_1 divides the boundary of $\mathcal{E}_A(V)$ into two parts. The part which consists entirely of divisors which are not big is contained in the interior of $\mathcal{L}_A(V)$. Consider tracing this boundary from Θ_0 to Θ_1. Then there are finitely many $2 \leq i \leq l$ points Θ_i which are contained in more than two polytopes $\mathcal{C}_{A, f_i}(V)$. (3.7) implies that for each such point there is a Sarkisov link $\sigma_i \colon X_i \dashrightarrow Y_i$ and σ is the composition of these links. $\qquad \square$

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