hep-th/0305048
NSF-KITP-03-27

Dibaryon Spectroscopy

Christopher P. Herzog[♯] and James M[c]Kernan[♭]
♯ KITP, UCSB, ♭ Dept. of Math., UCSB,
Santa Barbara, CA 93106, U.S.A.
herzog@kitp.ucsb.edu, mckernan@math.ucsb.edu

Abstract

The AdS/CFT correspondence relates dibaryons in superconformal gauge theories to holomorphic curves in Kähler-Einstein surfaces. The degree of the holomorphic curves is proportional to the gauge theory conformal dimension of the dibaryons. Moreover, the number of holomorphic curves should match, in an appropriately defined sense, the number of dibaryons. Using AdS/CFT backgrounds built from the generalized conifolds of Gubser, Shatashvili, and Nekrasov (1999), we show that the gauge theory prediction for the dimension of dibaryonic operators does indeed match the degree of the corresponding holomorphic curves. For AdS/CFT backgrounds built from cones over del Pezzo surfaces, we are able to match the degree of the curves to the conformal dimension of dibaryons for the nth del Pezzo surface, $1 \leq n \leq 6$. Also, for the del Pezzos and the A_k type generalized conifolds, for the dibaryons of smallest conformal dimension, we are able to match the number of holomorphic curves with the number of possible dibaryon operators from gauge theory.

May 2003

1 Introduction

AdS/CFT correspondence [1, 2, 3] asserts that type IIB string theory on $AdS_5 \times \mathbf{X}$ is equivalent to a superconformal quiver gauge theory. The geometric objects involved here are five dimensional anti-de Sitter space AdS_5 and a five dimensional Sasaki-Einstein manifold $\mathbf{X}$. A supersymmetric quiver gauge theory contains a collection of vector multiplets transforming under $SU(N)$ gauge groups for each node in the quiver and, for each line in the quiver, a collection of chiral multiplets transforming under bifundamental and adjoint representations of the gauge groups. Despite extensive efforts, AdS/CFT remains a conjecture and the present work was motivated largely by the need to find checks of the correspondence which do not require explicit knowledge of the metric on $\mathbf{X}$.

To review the origins of AdS/CFT, recall that the original correspondence [1, 2, 3] was motivated by comparing a stack of N elementary branes with the metric it produces (for reviews, see for example [4, 5]). For a stack placed in ten dimensional space, the theory on the branes is $\mathcal{N} = 4$ supersymmetric $SU(N)$ gauge theory. The quiver would have a single node, corresponding to a single $\mathcal{N} = 4$ vector multiplet. The gravitational back reaction from the stack causes the space to factorize into $AdS_5 \times \mathbf{S}^5$ close to the D3-branes; the gauge theory on the branes is conjectured to be equivalently described by type IIB string theory in this factorized background.

In order to break some of the supersymmetry (SUSY), we may place the branes at a conical singularity instead of in flat ten dimensional space [6, 7, 8, 9, 10, 11]. For example, branes placed at the orbifold singularity of $\mathbb{C}^2/\Gamma$ where Γ is a discrete subgroup of $SU(2)$ preserve $\mathcal{N} = 2$ SUSY [6, 7]. The gauge theory quivers correspond to simply laced Dynkin diagrams, as will be reviewed in greater detail in section 5. Geometrically, the ten dimensional space factorizes into $AdS_5 \times \mathbf{S}^5/\Gamma$. Only $\mathcal{N} = 1$ SUSY is preserved if the branes are placed at a conifold singularity [8, 10, 11]. The gauge theory here is $SU(N) \times SU(N)$ with bifundamental matter, while the geometry is $AdS_5 \times T^{1,1}$

Unfortunately, S^5, $T^{1,1}$, and their orbifolds just about exhaust the mathematical literature of SUSY preserving five dimensional Einstein spaces for which explicit metrics are known. In these cases, it was explicit knowledge of the metric which allowed for many tests of the AdS/CFT correspondence. Nevertheless, there are an infinite number of five dimensional $\mathbf{X}$ from which one can construct AdS/CFT correspondences, and it would be very useful to have techniques for dealing with more general $\mathbf{X}$.

In constructing these more general correspondences, we start with a non-compact Calabi-Yau cone $\mathbf{Y}$, whose base is the five dimensional Sasaki-Einstein manifold $\mathbf{X}$. Comparing the metric with the D-brane description leads one to conjecture that type IIB string theory on $AdS_5 \times \mathbf{X}$ is dual to the low-energy description of the worldvolume theory on the D3-branes at the conical singularity [10, 11]. It is known that for a stack of D3-branes placed at the conical singularity of $\mathbf{Y}$, the ten dimensional supergravity solution is

$$ds^2 = h(r)^{-1/2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + h(r)^{1/2}(dr^2 + r^2 ds^2_{\mathbf{X}}) ,$$

$$h(r) = 1 + \frac{L^4}{r^4} \ , \quad L^4 = \frac{4\pi^4 g_s N \alpha'^2}{\mathrm{Vol}(\mathbf{X})} \ , \tag{1}$$

$$F_5 = \mathcal{F}_5 + \star\mathcal{F}_5 \ , \quad \mathcal{F}_5 = 16\pi\alpha'^2 N \frac{\mathrm{Vol}(\mathbf{S}^5)}{\mathrm{Vol}(\mathbf{X})} \mathrm{vol}(\mathbf{X}) \ ,$$

where all the other field strengths vanish and N is the number of D3-branes. The constants g_s and α' are the string coupling and the Regge slope. With this notation, vol is the volume differential form. Thus

$$\int_{\mathbf{X}} \mathrm{vol}(\mathbf{X}) = \mathrm{Vol}(\mathbf{X}) \ .$$

To get the space to factorize into $AdS_5 \times \mathbf{X}$, we take the near horizon limit $r \ll L$.

As can be seen from (1), it would be good to have a metric independent expression for the Vol($\mathbf{X}$). Indeed, in [12], such an expression was derived for a large class of Sasaki-Einstein manifolds $\mathbf{X}$. This volume is also important for calculating the central charge c of the gauge theory, as was pointed out in [13, 14]. In this paper, we will calculate the minimal volumes of three-cycles in $\mathbf{X}$ in a metric independent way. These three-cycles are important for calculating the conformal dimension of dibaryonic operators and may have other uses as well.

In order to demonstrate that the AdS/CFT correspondence is a duality between string theory and gauge theory and not just between a supergravity theory and a gauge theory, objects known as dibaryons and giant gravitons have played an important role [15, 16]. On the string theory side, dibaryons and giant gravitons correspond to supersymmetric D3-branes wrapped on three cycles $\mathcal{H} \subset \mathbf{X}$. Roughly speaking, the mass m of these D3-branes is just the D3-brane tension τ times Vol($\mathcal{H}$). In fact, the mass suffers additional zero mode corrections [17], and instead the conformal dimension Δ is exactly

$$\Delta = L^4 \mathrm{Vol}(\mathcal{H}) \tau \ . \tag{2}$$

Geometrically, the conformal dimension is the eigenvalue of the $r\partial/\partial r$ operator acting on a wrapped D3-brane state. On the gauge theory side, these dibaryons correspond to antisymmetrized products of order N matter fields. The conformal dimension Δ of the dibaryon is the number of bifundamental matter fields in the antisymmetrized product multiplied by the conformal dimension of the individual bifundamental matter fields.

An important piece of evidence that dibaryons are alternately described either as wrapped D3-branes or as antisymmetric products of order N matter fields is that the conformal dimensions, when calculated in these two different ways, agree. The dibaryon dimension is calculated from wrapped D3-branes at strong 't Hooft coupling while the gauge theory calculation is naturally a weak coupling result. One may ask why these conformal dimensions are not a function of the 't Hooft coupling. The reason is that the dibaryons are BPS objects and their conformal dimension is protected by the SUSY algebra [15].

To understand the relationship between dibaryons and holomorphic curves mentioned in the abstract, we note that generically these Sasaki-Einstein manifolds $\mathbf{X}$ can be expressed as $U(1)$ fibrations of Kähler-Einstein surfaces $\mathbf{V}$. Similarly, for the simplest, time independent dibaryons, the wrapped three-cycles $\mathcal{H}$ correspond to $U(1)$ fibrations over holomorphic curves $C \subset \mathbf{V}$. The Kähler-Einstein relation allows us to relate $\mathrm{Vol}(\mathcal{H})$ to the degree $-K_{\mathbf{V}} \cdot C$ of the curve C. Moreover, there should be a relationship between the number of curves C of a given degree and the number of dibaryons of a given conformal dimension, which we will only begin to explore in this paper.

We begin by giving a precise geometric description of dibaryonic operators and derive a metric independent formula for the volumes of the wrapped three-cycles. Next, we show that our formula gives the correct dimension for giant gravitons in $\mathbf{S}^5$ and dibaryons in $T^{1,1}$ and $\mathbf{S}^5/\mathbb{Z}_3$. Previous derivations [16, 18] of these dimensions from geometry relied on an explicit knowledge of the $\mathbf{S}^5$ and $T^{1,1}$ metrics.

In the second half of the paper, we apply our formula to study dibaryons in new contexts. We apply our formula to and do some simple counting of dibaryons in U(1) bundles over smooth del Pezzo surfaces. Previously, only dibaryons in the third del Pezzo had been studied in detail [19]. These U(1) bundles over smooth del Pezzo surfaces are examples of Sasaki-Einstein manifolds $\mathbf{X}$ and have been heavily studied in the context of AdS/CFT correspondence [20, 21, 22, 23]. We take advantage of some recent progress in understanding the gauge theory duals of these manifolds [24, 25].

Finally, we consider the dibaryons in the generalized conifolds of Gubser, Nekrasov, and Shatashvili [26]. These generalized conifolds can be understood alternately as generalizations of the $AdS_5 \times T^{1,1}$ correspondence or as deformations of the $AdS_5 \times \mathbf{S}^5/\Gamma$ correspondences reviewed above. In order to apply our volume formula, we will need to develop some additional tools from algebraic geometry to deal with the quotient singularities that appear in studying these $\mathbf{X}$. For the dibaryons of smallest conformal dimension, we are able to match the number of holomorphic curves to the number of gauge theory dibaryons for some of the generalized conifolds.

2 The Geometry of the Dibaryon

To understand these wrapped D3-branes better, let us review an argument due to Mikhailov [27] and Beasley [28] that relates the D3-brane wrapping in $\mathbf{X}$ to holomorphic four cycles in the full Calabi-Yau cone $\mathbf{Y}$. We begin by thinking about Euclidean signature ten dimensional space rather than Minkowski signature. It is well known that in the standard compactification of ten dimensional space on a six dimensional Calabi-Yau manifold, D-branes that wrap holomorphic four cycles in $\mathbf{Y}$ preserve supersymmetry and hence will be stable. When we add the F_5 flux from the N D3-branes, the near horizon geometry factorizes into $\mathbf{X} \times H^5$ where H^5 is Euclidean signature hyperbolic space. While the D3-brane looked like a point in the Euclidean $\mathbb{R}^4$, it now looks like a line in H^5. The next step is to Wick rotate. Wick rotation will preserve the supersymmetry, and the hyperbolic space H^5 is highly symmetric so we can Wick rotate a coordinate that will cause the D3-brane's path in H^5 to be time-like in the resulting AdS_5.

From this construction of a giant graviton or dibaryon, it is clear that the time coordinate in AdS_5 will be paired holomorphically to a second coordinate in $\mathbf{X}$ to make what used to be a single complex coordinate in the cone $\mathbf{Y}$. To see how this pairing works, we need to investigate the structure of $\mathbf{X}$ in greater detail.

Let $\mathbf{X}$ be a Riemannian manifold of real dimension $2n+1$ and g_{ab} the associated Riemannian metric. $\mathbf{X}$ is defined to be Sasaki if the holonomy group of the cone $\mathbf{Y} \equiv \mathbb{R}^+ \times \mathbf{X}$ with metric

$$ds^2_{\mathbf{Y}} = dr^2 + r^2 g_{ab} dx^a dx^b \tag{3}$$

reduces to a subgroup of $U(n+1)$. Moreover, $\mathbf{X}$ is Sasaki-Einstein if and only if the cone $\mathbf{Y}$ is Calabi-Yau. Let us specialize to the case where $\mathbf{Y}$ is three complex dimensional. One finds $R_{ab} = 4g_{ab}$ where R_{ab} is the Ricci tensor on $\mathbf{X}$.

There is a natural $U(1)$ group action on $\mathbf{X}$, and a number of ways to understand where it comes from. From AdS/CFT correspondence, we know the gauge theories are conformal and have $\mathcal{N} = 1$ supersymmetry, provided $\mathbf{Y}$ is Calabi-Yau. Thus, there will be a $U(1)$ R-symmetry. Alternately, there is a complex structure on $\mathbf{Y}$. By taking the level surfaces $\mathbf{X}$ of $\mathbf{Y}$, we have essentially quotiented by the absolute value of one of the complex coordinates on $\mathbf{Y}$. This complex coordinate also has a phase, which corresponds to this $U(1)$ action.

There exist a large class of $\mathbf{X}$ called quasiregular Sasaki-Einstein manifolds for which the orbits of the $U(1)$ action are compact [29]. If $\mathbf{X}$ is quasiregular, then it can be expressed as a circle bundle $\pi : \mathbf{X} \to \mathbf{V}$ where $\mathbf{V}$ is a Kähler-Einstein orbifold. Let π^* be the pullback from $\mathbf{V}$ to $\mathbf{X}$. Let

$$\omega = i h_{a\bar{b}} dz^a \wedge d\bar{z}^{\bar{b}} \tag{4}$$

be the Kähler form on $\mathbf{V}$. Define a one-form η on the fibration with curvature $d\eta = 2\pi^*\omega$. The metric on $\mathbf{X}$ is

$$\mathbf{g} = \pi^* \mathbf{h} + \eta \otimes \eta \ . \tag{5}$$

If we let ψ be an angular coordinate on the circle bundle, this construction has hopefully made clear that the radius r and ψ get paired holomorphically as a single complex coordinate in $\mathbf{Y}$. From our construction of a wrapped D3-brane, we saw that r gets turned into the time coordinate t. Thus t and ψ are paired. The embedding of the D3-brane can depend on time only in the combination $t + \psi$. (The coordinate t can be thought of as global time on AdS.)

In this paper, we will restrict to dibaryon and giant graviton configurations which are time independent. (For more on time dependence, see [28, 17].) As a result, the configurations must also be ψ independent. In other words, the wrappings will be invariant under the U(1) action of the fiber bundle. The way to insure this invariance is to make sure that for every point $p \in \mathbf{V}$ where the D3-brane is present, the D3-brane also wraps completely the U(1) fiber at that point. Holomorphicity then requires that these points $p \in \mathbf{V}$ trace out a holomorphic curve in $\mathbf{V}$ which we shall call C.

Naively, one might have thought our D3-brane wrappings would correspond to topologically distinct three-cycles of $\mathbf{X}$. This intuition is not correct. There are many more holomorphic curves C in $\mathbf{V}$ then there are homology cycles although in general we can build C out of homology cycles in $\mathbf{V}$. The minimal volume curves should correspond to homology cycles in $\mathbf{V}$, but they do not, always, correspond to homology cycles in $\mathbf{X}$. Instead, the minimal volume curves correspond to equivariant homology cycles in $\mathbf{X}$.

Let ω be the Kähler form on $\mathbf{V}$. Let ω_i, $i = 1, \ldots, n$ correspond to the other harmonic forms on $\mathbf{V}$. (It is possible there are no harmonic forms besides ω.) We can insist that $\omega \wedge \omega_i = 0$. To construct harmonic forms on $\mathbf{X}$, we wedge with the one form η. The set of harmonic three forms on $\mathbf{X}$ is given by $\{\eta \wedge \pi^* \omega_i : i = 1, \ldots n\}$ where π^* is the pull back map of the fibration. Note that $\eta \wedge \pi^* \omega$ is not closed, and so the Betti number of $\mathbf{X}$ $b_3 = n$ will be one less than the Hodge number $h^{1,1} = n + 1$ of $\mathbf{V}$.

It turns out, however, that the dimension of the equivariant homology is exactly the dimension of $H^{1,1}(\mathbf{V})$. Consider the two-form $\pi^* \omega$. It is not harmonic because $d\eta = 2\pi^* \omega$. However, we can't use η in calculating the equivariant cohomology because η is not invariant under the $U(1)$ action. A more careful analysis shows that ω is a good representative of the equivariant cohomology, and so its dual cycle is a good representative of the equivariant homology.

Now to compute volumes of these submanifolds in $\mathbf{X}$, we shall make use of the Einstein condition. In particular, the fact that $\mathbf{V}$ is Kähler-Einstein means $\mathbf{R} = 6\mathbf{h}$, where $\mathbf{R}$ is the Ricci tensor on $\mathbf{V}$. The first Chern class of $\mathbf{V}$, denoted $c_1(\mathbf{V})$, is given locally by

$$c_1(\mathbf{V}) = i \frac{R_{a\bar{b}}}{2\pi} dz^a \wedge d\bar{z}^{\bar{b}} \ . \tag{6}$$

The critical step here is to use the Kähler-Einstein condition to write the Kähler form ω in terms of the first Chern class

$$\omega = \frac{\pi}{3} c_1(\mathbf{V}) \ . \tag{7}$$

It is this relation which obviates the need for explicit knowledge of the metric on $\mathbf{V}$. Volumes, which are proportional to integrals over ω, can now be expressed as integrals over the curvature, which usually have a topological interpretation.

Now the volume of $\mathbf{V}$, as was found in [12] is (see also [30])

$$\mathrm{Vol}(\mathbf{V}) = \frac{1}{2} \int_{\mathbf{V}} \omega \wedge \omega = \frac{\pi^2}{18} \int_{\mathbf{V}} c_1(\mathbf{V})^2 \ . \tag{8}$$

If we define $K_{\mathbf{V}}$ to be the canonical line bundle over $\mathbf{V}$, one finds

$$\mathrm{Vol}(\mathbf{V}) = \frac{\pi^2}{18} K_{\mathbf{V}} \cdot K_{\mathbf{V}} \ . \tag{9}$$

In other words, the volume of $\mathbf{V}$ is related in a simple way to the self-intersection number of the canonical line bundle.

The quasiregular condition means additionally that the length of the U(1) fiber does not vary as we move around in $\mathbf{V}$. Thus, the volume of the manifold $\mathbf{X}$ is the volume of the manifold $\mathbf{V}$ times the length of the U(1) fiber. Similarly, the volume of the three-cycle $\mathcal{H}$ will be the area of the corresponding holomorphic curve C in $\mathbf{V}$ times the length of the U(1) fiber.

To make the gauge theory comparison, we need to calculate the conformal dimension of a D3-brane wrapped on such a cycle $\mathcal{H}$. As was shown in [17] and mentioned in the introduction

$$\Delta = L^4 \mathrm{Vol}(\mathcal{H}) \tau \ ,$$

where $\tau = 1/8\pi^3 g_s \alpha'^2$ is the D3-brane tension. Then from the quantization condition (1) on L

$$\Delta = L^4 \mathrm{Vol}(C) \tau = \frac{\pi N}{2} \frac{\mathrm{Vol}(\mathcal{H})}{\mathrm{Vol}(\mathbf{X})} \ . \tag{10}$$

Both $\mathrm{Vol}(\mathcal{H})$ and $\mathrm{Vol}(\mathbf{X})$ include the same factor of the length of the U(1) fiber. We divide out to find

$$\Delta = \frac{\pi N}{2} \frac{\mathrm{Vol}(C)}{\mathrm{Vol}(\mathbf{V})} \ . \tag{11}$$

We can relate the $\mathrm{Vol}(C)$ to an intersection number calculation. In particular

$$\mathrm{Vol}(C) = \int_C \omega = \frac{\pi}{3} \int_C c_1(\mathbf{V}) = -\frac{\pi}{3} K_{\mathbf{V}} \cdot C \ . \tag{12}$$

Putting the pieces (9), (11), and (12) together, we arrive at our final formula for the dibaryon dimension

$$\Delta = -3N \frac{K_{\mathbf{V}} \cdot C}{K_{\mathbf{V}} \cdot K_{\mathbf{V}}} \ . \tag{13}$$

For smooth manifolds $\mathbf{V}$, the intersection numbers $K_{\mathbf{V}} \cdot K_{\mathbf{V}}$ and $K_{\mathbf{V}} \cdot C$ are integers. When $\mathbf{V}$ is an orbifold however, these intersections are generally rational numbers.

3 Dibaryons in $T^{1,1}$ and Giant Gravitons in $\mathbf{S}^5$

This formula (13) is rather powerful. To understand how to apply it, let us begin with examples of $\mathbf{V}$ where explicit metrics are known and where the calculation of Δ was performed originally in a more straightforward manner.

3.1 $AdS_5 \times \mathbf{S}^5$

We begin with $AdS_5 \times \mathbf{S}^5$. The sphere $\mathbf{S}^5$ can be thought of as a U(1) bundle over $\mathbb{P}^2$. The surface $\mathbf{V} = \mathbb{P}^2$ is clearly a Kähler-Einstein space. Let H be the hyperplane bundle on $\mathbb{P}^2$.

The canonical bundle on $\mathbb{P}^2$ is $K_{\mathbb{P}^2} = -3H$. The intersection $H \cdot H = 1$. This relation is just a formal statement of the fact that two lines intersect at a point. Let our holomorphic curve be H. From (13), one finds that $\Delta = N$ for the choice $C = H$.

This result $\Delta = N$ for $AdS_5 \times \mathbf{S}^5$ has two different possible interpretations. The first interpretation involves the giant gravitons of [16]. The gauge theory dual of $AdS_5 \times \mathbf{S}^5$ is $\mathcal{N} = 4$ $SU(N)$ super Yang Mills, which contains three scalars, X, Y, and Z, transforming in the adjoint of $SU(N)$. Each of these scalars has conformal dimension 1. There is clearly a set of operators of dimension N obtained by antisymmetrizing over a product of N of the X's, Y's, and Z's

$$\epsilon^{i_i i_2 \ldots i_N} \epsilon_{j_1 j_2 \ldots j_N} X^{j_1}_{i_1} \cdots X^{j_x}_{i_x} Y^{j_{x+1}}_{i_{x+1}} \cdots Y^{j_y}_{i_y} Z^{j_{y+1}}_{i_{y+1}} \cdots Z^{j_N}_{i_N} \tag{14}$$

where $x \leq y$ are some integers. These operators are the maximal giant gravitons of [16]. They are not topologically stable but rather dynamically stabilized. So despite the fact that it doesn't make sense to think about D3-branes wrapping topologically nontrivial cycles in $\mathbf{S}^5$, the topology of the underlying $\mathbb{P}^2$ is all that's needed to understand the conformal dimension of the maximal giant gravitons. There also exist smaller giants where some of the X, Y, and Z are replaced with the identity operator. While the analysis of [16] shows that the maximal giants are time independent, the sub-maximal giants spin in the transverse $\mathbf{S}^5$, are thus time dependent, and are beyond the scope of this paper.

The other interpretation involves the fact that there are actually two distinct U(1) fibrations over $\mathbb{P}^2$ that yield Sasaki-Einstein spaces. The naive fibration produces $\mathbf{S}^5$. However, if we shrink the length of the U(1) fiber by a factor of three, one gets the orbifold $\mathbf{S}^5/\mathbb{Z}_3$ which was well studied by [31].[1] This geometry has a dual gauge theory with gauge group $SU(N) \times SU(N) \times SU(N)$. There are three sets of three bifundamental matter fields X_i, Y_i, and Z_i, $i = 1$, 2, or 3, transforming in the bifundamental representations of each pair of the three $SU(N)$'s. The conformal dimension of each of the bifundamental matter fields is still one as all we have done is orbifold. This space $\mathbf{S}^5/\mathbb{Z}_3$ has nonvanishing homology class $H_3(\mathbf{S}^5/\mathbb{Z}_3) = \mathbb{Z}_3$, and so it makes sense to speak of nontrivially wrapped D3-branes. Indeed, $\Delta = N$ is the right prediction for an antisymmetric product of N of the bifundamental matter fields.

3.2 $AdS_5 \times T^{1,1}$

This calculation was done first by Gubser and Klebanov [18]. We repeat their calculation with our advanced technology. To produce the space $T^{1,1}$ from a U(1) fibration, one takes the underlying Kähler-Einstein space to be $\mathbf{V} = \mathbb{P}^1 \times \mathbb{P}^1$. There is also a second U(1) fibration which results in $T^{1,1}/\mathbb{Z}_2$, but as the dimension Δ remains the same, we will focus on $T^{1,1}$ in what follows.

[1] The different possible fibrations come from the Thom-Gysin sequence which implies that the first Chern class of the circle fibration c_1^* divides $c_1(\mathbf{V})$ [32, 33]. For $\mathbf{S}^5$, $c_1(\mathbf{V}) = 3H$ and so c_1^* is either H or $3H$.

The canonical class $K_{\mathbf{V}} = -2f - 2g$ where $f \cdot g = 1$, $f \cdot f = 0$, and $g \cdot g = 0$. One can think of the line bundles f and g as corresponding to the individual $\mathbb{P}^1$'s. Each $\mathbb{P}^1$ does not intersect with itself and intersects with the other $\mathbb{P}^1$ exactly once. The simplest holomorphic curve one can take is $C = f$ or equivalently $C = g$. Based on this construction, one sees easily that $\Delta = 3N/4$.

The gauge theory dual to $AdS_5 \times T^{1,1}$ is $\mathcal{N} = 1$ $SU(N) \times SU(N)$ with two bifundamental matter fields A and B. The A fields transform under $(\mathbf{N}, \bar{\mathbf{N}})$ while the B fields transform under $(\bar{\mathbf{N}}, \mathbf{N})$. A dibaryon D is a product of N A's or alternately of N B's, totally antisymmetrized with respect to both color indices:

$$D = \epsilon^{\alpha_1 \alpha_2 \cdots \alpha_N} \epsilon_{\beta_1 \beta_2 \cdots \beta_N} A_{\alpha_1}^{\beta_1} A_{\alpha_2}^{\beta_2} \cdots A_{\alpha_N}^{\beta_N} \ . \tag{15}$$

The conformal dimension of the A and B fields is 3/4. It follows that the total dimension $\Delta(D)$ is $3N/4$.

4 Smooth V

We now consider the case when $\mathbf{V}$ is a Kähler-Einstein manifold without orbifold singularities. In addition to $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ considered above, there are remarkably few such spaces. They are the del Pezzo surfaces where $\mathbb{P}^2$ has been blown up at n points, $3 \leq n \leq 8$.[2] Let H be the hyperplane bundle in $\mathbb{P}^2$. The canonical class of the nth del Pezzo surface is

$$K_n = -3H + \sum_{i=1}^{n} E_i \tag{16}$$

where E_i is the exceptional divisor of a blown up point. Again $H \cdot H = 1$. Moreover, $E_i \cdot E_j = -\delta_{ij}$, and $H \cdot E_i = 0$.

In general, we can pick C such that $-K_{\mathbf{V}} \cdot C$ is any positive integer k. From our construction, $K_n \cdot K_n = 9 - n$. Thus

$$\Delta = k \frac{3N}{9-n} \ . \tag{17}$$

As an example, take the line $C = E_i$ as our holomorphic curve inside $\mathbf{V}$. Thus $K_n \cdot E_i = -1$, and the curve C has $k = 1$.

In addition to matching the spectrum Δ of conformal dimensions from gauge theory, we can also do some elementary counting of dibaryons. In preparation for this counting, note that the curve C can be written

$$C = aH - \sum_{i=1}^{n} b_i E_i \tag{18}$$

[2] No three points should be collinear and no six points should lie on a conic.

where the a and b_i are integers. The genus formula tells us that

$$(K_n + C) \cdot C = 2g - 2 \tag{19}$$

where g is the genus of the curve. Moreover, $g \geq 0$. Let us count how many degree $k = 1$ curves there are for each del Pezzo, taking into account the constraint $g \geq 0$. It turns out that the only $k = 1$ curves also have $g = 0$:

$$\begin{array}{cccccccccc} n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \#\text{ of curves} & 1 & 3 & 6 & 10 & 16 & 27 & 56 & 240 \end{array} \tag{20}$$

The gauge theory duals of the del Pezzos are more complicated because of a phenomenon known variously as Seiberg duality, toric duality, and Picard-Lefschetz monodromy [21, 19, 20]. In simple words, a single Calabi-Yau cone over a del Pezzo has more than one gauge theory dual description. In fact, there are infinite families of gauge theories which all correspond to the same cone over a del Pezzo! In the following, we will for the most part content ourselves with looking at a single one of the possible gauge theory duals for each del Pezzo.

4.1 The Third del Pezzo

The third del Pezzo, $n = 3$, was studied in great detail by Beasley and Plesser [19]. For completeness, we give a brief summary of their discussion of dibaryon dimensions. Beasley and Plesser studied four of the Seiberg dual gauge theories which map to this third del Pezzo through AdS/CFT correspondence. These four gauge theories have complicated chiral quiver diagrams involving six gauge groups and a large number of bifundamental matter fields.

To check the calculation of Δ, we need not be concerned by these complications. Beasley and Plesser tell us that the bifundamental matter fields present in the four quiver theories they considered all have conformal dimension 1/2, 1, or 3/2. Moreover, the gauge groups are all $SU(N)$. In other words, Δ should be an integer multiple of $N/2$. To get $N/2$, we choose for example $C = -E_i$. To get these larger dimensions, we need to choose a different holomorphic curve. For example, $C = H - E_i$ yields $\Delta = N$, and $C = 2H - E_1 - E_2 - E_3$ yields $\Delta = 3N/2$.

As [19] did before us, we may also count the number of dibaryons with the smallest conformal dimension $N/2$. In each of the four quiver theories considered, there are six bifundamental matter fields with conformal dimension 1/2, corresponding to the six degree one curves in table (20).

The authors of [19] also count the dibaryons of dimensions N and $3N/2$. Counting higher degree curves is easy from the geometric point of view. However, the gauge theory story is more complicated. The naive number of these dibaryons from gauge theory is far too large and gets reduced by classical and "quantum" relations between the bifundamental matter fields. We refer the reader to [19] for a discussion of these counting complications for the third del Pezzo.

4.2 del Pezzos Five and Six

The gauge theory duals of the fifth and sixth del Pezzos were considered by [22, 23, 24]. The conformal dimension of the dibaryon operators in both cases can be understood from the $AdS_5 \times \mathbf{S}^5$ and $AdS_5 \times T^{1,1}$ examples studied above. There are quivers for the fifth and sixth del Pezzo which are identical to the quivers for orbifolds of $T^{1,1}$ and $\mathbf{S}^5$. In particular, there is a quiver for the fifth del Pezzo which is identical to the quiver for a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold of $T^{1,1}$. The smallest dibaryons for the fifth del Pezzo have dimension $3N/4$, just like the $T^{1,1}$ case. There is a quiver for the sixth del Pezzo which is identical to the quiver for a $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold of $\mathbf{S}^5$. Not surprisingly, we find that the smallest dibaryon for the sixth del Pezzo has dimension N, just like the $\mathbf{S}^5$ case.

Geometrically, one can understand roughly how these relations to orbifold quivers arise. There are limits of the del Pezzo surfaces where three of the blown up points lie on a line or six lie on a conic. In these limits, the surface may admit a simpler orbifold description. We know there is a limit of the fifth del Pezzo which gives rise to this $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold and a limit of the cone over the sixth del Pezzo which gives $\mathbb{C}^3/\mathbb{Z}_3 \times \mathbb{Z}_3$.

We can count the number of smallest dibaryons. The quiver for the fifth del Pezzo (see figure 1) has 16 lines, corresponding to 16 bifundamental matter fields with conformal dimension 3/4. From table (20), we see that there are exactly 16 degree one curves in the fifth del Pezzo. Similarly, for the sixth del Pezzo, there are 27 bifundamentals with conformal dimension 1, corresponding both to the 27 degree one curves in the sixth del Pezzo and the 27 lines in the quiver of figure 1.

4.3 The Fourth del Pezzo

A less trivial example is the fourth del Pezzo. Wijnholt has produced a quiver and superpotential for the gauge theory dual (Eq. 3.15 of [24]). There are five fields X_i transforming under the bifundamental of $SU(3N) \times SU(N)$, five fields Z_i transforming in the bifundamental of $SU(N) \times SU(3N)$, and fifteen fields Y_i transforming in the bifundamental of $SU(N) \times SU(N)$, as shown in figure 1. The superpotential is constructed out of sums of the $\mathrm{Tr}(X_i Y_j Z_k)$. Vanishing of the beta functions and the fact that the superpotential has R-charge two completely specify the conformal dimension of the bifundamental matter fields. In particular, from the supersymmetry algebra, we know that the conformal dimension of the matter fields is 3/2 their R-charge. Thus from the superpotential constraint, it follows that

$$\Delta_X + \Delta_Y + \Delta_Z = 3 \ . \tag{21}$$

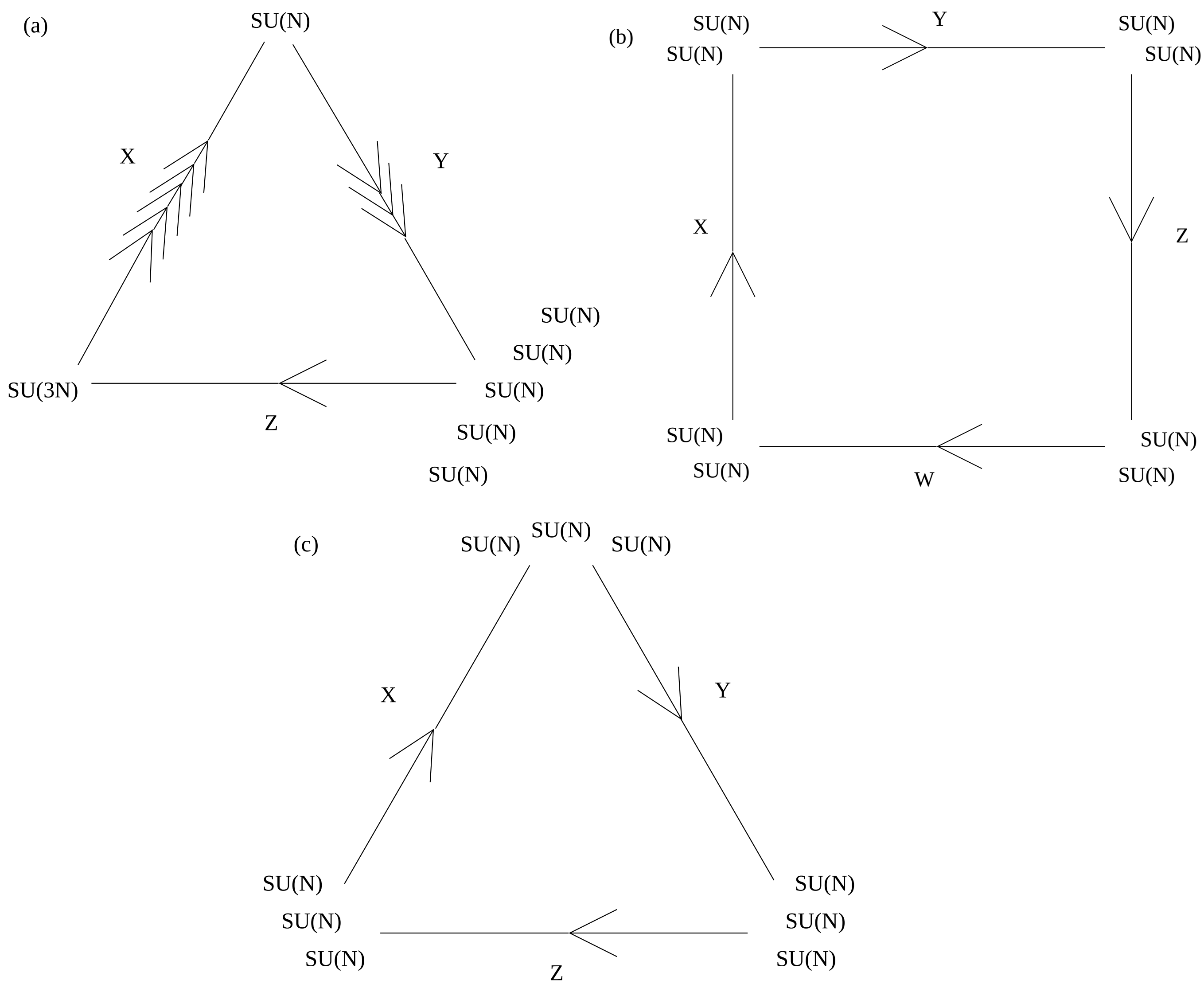

Figure 1: Quivers of [24] for the (a) fourth, (b) fifth, and (c) sixth del Pezzos. In this condensed notation, each $SU(N)$ represents a node. For example, for the fourth del Pezzo, a pair of bifundamentals Y and Z attaches to each $SU(N)$ node in the lower right hand corner of the quiver.

For each node, we get an additional constraint from the vanishing of the NSVZ beta function:

$$0 = 3C_2(G) - \sum_i T(R_i)(3 - 2\Delta_i) \ , \tag{22}$$

where $C_2(G)$ is the Casimir of the gauge group and $T(R_i)$ is the index of representation of the matter field. For $SU(N)$, $C_2 = N$ and the index of the fundamental representation is 1/2. From these constraints, we learn that the X have conformal dimension 1, the Y have conformal dimension 9/5, and the Z have conformal dimension 1/5.

This quiver for the fourth del Pezzo is the first example we have come across where the gauge groups are not all $SU(N)$, and we have to be a little careful about constructing dibaryons. In general, if we have operators $(\mathcal{O}_i)^\alpha_\beta$ transforming in the bifundamental of $SU(aN) \times SU(bN)$ where a and b are integers, we will need $N\,\mathrm{lcm}(a,b)$ copies of $\mathcal{O}$ in order to be able to antisymmetrize completely over both color indices. Moreover, for each antisymmetrization over $SU(aN)$ or $SU(bN)$, the aN or bN $\mathcal{O}_i$ need to be distinct in some way. In our case, $a = 1$ and $b = 3$; the smallest dibaryon that can be constructed from the X_i looks like

$$\epsilon^{\beta_1\cdots\beta_N}\epsilon^{\gamma_1\cdots\gamma_N}\epsilon^{\delta_1\cdots\delta_N}\epsilon_{\alpha_1\cdots\alpha_{3N}}X^{\alpha_1}_{\beta_1}\cdots X^{\alpha_N}_{\beta_N}X^{\alpha_{N+1}}_{\gamma_1}\cdots X^{\alpha_{2N}}_{\gamma_N}X^{\alpha_{2N+1}}_{\delta_1}\cdots X^{\alpha_{3N}}_{\delta_N} \tag{23}$$

and has conformal dimension $3N$. The conformal dimensions of the Y and Z dibaryons are correspondingly $9N/5$ and $3N/5$. All of these numbers are integer multiples of $3N/5$ as predicted by (17). To get a dibaryon of dimension $6N/5$, one could take the bifundamental field ZX, where we have traced over the internal $SU(3N)$ color indices. An antisymmetric product of N copies of the ZX does indeed have conformal dimension $6N/5$.

From table (20), there are 10 degree one curves in the fourth del Pezzo, and hence there should be 10 dibaryons with conformal dimension $3N/5$. Note that there are five bifundamental fields X_i. In constructing a dibaryon, we get to choose any three of them, and five choose three is indeed 10. If we choose the same X_i twice, the antisymmetrization gives zero.

4.4 The First and Second del Pezzos

Despite the fact that the first and second del Pezzo are not Kähler-Einstein, let us try applying our formula (17) anyway. For the first and second del Pezzo, the vanishing of the NSVZ beta functions and the R-charge constraint from the superpotential are not sufficient to specify completely the conformal dimension of the bifundamental matter fields. Intriligator and Wecht [25] proposed recently an additional constraint on these conformal dimensions. They demonstrated that the conformal dimensions or equivalently the R-charges should be chosen in a way that maximize the conformal anomaly a. In particular, the conformal anomaly a is proportional to

$$a \sim 3\sum_i r(\psi_i)^3 - \sum_i r(\psi_i) \tag{24}$$

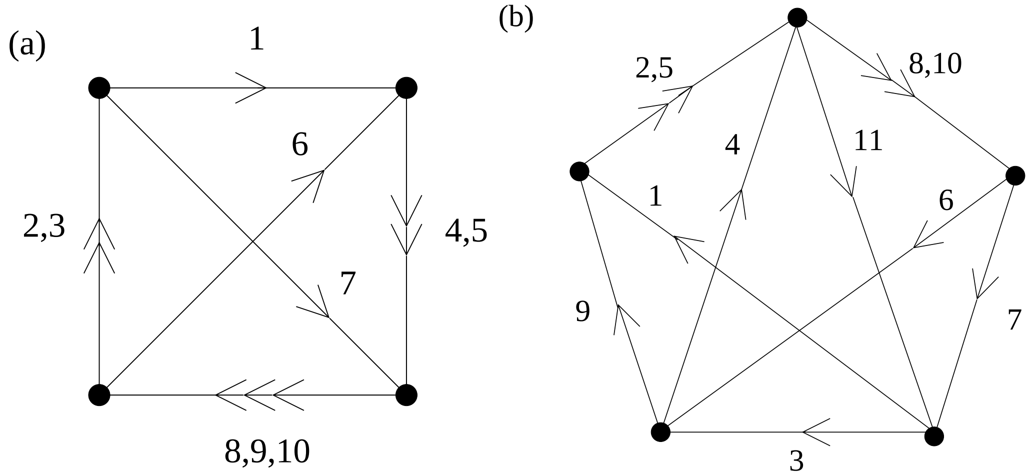

Figure 2: a) The quiver of [25] for the first del Pezzo. b) The Model II quiver of [20] for the second del Pezzo. The nodes correspond to $SU(N)$ gauge groups.

where $r(\psi_i)$ is the R-charge of the fermion ψ_i and the sum runs over all species of fermion in the gauge theory.

Using this additional constraint, Intriligator and Wecht [25] have calculated the conformal dimensions of the bifundamental matter fields for a particular gauge theory dual to the first del Pezzo (see figure 2a). The gauge groups are all $SU(N)$, and there are 10 bifundamental matter fields X_i, $i = 1, 2, \ldots, 10$. The allowed dimensions of these X_i are 3/8, 3/4, and 9/8, as predicted by (17). Moreover, there is exactly one bifundamental matter field, X_1, with conformal dimension 3/8, corresponding to the single degree one holomorphic curve for the first del Pezzo in table (20).

The conformal dimensions of the bifundamental matter fields have not yet been calculated for the second del Pezzo. Let us take the quiver and superpotential from Model II, Eq. 4.4 of [20]. All the gauge groups in this model are $SU(N)$. We have reproduced their quiver as figure 2b. As can be seen from the quiver, there are 11 bifundamental matter fields, X_i, $i = 1, 2, \ldots 11$. The superpotential for this theory is

$$X_5X_8X_6X_9 + X_1X_2X_{10}X_7 + X_{11}X_3X_4 - X_4X_{10}X_6 - X_2X_8X_7X_3X_9 - X_{11}X_1X_5 \ . \qquad (25)$$

The constraints from the vanishing of the beta functions for each $SU(N)$ gauge group and the fact that the superpotential have R-charge 2 yield a two parameter family of solutions for the dimensions of the bifundamental matter fields. We then maximize the conformal anomaly a, as described by [25]. We find that the conformal dimensions of X_3, X_7, and X_9 are 3/7. The conformal dimensions of X_1, X_2, X_5, X_6, X_8, and X_{10} are all 6/7. Finally, the conformal dimensions of X_4 and X_{11} are each 9/7. These numbers, 3/7, 6/7, and 9/7, are exactly what one expects from (17). Moreover, there are three fields with conformal dimension 3/7, as predicted by table (20).

This calculation can be repeated for the slightly more complicated Model I gauge theory dual to the second del Pezzo of [20]. The quiver and superpotential of Model I are related to Model II through a Seiberg duality on one of the nodes of the quiver. The gauge groups are all $SU(N)$. There are thirteen matter fields instead of eleven, but their conformal dimensions follow the same sequence 3/7, 6/7, 9/7, ... of rational numbers. Moreover, only three of the thirteen fields have conformal dimension 3/7. Thus, we find again the dibaryon spectrum of (17) and table (20).

4.5 del Pezzos Seven and Eight

The cases $n = 7, 8$ are more troublesome because the gauge theory descriptions are still incomplete. Quivers but no superpotentials have been proposed by [22]. We can make a prediction, however, for a quiver gauge theory dual to the seventh or eighth del Pezzo.

Let us start with the most troublesome case, $n = 8$. The geometric calculation tells us that Δ is an integer multiple of $3N$. If the gauge groups in the quiver were all SU(N), we would conclude that the smallest conformal dimension of a bifundamental matter field is 3 and thus the smallest R-charge 2. We could then conclude that the superpotential vanishes because a loop in the quiver needs at least two bifundamental matter fields to close on itself. We thus have two possibilities. Either the superpotential vanishes or the gauge groups are not all $SU(N)$.

Let us assume for a moment that some of the gauge groups are not $SU(N)$ but $SU(\alpha_i N)$ where the α_i are some integers. As we discussed in the case of the fourth del Pezzo surface, for a collection of bifundamental operators $(\mathcal{O}_k)^\alpha_\beta$ transforming under $SU(\alpha_i N) \times SU(\alpha_j N)$ one needs now $N\,\mathrm{lcm}(\alpha_i, \alpha_j)$ copies of $\mathcal{O}_k$ to antisymmetrize properly. For the eighth del Pezzo, the minimum conformal dimension of such an $\mathcal{O}_k$ is reduced from 3 to $3/\mathrm{lcm}(\alpha_i, \alpha_j)$.

For the case $n = 7$, if we assume all the gauge groups are SU(N), the minimal naive dimension of the bifundamental matter fields is 3/2. So for $n = 7$, it might be possible to have a quadratic superpotential. However, the quiver in [22] is chiral and has no loops with only two bifundamental matter fields. Thus it seems reasonable to conjecture that some of the gauge groups are not pure $SU(N)$ but $SU(\alpha_i N)$ for α_i integer.

That exhausts the collection of smooth Kähler-Einstein manifolds. Next we turn to **V** with quotient singularities.

5 The Generalized Conifolds

Consider a weighted homogenous polynomial in $\mathbb{C}^4$, by which we mean a polynomial $F(\mathbf{z})$ which satisfies

$$F(\lambda^{w_0} z_0, \lambda^{w_1} z_1, \lambda^{w_2} z_2, \lambda^{w_3} z_3) = \lambda^d F(z_0, z_1, z_2, z_3) \ ,$$

where $\lambda \in \mathbb{C}^*$ and $w_i \in \mathbb{Z}^+$, and the degree d is a positive integer. There is a theorem due to Tian and Yau which states that as long as the index $I = \sum w_i - d$ is positive, the cone $\mathbf{Y}$ cut out by $F = 0$ is Calabi-Yau [34].[3] We also insist that the only singularity of $\mathbf{Y}$ is at the tip of the cone $r = 0$. This requirement on the singularities means the only solution to the system of equations $\{\partial_i F = 0 : i = 0, 1, 2, 3\}$ is the point $\mathbf{z} = 0$.

In the previous section, we thought of $\mathbf{Y}$ as a fibration over a Kähler-Einstein manifold $\mathbf{V}$. In terms of F, we can think of $\mathbf{V}$ as the corresponding variety cut out by $F = 0$ in weighted projective space $\mathbb{P}(w_0, w_1, w_2, w_3)$ rather than in affine $\mathbb{C}^4$. Weighted projective space is defined in analogy to ordinary projective space: instead of the uniform weighting, the $\mathbb{C}^*$-action on $\mathbb{C}^4$ is weighted by a vector of weights (w_0, w_1, w_2, w_3). The point $\mathbf{z} = 0$ is not included in $\mathbb{P}(w_0, w_1, w_2, w_3)$.

In general, the space $\mathbf{V}$ is not a smooth manifold but rather has cyclic quotient singularities inherited from $\mathbb{P}(w_0, w_1, w_2, w_3)$. In the coordinate patch $z_i \neq 0$, weighted projective space looks like a quotient of $\mathbb{C}^3$ by $\mathbb{Z}_{w_i}$. So there will be in general a quotient singularity at the point $(0, 0, z_i \neq 0, 0)$. If the weights have common factors, there may also be singular lines, planes, etc. If the variety $\mathbf{V}$ intersects any of these singular regions, the variety will also have quotient singularities. To insure that the singularities of $\mathbf{V}$ are of codimension 2 or less, one usually assumes that the weighted projective space is well-formed:

$$\gcd(w_0, \ldots, \hat{w}_i, \ldots, \hat{w}_j, \ldots, w_n) \mid d \tag{26}$$

$$\gcd(w_0, \ldots, \hat{w}_i, \ldots, w_n) = 1 \tag{27}$$

The second condition is less stringent because any projective space that does not satisfy condition (27) is isomorphic to one that does [35, 36].

Although it is true that if $\mathbf{V}$ is Kähler-Einstein, then $\mathbf{Y}$ is Calabi-Yau, it does not appear to be true in general that if $\mathbf{Y}$ is a Calabi-Yau cone that there exists such a Kähler-Einstein $\mathbf{V}$. Thousands of Kähler-Einstein $\mathbf{V}$ have been cataloged. See for example [37, 38]. In this paper, we will be interested with a class of $\mathbf{V}$ where the corresponding $\mathbf{Y}$ are called generalized conifolds and for which the cone $\mathbf{Y}$ is Calabi-Yau and smooth except at the tip, but where it is not yet known whether $\mathbf{V}$ admits a Kähler-Einstein metric. We will work under the hypothesis that these $\mathbf{V}$ are indeed Kähler-Einstein and we will get sensible results for dibaryon masses.

5.1 Gauge Theory Duals for Generalized Conifolds

To introduce these generalized conifolds, let us begin by reviewing the gauge theory on the world volume of a collection of N D-branes placed at the orbifold singularity of $\mathbb{C}^2/\Gamma$, where Γ is a discrete subgroup of $SU(2)$ of ADE type.[4] The field theory has $\mathcal{N} = 2$ supersymmetry.

[3] More precisely, the cone minus the apex $\mathbf{z} = 0$ is Calabi-Yau.

[4] Much of this discussion is drawn from [26].

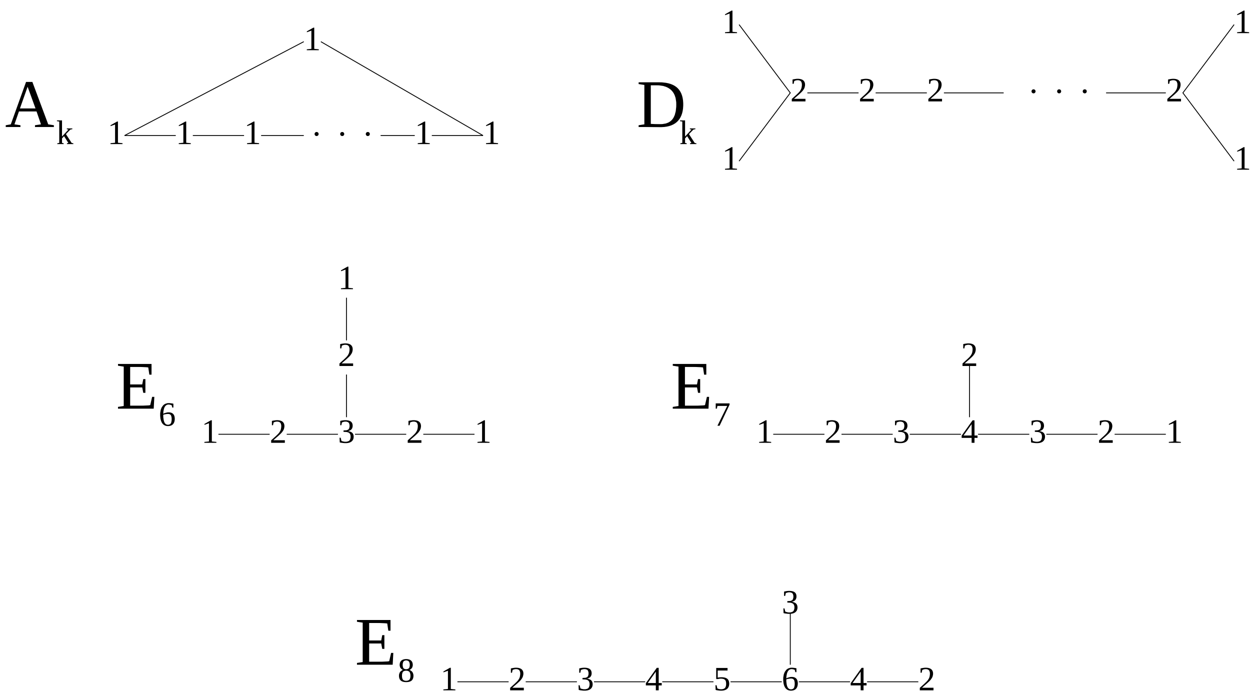

Figure 3: The extended Dynkin diagrams of ADE type, including the indices n_i of each vertex.

Its gauge group is the product

$$G = \prod_{i=0}^{r} U(N_i)$$

where i runs through the set of vertices of the extended Dynkin diagram of the corresponding ADE type (see figure 3) [39]. We have also introduced $N_i = Nn_i$, where n_i is the index of the ith vertex of the Dynkin diagram. Equivalently, one may think of i as running through the irreducible representations $\mathbf{r}_i$ of Γ, in which case n_i can be thought of as the dimension of $\mathbf{r}_i$.

The field content of the gauge theory can be summarized conveniently with a quiver diagram which is in fact the corresponding extended Dynkin diagram. For each vertex in the Dynkin diagram, we have an $\mathcal{N} = 2$ vector multiplet transforming under the adjoint of $U(N_i)$. For each line in the diagram, there is a bifundamental hypermultiplet a_{ij} in the representation $(N_i, \bar{N}_j)$.

To write a superpotential for this gauge theory, it is convenient to decompose the fields into $\mathcal{N} = 1$ multiplets. Each a_{ij} will give rise to a pair of chiral multiplets, (B_{ij}, B_{ji}), where B_{ij} is a complex matrix transforming in the $(N_i, \bar{N}_j)$ representation. Moreover, there is a chiral multiplet ϕ_i for each vector multiplet in the theory.

The superpotential is then

$$W = \sum_i \mathrm{Tr}\mu_i \phi_i \tag{28}$$

where μ_i is the "complex moment map"

$$\mu_i{}^{\alpha_i}{}_{\beta_i} = \sum_j s_{ij} B_{ij}{}^{\alpha_i}{}_{\gamma_j} B_{ji}{}^{\gamma_j}{}_{\beta_i} \ . \tag{29}$$

Although the indices are confusing, essentially all we have done is construct a cubic polynomial in the $\mathcal{N}=1$ superfields consistent with $\mathcal{N}=2$ SUSY and the gauge symmetry. The factor s_{ij} is the antisymmetric adjacency matrix for the Dynkin diagram: $s_{ij} = \pm 1$ when i and j are adjacent nodes and zero otherwise. The upper index α_i indicates a fundamental representation of $U(N_i)$, while a lower index β_i indicates an anti-fundamental representation of $U(N_i)$. There is a relation among the μ_i

$$\sum_i \mathrm{Tr}\mu_i = 0 \ . \tag{30}$$

which holds because the trace gives something symmetric in i and j summed against s_{ij} which is antisymmetric.

This $\mathcal{N}=2$ gauge theory is superconformal and thus must have an R symmetry. W must have R charge 2. Conveniently, the B_{ij} and the ϕ_i have R charge $2/3$ and the superpotential, as noted above, is cubic.

In the large N limit in the case of D3-branes, we can invoke the AdS/CFT correspondence for this gauge theory [6, 7]. The correspondence tells us that the gauge theory described above is dual to type IIB supergravity (SUGRA) on an $AdS_5 \times \mathbf{S}^5/\Gamma$ background. To see how the orbifolding works, consider $\mathbf{S}^5 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : \sum_i |z_i|^2 = 1\}$. The group Γ acts only on $(z_1, z_2) \in \mathbb{C}^2$. As a result, there is an $\mathbf{S}^1$ of the $\mathbf{S}^5$ which is left invariant under Γ.

We can add a term to the superpotential (28) that will give masses m_i to the ϕ_i. Such a term will eliminate the ϕ_i from the theory at energies below the mass scale set by the m_i and break the supersymmetry from $\mathcal{N}=2$ to $\mathcal{N}=1$. In particular, we add the term

$$W' = W - \frac{1}{2}\sum_i m_i \mathrm{Tr}\phi_i^2 \ .$$

To see what happens at low energies, let us look at the equations of motion $dW' = 0$. By varying with respect to the matter fields that define μ_i, we see that

$$\phi_i = \phi \mathrm{Id}_{N_i} \tag{31}$$

where ϕ is the Lagrange multiplier used to ensure that the constraint (30) is satisfied. Varying with respect to ϕ_i and employing (31), one gets

$$\mu_i = m_i \phi \mathrm{Id}_{N_i} \ .$$

From (30), it follows that

$$\sum_i n_i m_i = 0 \ .$$

Assuming that none of the $m_i = 0$, we can eliminate ϕ_i from the action to find an effective low energy superpotential:

$$W_{eff} = \sum_i \frac{1}{2m_i} \mathrm{Tr}\mu_i^2 \ .$$

Notice that W_{eff} is quartic in the superfields B_{ij} (see (29)). We would like the endpoint of the RG flow generated by adding these mass terms to be an IR conformal fixed point. However, if the fields B_{ij} are given their naive R charges of 2/3, this quartic superpotential will explicitly break our R symmetry. By giving the B_{ij} anomalous dimensions, we find that after flowing to the IR, the R charge of the B_{ij} can be adjusted to 1/2, and the R symmetry is preserved. Thus, the conformal dimension of the B_{ij} is always 3/4.

5.2 The Geometry of the Generalized Conifolds

In the context of the AdS/CFT correspondence, the authors of [26] generalized an argument of [8] for the A_1 case, arguing that the IR endpoint of this RG flow is dual to type IIB SUGRA in an $AdS_5 \times \mathbf{X}_\Gamma$ background where the $\mathbf{X}_\Gamma$ are the level surfaces of certain "generalized conifolds".[5] The generalized conifolds are three complex dimensional Calabi-Yau manifolds with a conical scaling symmetry. The conifolds can be described by a polynomial embedding relation $F_\Gamma = 0$ in $\mathbb{C}^4$. To conform with the notation of [26], we use the coordinates $(\phi, x, y, z) \in \mathbb{C}^4$. The polynomial F_Γ is invariant under a $\mathbb{C}^*$-action, the real part of which is the conical scaling symmetry while the imaginary part corresponds to an R symmetry transformation in the dual gauge theory. F_Γ transforms under this $\mathbb{C}^*$-action with weight d. The coordinates ϕ, x, y, and z transform with weights

$$\begin{array}{cccccc} \Gamma & [\phi] & [x] & [y] & d/2=[z] & d \\ A_{2q} & 2 & 2 & 2q+1 & 2q+1 & 4q+2 \\ A_{2q+1} & 1 & 1 & q+1 & q+1 & 2q+2 \\ D_k & 1 & 2 & k-2 & k-1 & 2(k-1) \\ E_6 & 1 & 3 & 4 & 6 & 12 \\ E_7 & 1 & 4 & 6 & 9 & 18 \\ E_8 & 1 & 6 & 10 & 15 & 30 \end{array} \qquad (32)$$

To make things more concrete, we give the polynomials

$$F_{A_k} = \prod_{i=0}^{k} (x - \xi_i \phi) + y^2 + z^2 \ , \qquad (33)$$

$$F_{D_k} = \prod_{i=0}^{k-2} (x - \xi_i \phi^2) + t_0 \phi^k y + x y^2 + z^2 \ , \qquad (34)$$

[5]Similar conclusions were reached for the A_k type generalized conifolds in [40].

$$F_{E_6} = y^3 + t_0\phi^2(x - a_1\phi^3)(x - a_2\phi^3)y + \prod_{i=1}^{4}(x - b_i\phi^3) + z^2 \ , \tag{35}$$

$$F_{E_7} = y^3 + (x - a_1\phi^4)(x - a_2\phi^4)(x - a_3\phi^4)y + t_0\phi^2\prod_{i=1}^{4}(x - b_i\phi^4) + z^2 \ , \tag{36}$$

$$F_{E_8} = y^3 + t_0\phi^2(x - a_1\phi^6)(x - a_2\phi^6)(x - a_3\phi^6)y + \prod_{i=1}^{5}(x - b_i\phi^6) + z^2 \ , \tag{37}$$

where ξ_i, t_0, a_i, and b_i are free constants transforming with weight zero.

Our object is to calculate geometrically the quantity given in (13),

$$\Delta = -3N\frac{K_V \cdot C}{K_V \cdot K_V} \ .$$

In fact we will see below that it suffices to minimize Δ. Thus it suffices to determine

$$-K_V \cdot K_V \tag{38}$$

and to minimise

$$-K_V \cdot C, \tag{39}$$

where C ranges over all holomorphic curves in V. We we will refer to any such curve on W as a curve of minimal degree. We note that in each case V admits a double cover of a weighted projective space W of dimension two,

$$\pi \colon V \longrightarrow W$$

which corresponds to projection onto the (ϕ, x, y)-coordinates from the point $[0 : 0 : 0 : 1]$ of the corresponding weighted projective space. The fact that π is a double cover corresponds to the fact that if we fix the value of (ϕ, x, y), then z takes on two possible values, corresponding to the two choices of square root, positive and negative.

Let B be the branch locus of π. Let G_Γ be the polynomial in (ϕ, x, y), obtained by setting $z = 0$ in F_Γ. Clearly the zero locus Σ of G_Γ is part of the branch locus of π. However, in some of the cases when we drop the coordinate z, the resulting weights are not well-formed. In fact sometimes two of the first three entries share a common factor of 2. The weights for W are obtained by canceling the common factor. However it is easy to see, from the description of weighted projective space as a quotient of $\mathbb{C}^4$, that in fact the coordinate axis corresponding to the vanishing of the other coordinate is also a component of B.

We let e denote the degree of B. Clearly the degree of Σ is equal to the degree of G, which is of course d. It will be easy to calculate the degree of the extra component, should it be present, and of course e is nothing but the sum of these degrees.

Our aim is to reduce the calculation of the relevant intersection numbers on V to a calculation on W, which it will turn out is considerably easier. To this end, we want to apply the push-pull formula. Given a cohomology class α on V and a cohomology class β on W, push-pull reads as

$$\pi_*(\alpha \cdot \pi^*\beta) = \pi_*\alpha \cdot \beta.$$

To apply push-pull then, we need to express K_V as the pull-back of some class from W. In fact it is easy to do so, using the Riemann-Hurwitz formula, which in our case reads as

$$K_V = \pi^*(K_W + 1/2B). \tag{40}$$

Given this, it is easy to compute the number $K_V \cdot K_V$ on W,

$$\begin{aligned}
\pi_*(K_V \cdot K_V) &= \pi_*(\pi^*(K_W + 1/2B) \cdot \pi^*(K_W + 1/2B) \cdot [V]) \\
&= \pi_*(\pi^*((K_W + 1/2B) \cdot (K_W + 1/2B)) \cdot [V]) \\
&= (K_W + 1/2B) \cdot (K_W + 1/2B) \cdot \pi_*[V] \\
&= (K_W + 1/2B) \cdot (K_W + 1/2B) \cdot 2[W] \\
&= 2(K_W + 1/2B) \cdot (K_W + 1/2B),
\end{aligned}$$

where we use the obvious geometric fact that $\pi_*[V] = 2[W]$ (note also that the class of V is the identity in cohomology). Thus

$$K_V \cdot K_V = 2(K_W + 1/2B) \cdot (K_W + 1/2B). \tag{41}$$

Suppose that we are given a curve C in V. We want to do the same thing with $K_V \cdot C$. Let C' be the image of C inside W. By definition the push-forward of C is a multiple of C',

$$\pi_*C = fC',$$

where f is the degree of the map $\pi|_C \colon C \longrightarrow C'$. As π itself has degree two, either $f = 1$ or 2, and we may distinguish the two cases by considering the inverse image of C'. $f = 1$ if either the inverse image of C' is the union of two irreducible components (possibly connected), or C' is a component of the branch locus B. Otherwise $f = 2$, in which case C is the inverse image of C', but C' is not a component of the branch locus. Now we can compute the intersection number $K_V \cdot C$ by push-pull

$$\begin{aligned}
\pi_*(K_V \cdot C) &= \pi_*(\pi^*(K_W + 1/2B) \cdot C) \\
&= (K_W + 1/2B) \cdot \pi_*C \\
&= f(K_W + 1/2B) \cdot C'.
\end{aligned}$$

Thus

$$-K_V \cdot C = -f(K_W + 1/2B) \cdot C'. \tag{42}$$

Of course one tricky thing about this formula is that the value of f depends on C. In this way, we reduce the problem of minimizing the intersection number $-K_V \cdot C$, where C ranges over all holomorphic curves in V, to minimizing the intersection number

$$-f(K_W + 1/2B) \cdot C \tag{43}$$

where now C ranges over all holomorphic curves in W. In each case, it is not too hard to prove that given C, we may find a curve D numerically equivalent to kC such that the inverse of D represents $k\Delta$. Thus it does indeed suffice to minimize (43). Consider the following table:

Γ	$[\phi]$	$[x]$	$[y]$	d	e	f	$\Delta/3N$	$\#$
A_{2q}	1	1	$2q+1$	$2q+1$	$4q+2$	1	1/4	$2(k+1)$
A_{2q+1}	1	1	$q+1$	$2q+2$	$2q+2$	1	1/4	$2(k+1)$.
D_{2q}	1	1	$q-1$	$2q-1$	$2q$	1	1/2	
D_{2q+1}	1	2	$2q-1$	$4q$	$4q$	2	1/2	
E_6	1	3	4	12	12	2	1/2	
E_7	1	2	3	9	10	1	1/2	
E_8	1	3	5	15	16	1	1/2	

(44)

We next explain how we got the last four columns. To proceed further, observe that any weighted projective space is in fact an example of a toric variety. Recall that a variety is said to be a toric variety, if it contains a dense open subset isomorphic to a torus, that is a copy of $(\mathbb{C}^*)^n$, where, moreover, the natural action of the torus extends to an action on the whole of the toric variety. Suppose that X is a toric variety and that V is any subvariety. We claim that the class $[V]$ of V (either in cohomology or better yet in the Chow ring) is an integral linear combination of classes of invariant subvarieties (under the action of the torus), that is

$$[V] = \sum_i a_i [Z_i]$$

where each a_i is a non-negative integer and Z_i ranges over the invariant subvarieties. Indeed if V is not already invariant, then we may find a one dimensional torus $\mathbb{C}^*$ inside the big torus, which moves V inside X. Taking the limit, we obtain a degeneration of V to a collection of subvarieties that are invariant under a subgroup of the torus of larger dimension. Continuing in this way, we finally degenerate V to a sum of cycles, all of which are invariant under the action of the whole torus.

In our case C is a curve and W is a toric surface of Picard number one. On a toric surface, the only invariant subvarieties are the surface itself, a finite union of invariant curves and their intersection points. As C is a curve, we only need worry about the invariant curves, and as W has Picard number one, there are only three invariant curves, B_0, B_1 and B_2 say, which form a triangle, where each invariant curve is given as the vanishing of one of the coordinates, ϕ, x or y. Thus (5.2) reduces to

$$[C] = \sum a_i [B_i], \tag{45}$$

where each a_i is a non-negative integer. Thus to minimise $-(K_W + 1/2B) \cdot C$, it suffices to find the minimum of $-f(K_W + 1/2B) \cdot B_i$, for $i = 0$, 1 and 2. The condition that the Picard number is one means that the cohomology classes of any two curves are proportional. In particular

$$B_i = \lambda_{i,j} B_j,$$

where summation notation has been suppressed. Clearly we want to choose j, so that for all i, $\lambda_{i,j} \geq 1$. To determine the constants, it suffices to compare the intersection numbers $B_i \cdot B_j$. Suppose the weights are (w_0, w_1, w_2), so that $W = \mathbb{P}(w_0, w_1, w_2)$. By symmetry it suffices to consider the case $B_0 \cdot B_1$. Now W has three invariant points, the vertices of the triangle, that is the points $p_2 = B_0 \cap B_1$, $p_1 = B_0 \cap B_2$ and $p_0 = B_1 \cap B_2$. Thus to calculate the intersection number $B_0 \cdot B_1$ it suffices to compute the local intersection number at the point p_2. Locally we have

$$f \colon \mathbb{C}^2 \longrightarrow \mathbb{C}^2/\mathbb{Z}_r,$$

where $r = w_2$ is the index of the singular point. Let C_0 and $C_1 \subset \mathbb{C}^2$ be the axes upstairs. Then the local intersection number $C_0 \cdot C_1 = 1$ and of course $C_1 = f^* B_1$. Hence, by push-pull

$$\begin{aligned} 1 &= f_*(C_0 \cdot C_1) \\ &= f_*(C_0 \cdot f^* B_1) \\ &= f_*(C_0) \cdot B_1) \\ &= r B_0 \cdot B_1, \end{aligned}$$

where we use the fact that $f_* C_0 = r B_0$, as $C_0 \longrightarrow B_0$ is an r-fold cover.

Thus

$$B_0 \cdot B_1 = 1/w_2 \qquad B_0 \cdot B_2 = 1/w_1 \qquad \text{and} \qquad B_1 \cdot B_2 = 1/w_0. \tag{46}$$

Putting all of this together we get

$$\lambda_{i,j} = \frac{w_i}{w_j}. \tag{47}$$

In our case, $w_0 = 1$, and the weights are increasing. Thus every curve C in W is numerically equivalent to an integral multiple of B_0. Thus it is natural to express all curves as multiples of B_0.

As $f = 1$ or 2, it follows that a curve C numerically equivalent to B_0 minimizes (43), so that we are reduced to finding the minimum of

$$f(K_W + 1/2B) \cdot C, \tag{48}$$

where C is a curve numerically equivalent to B_0. For D_{2q}, E_7 and E_8, B_0 is a component of B. Thus $f = 1$, by definition. For A_k, B_0 and B_1 are numerically equivalent. In fact W is a cone over a rational normal curve, and B_0 and B_1 correspond to lines of the ruling. Let L be any line of this ruling and let M be the inverse image of L. L is determined by setting a linear combination of ϕ and x to zero (for example B_0 corresponds to $\phi = 0$ and B_1 to $x = 0$), so that we fix the ratio between ϕ and x. Having fixed this ratio, then M is a conic in a weighted projective plane (in fact the cone over L and the point of projection), given on an affine piece by an equation of the form

$$z^2 + y^2 = l,$$

where l is a constant. For us the only important thing is whether l is zero or not. If l is zero, then $z^2 + y^2$ factors as the product of two linear polynomials, and M consists of two lines. Otherwise M is a smooth conic. Of course l is zero iff $x = \xi_i \phi$, so that this happens $k+1$ times. In this case $f = 1$ as well. Moreover there are two $2(k+1)$ curves of minimal degree, two for each i, giving the last column of the first and second rows of (44). In the other two cases, D_{2q+1} and E_6, B_0 is the only (effective) curve in its numerical equivalence class, and the inverse image of L is easily seen to be irreducible and so $f = 2$ in these two cases. These values for f give the second from last column of (44).

We turn to the problem of expressing the classes of K_W and B in terms of B_0. It is a basic property of any toric variety X that $-K_X$ is equivalent to a sum of all the invariant divisors. In our case, it is particularly easy to see that this is true; the invariant curves $D = B_0 + B_1 + B_2$ form a triangle, and each is a copy of $\mathbb{P}^1$. Taken together they are then a curve of genus one, and so $K_D = 0$. But by adjunction

$$(K_W + D)|_D = K_D = 0.$$

As we are on a surface of Picard number one, the fact that the restriction is zero implies that $K_W + D = 0$. Thus

$$\begin{aligned} -K_W &= B_0 + B_1 + B_2 \\ &= \left(1 + \frac{w_1}{w_0} + \frac{w_2}{w_0}\right) B_0, \end{aligned}$$

so that

$$-K_W = \frac{1}{w_0}(w_0 + w_1 + w_2) B_0. \tag{49}$$

To compute the class of B note that in all cases, B_0 has weight one. It follows that

$$B = eB_0, \tag{50}$$

where e is the degree of B. In all cases when $e \neq d$, excepting the case A_{2q}, the extra component of the branch locus is B_0 itself. In these cases $e = d + 1$. Otherwise in the case of A_{2q}, the extra component is B_2 and $B_2 = (2q+1)B_0$, so that $e = d + (2q+1) = 4q + 2$.

Finally we note one further advantage of working on W, a surface of Picard number one. To compute the ratio

$$\frac{-f(K_W + 1/2B) \cdot C}{2(K_W + 1/2B) \cdot (K_W + 1/2B)}$$

we note that when we express everything in terms of B_0, there will be a lot of canceling. In fact it is easy to see that this ratio reduces to

$$\frac{f}{2\lambda}$$

where $-(K_W + 1/2B) = \lambda C$. In our case, we take $C = B_0$. Now

$$-(K_W + 1/2B) = (1 + w_1 + w_2 - e/2)B_0,$$

where we use the fact that $w_0 = 1$, so that

$$2\lambda = 2(1 + w_1 + w_2 - e/2) = 2 + 2w_2 + 2w_2 - e.$$

Putting all of this together, we obtain that $\Delta/3N$ will always be an integer multiple of the penultimate column of (44).

5.3 Dibaryons from Gauge Theory

Paths in the simply laced Dynkin diagrams of figure 3 correspond to dibaryonic operators. In particular, pick two nodes of a simply laced Dynkin diagram corresponding to gauge groups $SU(jN)$ and $SU(kN)$. To construct the dibaryon, we first need to construct a smaller object transforming in the bifundamental of $SU(jN) \times SU(kN)$. Choose a path along the Dynkin diagram that joins these two nodes. The path can double back on itself. For each link joining two nodes in the path, write down the corresponding bifundamental matter field. Which bifundamental we choose depends on which direction we move between the nodes. At this point, we will have some number s of bifundamental matter fields where we can trace over every index save the fundamental of $SU(jN)$ and the antifundamental of $SU(kN)$. Call this object of conformal dimension $3s/4$ $\mathcal{O}^{\alpha}_{\beta}$.

Now if $j = k$, then we can antisymmetrize over jN copies of $\mathcal{O}$ and the conformal dimension of the dibaryon will simply be

$$\Delta = \frac{3sN}{4} j \ . \tag{51}$$

If $j \neq k$, we have to be more careful, as we saw in the case of the fourth del Pezzo. We need to choose some more paths in the Dynkin diagram, corresponding to some collection of bifundamental fields $\mathcal{O}_i$. If we choose the paths correctly, we can antisymmetrize over the collection to form a (non-zero) gauge invariant dibaryon. For $j \neq k$, one bad idea is to antisymmetrize over $\mathrm{lcm}(j,k)N$ copies of the original field $\mathcal{O}$. Such an antisymmetric sum is zero. However, if we can construct a collection of fields with the same path length s such that all transform under $SU(jN) \times SU(kN)$ and such that the antisymmetrization over each $SU(jN)$ and $SU(kN)$ is nonzero, then we find the rather remarkable formula

$$\Delta = \frac{3sN}{4} \mathrm{lcm}(j,k) \ . \tag{52}$$

Staring at the Dynkin diagrams, it is straightforward to compute the different possible values of the conformal dimension. For the A_k types, the dimension is an integer multiple of $3N/4$. For the D_k and E_k types, the dimension is an integer multiple of $3N/2$. These numbers are exactly as predicted by the geometric calculation presented above.

We may also count the number of dibaryonic operators of smallest conformal dimension. For the A_k type quivers, there is a smallest dibaryon for each elementary bifundamental matter field B_{ij}, or equivalently for each path of length $s = 1$. Thus, there are $2(k+1)$ smallest dibaryons for the A_k quiver, in precise agreement with the geometric calculation presented above in the last column of table (44).

6 Remarks

This paper marks a step forward in the authors' understanding of the relation between dibaryons in superconformal gauge theories and holomorphic curves in Kähler-Einstein surfaces all in the context of the AdS/CFT correspondence. The formula (13) is a powerful way of calculating geometrically the conformal dimension of time independent dibaryons in a wide variety of contexts without having explicit knowledge of the metric. We applied this formula to some well studied examples of AdS/CFT correspondence, namely the del Pezzos and the generalized conifolds, where previously lack of a metric had hampered progress. In all cases, we found good agreement with previously established gauge theory results.

Having remarked on the progress, there are a large number of questions which still need to be addressed. For example, how does the number of holomorphic curves C with a given value of $-K_{\mathbf{V}} \cdot C$ compare to the number of dibaryons of a given conformal dimension. For the del Pezzos and the A_k type generalized conifolds, we were able to count the number of smallest dibaryons and compare successfully with the number of lowest degree holomorphic curves. However, for the del Pezzo gauge theories, there seem to be too many larger dibaryons. The superpotential provides some classical relations between the bifundamental matter fields that partially reduce this number. Beasley and Plesser demonstrated the existence of additional "quantum" relations between bifundamental matter fields that reduced this naive number even further for the third del Pezzo [19]. Taking into account both classical and "quantum" relations, Beasley and Plesser were able to match the number of gauge theory dibaryons with the number of holomorphic curves. It would be interesting to reproduce their calculation for the other del Pezzos. It would also be interesting to count dibaryons more generally for the generalized conifolds.

A bizarre development is the agreement between gauge theory and geometry for the conformal dimension of dibaryons in the first and second del Pezzo. The first and second del Pezzo are known not to admit a Kähler-Einstein metric, and it is a little difficult to imagine how (13) remains meaningful in this case. Along the same lines, no one has yet demonstrated that the $\mathbf{V}$ of the generalized conifolds admit a Kähler-Einstein metric. It would be interesting to know whether these $\mathbf{V}$ are Kähler-Einstein or whether they fall into the same category as the first and second del Pezzos.

Another interesting question is how Seiberg duality affects, or rather fails to affect, the spectrum of dibaryons. Seiberg dual theories can have very different quivers and superpotentials. Somehow, one continues to find the same set of conformal dimensions for the dibaryons.

Finally, given the geometry $AdS_5 \times \mathbf{X}$, is it possible that the dibaryon spectrum can be used to construct the gauge theory dual? In all the examples considered here, except the seventh and eighth del Pezzo, the gauge theory duals had been constructed using independent considerations. In general, constructing these gauge theory duals is nontrivial, and dibaryons may prove to be a useful additional tool.

We hope to return to some of these questions in the future.

Note: While this letter was being prepared, we learned of [41] which overlaps this work to some extent.

Acknowledgments

C. H. would like to single out Igor Dolgachev for special thanks. C. H. would also like to thank the MCTP in Ann Arbor, where this project was resurrected, for hospitality. We would like to thank C. Beasley and A. Bergman for collaboration in the early stages of this project. We would also like to thank A. Iqbal, A. Mikhailov, and J. Walcher for useful conversations. This research was supported in part by the National Science Foundation under Grant No. PHY99-07949.

References

[1] J. Maldacena, "The Large N limit of superconformal field theories and supergravity," *Adv. Theor. Math. Phys.* **2** (1998) 231, `hep-th/9711200`.

[2] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, "Gauge theory correlators from noncritical string theory," *Phys. Lett.* **B428** (1998) 105, `hep-th/9802109`.

[3] E. Witten, "Anti-de Sitter space and holography," *Adv. Theor. Math. Phys.* **2** (1998) 253, `hep-th/9802150`.

[4] O. Aharony, S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, "Large N Field Theories, String Theory and Gravity," *Phys. Rept.* **323** (2000) 183, hep-th/9905111

[5] I. R. Klebanov, "TASI Lectures: Introduction to the AdS/CFT Correspondence," hep-th/0009139.

[6] A. Lawrence, N. Nekrasov, and C. Vafa, "On conformal field theories in four-dimensions," *Nucl. Phys.* **B533** (1998) 199, hep-th/9803015.

[7] S. Kachru and E. Silverstein, "4-D conformal theories and strings on orbifolds," *Phys. Rev. Lett.* **80** (1998) 4855, hep-th/9802183.

[8] I. R. Klebanov and E. Witten, "Superconformal field theory on three-branes at a Calabi-Yau singularity," *Nucl. Phys.* **B536** (1998) 199, hep-th/9807080.

[9] A. Kehagias, "New Type IIB Vacua and Their F-Theory Interpretation," *Phys. Lett.* **B435** (1998) 337, hep-th/9805131.

[10] D. Morrison and R. Plesser, "Non-Spherical Horizons, I," *Adv. Theor. Math. Phys.* **3** (1999) 1, hep-th/9810201.

[11] B. S. Acharya, J. M. Figueroa-O'Farrill, C. M. Hull and B. Spence, "Branes at conical singularities and holography," *Adv. Theor. Math. Phys.* **2** (1999) 1249, hep-th/9808014.

[12] A. Bergman and C. P. Herzog, "The volume of some non-spherical horizons and the AdS/CFT correspondence," *JHEP* **0201** (2002) 030, hep-th/0208020

[13] S. S. Gubser, "Einstein Manifolds and Conformal Field Theories," *Phys. Rev. D* **59** (1999) 025006, hep-th/9807164.

[14] M. Henningson and K. Skenderis, "The holographic Weyl anomaly," JHEP **9807**, 023 (1998) hep-th/9806087.

[15] E. Witten, "Baryons and Branes in Anti-de Sitter Space," *JHEP* **9807** (1998) 006, hep-th/9805112.

[16] J. McGreevy, L. Susskind and N. Toumbas, "Invasion of the giant gravitons from anti-de Sitter space," *JHEP* **0006** (2000) 008, hep-th/0003075.

[17] D. Berenstein, C. P. Herzog and I. R. Klebanov, "Baryon spectra and AdS/CFT correspondence," *JHEP* **0206** (2002) 047, hep-th/0202150.

[18] S. S. Gubser and I. R. Klebanov "Baryons and Domain Walls in an N=1 Superconformal Gauge Theory," *Phys. Rev.* **D58** (1998) 125025, hep-th/9808075.

[19] C. E. Beasley and M. Ronen Plesser, "Toric Duality is Seiberg Duality," *JHEP* **0112** (2001) 001, hep-th/0109053

[20] B. Feng, A. Hanany, Y.-H. He, and A. M. Uranga, "Toric Duality as Seiberg Duality and Brane Diamonds," `hep-th/0109063`.

[21] F. Cachazo, B. Fiol, K. A. Intriligator, S. Katz and C. Vafa, "A geometric unification of dualities," *Nucl. Phys.* **B 628** (2002) 3, `hep-th/0110028`.

[22] A. Iqbal and A. Hanany, "Quiver Theories from D6-branes via Mirror Symmetry," *JHEP* **0204** (2002) 009, `hep-th/0108137`

[23] B. Feng, S. Franco, A. Hanany, Y.-H. He, "Unhiggsing the del Pezzo," `hep-th/0209228`.

[24] M. Wijnholt, "Large Volume Perspective on Branes at Singularities," `hep-th/0212021`.

[25] K. Intriligator and B. Wecht, "The Exact Superconformal R-Symmetry Maximizes a," `hep-th/0304128`.

[26] S. G. Gubser, N. Nekrasov, and S. Shatashvili, "Generalized Conifolds and Four Dimensional $\mathcal{N}=1$ Superconformal Theories," *JHEP* **9905** (1999) 003, `hep-th/9811230`.

[27] A. Mikhailov, "Giant Gravitons from Holomorphic Surfaces," `JHEP` **0011** (2000) 027, `hep-th/0010206`.

[28] C. E. Beasley, "BPS Branes from Baryons," *JHEP* **0211** (2002) 015, `hep-th/0207125`

[29] C. P. Boyer and K. Galicki, "On Sasakian-Einstein Geometry," *Int. J. Math.* **11** (2000) 873, `math.dg/9811098`.

[30] A. L. Besse, *Einstein Manifolds*, Springer Verlag, 1987.

[31] S. Gukov, M. Rangamani and E. Witten, "Dibaryons, strings, and branes in AdS orbifold models," *JHEP* **9812** (1998) 025, `hep-th/9811048`.

[32] T. Friedrich and I. Kath, "Einstein Manifolds of Dimension Five with Small First Eigenvalue of the Dirac Operator", *J. Diff. Geom* **29** (1989) 263-279.

[33] T. Friedrich and I. Kath, "Compact Seven-Dimensional Manifolds with Killing Spinors", *Comm. Math. Phys.* **133** (1990) 263-279.

[34] G. Tian and S.-T. Yau, "Complete Kähler Manifolds with Zero Ricci Curvature II," *Inv. Math.* **106** (1991) 27.

[35] I. Dolgachev, "Weighted Projective Varieties" in *Proceedings, Group Actions and Vector Fields* (1981) LNM 956, 34-71.

[36] A. R. Fletcher, "Working with Weighted Complete Intersections", Preprint MPI/89-85, available at `http://cbel.cit.nih.gov/~arif/CV.html`.

[37] J. M. Johnson and J. Kollár, "Kähler-Einstein Metrics on log del Pezzo Surfaces in Weighted Projective 3-spaces," *Ann. Inst. Fourier (Grenoble)* **51** (2001), no. 1 69–79, math.AG/0008129.

[38] J. M. Johnson and J. Kollár, "Fano Hypersurfaces in Weighted Projective 4-spaces," *Experiment. Math.* **10** (2001), no. 1, 151–158, math.AG/0008189.

[39] M. Douglas and G. Moore, "D-Branes, Quivers and ALE Instantons," hep-th/9603167.

[40] E. Lopez, "A family of N = 1 SU(N)**k theories from branes at singularities," JHEP **9902**, 019 (1999) hep-th/9812025.

[41] K. Intriligator and B. Wecht, "Baryon Charges in 4d Superconformal Field Theories and Their AdS Duals," hep-th/0305046.