CONTRACTIBLE EXTREMAL RAYS ON $\overline{M}_{0,n}$.

Seán Keel and James McKernan

§1 Introduction and statement of results

One of the richest objects of study in higher dimensional algebraic geometry is the Mori-Kleiman (closed) cone of curves, $\overline{NE_1}(M)$, defined as the closed convex cone in $H_2(M, \mathbb{R})$ generated by classes of irreducible curves on M. A lot of geometric information about M is encoded in the cone of curves. For example the possibilities for maps with connected fibres are determined by the cone's faces. Not surprisingly, $\overline{NE_1}(M)$ is difficult to compute. Consider the problem of finding generators, that is, determining all of the "edges", or to use the technical term, "extremal rays" ("edge" is potentially misleading as portions of the cone may be circular). Extremal rays on which $-c_1(M) = K_M$ (or more generally log terminal $K_M+\Delta$) are negative are described by the powerful cone and contraction theorems of Mori-Kawamata-Shokurov: each is generated by a smooth rational curve, and can be "contracted", i.e. there is a map (with domain M) whose fibral curves are precisely the curves whose homology class lies on the extremal ray. Even in concrete examples, identifying contractible rays can be very challenging.

Here we consider $\bar{M}_{0,n}$, the moduli space of stable n-pointed rational curves, as well as $\overline{M}_{0,n}$ the quotient of $\bar{M}_{0,n}$ by the natural symmetric group action, which is (an irreducible component of) the moduli space of log pairs (see [1]).

The locus of points in $\bar{M}_{0,n}$ corresponding to a curve with at least $k+1$ components has pure codimension k; we call its irreducible components the **vital codimension k-cycles**. Vital divisors, curves, k-cycles etc. are analogously defined. By a vital cycle in $\overline{M}_{0,n}$ we mean the

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX image of a vital cycle in $\bar{M}_{0,n}$. It is relatively easy to check that the vital cycles generate the Chow group. It is natural to wonder if much more is true:

1.1 Question. *Is every effective cycle linearly equivalent to an effective sum of vital cycles?*

This was first posed to us by William Fulton. In the interest of drama, we will refer to (1.1) as Fulton's conjecture. Here we consider only the cases of curves and divisors. As homological and linear equivalence are the same on $\bar{M}_{0,n}$, the conjecture in these cases is equivalent to the statement that vital cycles generate all extremal rays of $\overline{NE}_1$ and $\overline{NE}^1$, the cones of curves and divisors. We prove this for $\overline{NE}^1(\overline{M}_{0,n})$ and for contractible extremal rays of $\overline{NE}_1(\bar{M}_{0,n})$.

Let $D \subset \bar{M}_{0,n}$ be the boundary, i.e. the sum of the vital divisors. Let $D = \sum B_i$ be its decomposition into S_n orbits (there are $[n/2]$ such orbits). For a subvariety $Z \subset \bar{M}_{0,n}$, let $\tilde{Z}$ be its image (with reduced structure) in $\overline{M}_{0,n}$.

Here are precise statements of our results:

1.2 Theorem. *Let R be an extremal ray of the cone of curves $\overline{NE}_1(\bar{M}_{0,n})$. Then R is spanned by a vital curve under any of the following conditions*

(1) *There is a morphism $f : \bar{M}_{0,n} \longrightarrow Y$, contracting R, with $\rho(Y) = \rho(\bar{M}_{0,n}) - 1$, and such that the exceptional locus of f is not a curve.*

(2) *$(K_{\bar{M}_{0,n}} + G) \cdot R \leq 0$, where G is an effective boundary whose support is contained in D.*

(3) *$n \leq 7$.*

Of course (1.2.3) says Fulton's conjecture holds for curves, provided $n \leq 7$. We were able to prove much stronger results for $\overline{M}_{0,n}$ (especially (1.3.1-2)):

1.3 Theorem.

(1) *The cone of effective divisors $NE^1(\overline{M}_{0,n})$ is a simplex, generated by the $\tilde{B}_i$.*

(2) *An effective divisor on $\overline{M}_{0,n}$ fails to be big iff its support is a proper subset of $\tilde{D}$, and in particular any non-trivial nef divisor is big.*

(3) *The cone of curves of $\overline{NE}_1(\overline{M}_{0,n})$ is generated by curves in $\tilde{D}$.*

Now suppose $n \leq 11$.

(4) *$NE_1(\overline{M}_{0,n})$ is a finite rational polyhedron, with edges spanned by images of vital curves,*

(5) *Every proper face is contractible by a log Mori fibre space. In particular every nef divisor is eventually free.*

(6) *The divisor $\sum_{i=2}^{[n/2]} r_i \tilde{B}_i$ is nef (resp. ample) iff*

$$r_{a+b} + r_{a+c} + r_{b+c} - r_a - r_b - r_c - r_d$$

is non-negative (resp. strictly positive), for all positive integers a, b, c and d, with $n = a + b + c + d$ (where we define $r_1 = 0$ and $r_i = r_{n-i}$ for $i > [n/2]$).

The spaces $\bar{M}_{0,n}$ and $\overline{M}_{0,n}$ are interesting from a number of viewpoints. They are closely related to the moduli space of curves, $\mathcal{M}_g$. A finite quotient of $\bar{M}_{0,n}$ occurs as a locus of degenerate curves in the boundary of $\mathcal{M}_g$, while $\bar{M}_{0,n}$ is the base of the complete Hurwitz scheme (see [2]) which can be used, for example, to prove that $\mathcal{M}_g$ is irreducible. By [4], $\bar{M}_{0,n}$ parameterises degenerations of rational normal curves. Generalisations of $\bar{M}_{0,n}$ are important for Quantum Cohomology calculations, see [10]. $\bar{M}_{0,n}$ is useful for studying fibrations with general fibre $\mathbb{P}^1$, as in particular it can sometimes be used in lieu of a minimal model program. Kawamata exploits this in [5] to prove additivity of log Kodaira dimension for one dimensional fibres, and in [6] to prove a codimension two subadjunction formula.

Another reason to study these spaces is their rich geometry. In fact there is a deep connection between the combinatorics of the vital subvarieties and their geometry, which is partly revealed by this paper. See also [11], where the combinatorics of the vital subvarieties plays a crucial rôle, and [8] where explicit generators and relations are given for the intersection ring of $\bar{M}_{0,n}$. Note that $\bar{M}_{0,n}$ is also a very natural compactification of C^{n-3}, where $C = \mathbb{P}^1 \setminus \{0, 1, \infty\}$. This gives a reason to believe (1.1), as the corresponding statement for toric varieties is true, and toric varieties provide natural compactifications of C^n where $C = \mathbb{P}^1 \setminus \{0, \infty\}$. Note that $\overline{M}_{0,n}$ is a natural compactification of $\mathbb{P}^{n-3} \setminus \Delta$, where Δ is the discriminant hypersurface.

We note that there is an explicit construction of $\bar{M}_{0,n}$, as a blow up of $\mathbb{P}^{n-3}$ along a sequence of simple centres (see (3.1)). In particular $\overline{M}_{0,5}$ is a del Pezzo of degree five, $\overline{M}_{0,6}$ is log Fano, and $\overline{M}_{0,7}$ is nearly log Fano, in the sense that $-K_{\overline{M}_{0,7}}$ is effective. It seems much harder to give an explicit construction of $\overline{M}_{0,n}$. Indeed the geometry of $\overline{M}_{0,n}$ seems much less well behaved inductively. By (1.3.3) it admits no non-trivial fibrations (see also (3.7)).

Despite the fact that this construction gives an easy computation of some invariants of $\bar{M}_{0,n}$, one cannot expect the same for the cone of curves. For example the blow up of $\mathbb{P}^2$ in eight points has a finite polyhedral cone of curves, but one can choose a ninth point in such a way that the blow up has a cone with infinitely many edges. We do not use the blow up description in any significant way in our proof of (1.2-3).

As we note in (3.5) $-K_{\overline{M}_{0,n}}$ and $-K_{\bar{M}_{0,n}}$ are not effective for $n \geq 8$. In view of this, the cases of (1.3) for $8 \leq n \leq 11$ are interesting in that they give examples of non log Fano varieties, for which every face of the cone of curves is none the less contractible.

From this perspective, (1.1), if true, would really be rather surprising. Each vital curve (indeed every vital cycle) is smooth and rational. The cone of curves of a log Fano is generated by rational curves, but one does not expect this in general, even for a rational variety. For example, let S be the blow up of $\mathbb{P}^2$ in a large number of general points. As observed by Kollár, and independently by Caporaso and Harris, K_S is strictly negative on rational curves, but of course $K_S^2 < 0$, so K_S must be positive on some curves (but see the remark after (2.4)).

If (1.1) holds for curves, then one can describe the ample cones of $\bar{M}_{0,n}$ or $\overline{M}_{0,n}$ by a series of inequalities analogous to those in (1.3), using (4.3). One can then describe, at least in theory, the cone of curves, since it is dual to the ample cone. As an example of the complexity of these cones, $NE_1(\overline{M}_{0,7})$ is a polyhedral cone of dimension 42 with 350 edges (see (4.3) and (4.6)).

Fulton's conjecture implies every vital curve spans an extremal ray and each is $K_{\bar{M}_{0,n}} + G$ negative for some G as in (1.2.2) (see (4.6)). So by the contraction theorem [7] each vital curve is contracted by a map of relative Picard number one. For $n \geq 9$ every vital curve deforms. So if (1.1) holds, then (1.2) contains all the possibilities for extremal rays, and (1.2.1) has all the possibilities for $n \geq 9$.

Here is a brief outline of our proofs of (1.2-3). It turns out each component of D is a product $\overline{M}_{0,i} \times \overline{M}_{0,j}$, for $i,\ j < n$ (see §3). For (1.2) we proceed by induction, the main work is to show the extremal ray R is in the subcone generated by curves in D. For this our main tool is (2.2). (1.3) follows from (1.2) and some simple intersection calculations: one set to show that $NE^1(\overline{M}_{0,n})$ is a simplex, and a second to show that for $n \leq 11$ every face of the cone contracts to a log Mori fibre space.

§2 contains some results about the cone spanned by curves lying in a divisor. In fact most of the results of §2 are of independent interest (see (2.4)) hopefully applicable to other moduli spaces (see (2.3) and (2.6)). For this reason we work in more generality. §3 then contains the necessary ingredients to apply some of the results of §2 to $D \subset \bar{M}_{0,n}$. Intersection products of various vital cycles are easy to compute, and the pairing between divisors and curves is described in §4. §5 contains a finishes the proof of (1.3).

We would like to say a few words about other seemingly natural approaches to (1.1). For curves, it is enough (in fact equivalent) to show that if a divisor intersects all vital curves non-negatively, then it is nef. By induction it is sufficient to show that such a divisor is linearly equivalent to an effective sum of vital divisors. As the vital divisors generate the Picard group, the intersection conditions give a finite collection of simple inequalities on the coefficients. Unfortunately the combinatorics are intimidating, and we were not able to make any progress in this direction, even for $n = 6$. Another way to approach (1.1) is to try to deform any cycle in $\bar{M}_{0,n}$, other than a vital cycle, and to break it up á la Mori. Even though this seems extremely hard in general it does at least give another indication why one might believe (1.1).

Throughout we will use the main results of the minimal model program, the contraction theorem, the cone theorem etc., as well as the established notation as set out in [7]. We also use elementary properties and notions of cones from Chapter II.4 of [9]. In particular, by an extremal ray R of a closed convex cone W we mean a one dimensional subcone with the property that if $x+y \in R$ for $x, y \in W$ then $x, y \in R$. We note that every closed convex cone is the convex hull of its extremal rays. All spaces are assume to be of finite type over $\mathbb{C}$. Unless otherwise stated, by a divisor we mean an $\mathbb{R}$-divisor. In [13] the main results of the MMP are extended to $\mathbb{R}$-divisors. However we only need one such result (see (2.2)).

§2 The cone spanned by curves inside a divisor

We first introduce some notation and definitions. Let D be a reduced Weil divisor inside the projective $\mathbb{Q}$-factorial klt variety M of dimension n. Let W be the closed subcone of $\overline{NE}_1(M)$ generated by curves lying in D.

We are interested in extremal rays that lie outside of W and moreover under what conditions $W = \overline{NE}_1(M)$.

As many of the results work inductively, we also define W_i to be the cone generated by curves lying in subvarieties Z, where Z is the intersection of components of D, Z has dimension at least i and the intersection of Z with any component of D not containing Z has dimension strictly less than i. Note that $W = W_{n-1}$.

2.1 Definition. We say that an effective divisor has **ample support** if it has the same support as some effective ample divisor.

We say that D has **anti-nef normal bundle** if for every curve $C \subset D$, $C \cdot D \leq 0$.

We will say an extremal ray R of $\overline{NE}_1(M)$ is **log extremal** if there exists a klt divisor $K_M + \Delta$ such that $(K_M + \Delta) \cdot R < 0$.

Note that log extremal rays are very special. In fact by the cone and contraction Theorems (see for example [7]) they are spanned by rational curves C, and there is a morphism $f : M \longrightarrow Y$ contracting C such that $f_*(\mathcal{O}_M) = \mathcal{O}_Y$ and $\rho(Y) = \rho(M) - 1$.

The following recent result of Shokurov will prove useful:

2.2 Lemma(Shokurov). *Let X be a projective variety, and let $L \in N^1(X)$ be a nef class (not necessarily rational) with $L^{dim(X)} > 0$. Then L is in the interior of $NE^1(X)$.*

Proof. This is implied by the proof of (6.17) of [13]. $\square$

(2.2) has some interesting corollaries:

2.3 Corollary. *If the components of D span $\overline{NE}^1(M)$ then $W = \overline{NE}_1(M)$.*

Proof. Let $D = \sum D_i$ be the decomposition of D into irreducible components.

Let A be an ample divisor with support in D, and let $R \subset \overline{NE}_1(M)$ be an extremal ray. Assume $R \notin W$. Let L be a nef class supporting R. $L|D$ is ample. Since L is an effective sum of D_i, $L^{\dim M} > 0$ thus by (2.2), R cannot be numerically effective. Since the D_i generate $\overline{NE}^1(M)$, $R \cdot D_i < 0$ for some i. But this implies $R \in W$, a contradiction. $\square$

2.4 Proposition. *Let G be an effective $\mathbb{Q}$-divisor, with non-empty support D.*

Let R be an extremal ray of $\overline{NE}_1(M)$, which does not lie in W. If $(K_M + G) \cdot R \leq 0$ then R is log extremal and $K_M \cdot R \leq 0$.

In particular, if $-(K_M + G)$ is nef then $\overline{NE}_1(M)$ is spanned by W and log extremal rays R, such that $K_M \cdot R \leq 0$.

Proof. Let R be an extremal ray of $\overline{NE}_1(M)$, not lying in W. In particular $R \cdot D_i \geq 0$ and so $K_M \cdot R \leq 0$.

On the other hand we are done if $K_M \cdot R < 0$. Thus we may assume $K_M \cdot R = 0$. Let $L \in N^1(M)$ be a nef class supporting R. Then L is strictly positive on $W \setminus 0$ and so by compactness of a slice of W, $L + \epsilon D$ is nef and supports R for $0 < \epsilon \ll 1$. As L is ample on D $L^{n-1} \cdot D > 0$. In particular we can replace L by $L + \epsilon D$ and assume $L^n > 0$. Then by (2.2) $R \cdot V < 0$ for some effective Weil divisor V. But $(K_M + \epsilon V) \cdot R < 0$ and $(K_M + \epsilon V)$ is klt for $0 < \epsilon \ll 1$. $\square$

Remark. (2.4) is interesting even in the case of a surface. For example pick a cubic in $\mathbb{P}^2$ and blow up as many points as you like along the cubic. Let M be the resulting surface and D the strict transform of the cubic. (2.4) then says that D union all the -2-curves and -1-curves generate the cone of curves of M.

2.5 Proposition. *Let $f : M \longrightarrow Y$ be a proper surjection from a smooth projective variety M to a normal variety Y with $f_*(\mathcal{O}_M) = \mathcal{O}_Y$, and $\rho(Y) = \rho(M) - 1$. Suppose D has ample support and each irreducible component of D has anti-nef normal bundle.*

If $f|D$ is finite then f is birational, and its exceptional locus is a curve.

Proof. Suppose on the contrary that there is an irreducible surface E whose image has dimension at most one.

Let $D = \sum_i D_i$ be the decomposition of D into irreducible components. Note the assumptions on Picard number imply that any class in $N^1(M)$ which is zero on some fibral curve, is pulled back from $N^1(Y)$.

Since D has ample support $I = D \cap E$ is non-empty. As $f|D$ is finite, I and each $D_i \cap E = D_i \cap I$, is an effective $\mathbb{Q}$-Cartier divisor of E, and in particular, is purely one dimensional. Thus if I meets D_i, it has an irreducible component contained in D_i. Since D has ample support, and $f|D$ is finite, E contracts to an irreducible curve $C \subset f(D) = f(I)$ and $f|I$ is finite.

Claim. We can find two irreducible components B_1, B_2 of I and (after renaming) two divisors D_1, D_2 with $B_i \subset D_i$ such that $B_i \cdot D_j \geq 0$ (for $i \neq j$) and at least one inequality is strict:

Choose an irreducible component B_1 of I contained in a maximal number of D_i. Suppose (after reordering) $D_1, D_2, \dots, D_k$ are the components of D containing B_1. Since the D_i have anti-nef normal bundles, and D has ample support, for some $j > k$ we have $D_j \cdot B_1 > 0$. Let B_2 be an irreducible component of $D_j \cap I$. By the choice of B_1 we can assume (after reordering) that $B_2 \not\subset D_1$. Now set $D_2 = D_j$.

This establishes the claim.

Since D_1, D_2 each meet a fibre, we can choose $\lambda > 0$ such that $D_1 - \lambda D_2$ is pulled back from Y. Let $J = (D_1 - \lambda D_2)|E$. Then $J \cdot B_1 \leq 0$ and $J \cdot B_2 \geq 0$, and one inequality is strict. Since J is pulled back from C, and the B_i are multi-sections, this is a contradiction. $\square$

Remarks.

(1) The assumption on the relative Picard number in (2.5) is necessary; it cannot be replaced by the weaker assumption that f is the contraction of an extremal ray. For example consider $M = E \times E$ for an elliptic curve E, $D = F_1 + F_2$ the sum of the two fibres and $f : M \longrightarrow E$ the addition map.

(2) The assumption on Picard number is not as restrictive as it looks, however, as it always holds when f is the contraction of a log extremal ray.

(3) One can not rule out the final possibility. For example: Let M be a del Pezzo surface whose cone of curves is not a simplex (e.g. blow up $\mathbb{P}^2$ at three non-collinear points). Let D be a sum of $\rho(M)$ -1-curves with ample support (any effective class is a sum of at most $\rho(M)$ extremal rays, and all the extremal rays are -1-curves). Let f blow down some other -1-curve.

2.6 Lemma. *Suppose M is smooth, every component of D has anti-nef normal bundle, and D has ample support. Let G be an effective $\mathbb{Q}$-divisor whose support lies in D.*

(1) *Let R be an extremal ray of $\overline{NE}_1(M)$. If either the dimension of M is three and $-(K_M + G) \cdot R < 0$, or the dimension of M is at least four and $-(K_M + G) \cdot R \leq 0$ then $R \in W$.*

(2) *If $-(K_M + G)$ is nef, and either the support of G is exactly D or the dimension of M is at least four, then $W = \overline{NE}_1(M)$.*

Moreover if $K_M + D$ is lt and G is a boundary then we may replace W by W_3 (and even W_2 in the case of strict inequality) in the statements above.

Proof. Let R be an extremal ray of $\overline{NE}_1(M)$, and suppose $R \notin W$ but $(K_M + G) \cdot R \leq 0$. Then by (2.4) we know that R is spanned by a contractible rational curve C. (1) and (2) now follow easily from (2.5) and the observation that if $K_M \cdot C < 0$ and M is a threefold (resp. $K_M \cdot C \leq 0$ and M has dimension at least four) then C deforms inside M (see II.1.13 of [9]).

Now suppose that $K_M + D$ is lt and G is a boundary. Let R be an extremal ray of $\overline{NE}_1(M)$, such that $-(K_M + G) \cdot R \leq 0$. Now R belongs to one of the components of D, say G' and we can increase the coefficient of G' in G to one, restrict to G' and apply induction. Thus the last statement also holds. $\square$

We will use the following technical result in the next section.

2.7 Lemma. *Let $N \subset M$ be a normal divisor and suppose that $\mathrm{Pic}\,(M)_{\mathbb{Q}} \longrightarrow \mathrm{Pic}\,(N)_{\mathbb{Q}}$ is surjective. Let $f : M \longrightarrow Y$ be a map to a normal projective variety with $f_*(\mathcal{O}_M) = \mathcal{O}_Y$, and $\rho(Y) = \rho(M) - 1$. Let $g : N \longrightarrow Z$ be the Stein factorisation of $f|_N$. If $f|N$ is not finite, then $\rho(Z) = \rho(N) - 1$.*

Proof. f contracts an extremal ray R. Suppose $f|N$ is not finite. Then $R \in N_1(N)$. If $L \in \mathrm{Pic}\,(M)_{\mathbb{Q}}$ and $L \cdot R = 0$, then $L|N$ is pulled back from Z. Since every class in $N^1(D)$ extends to M, the result follows. $\square$

§3 Geometry of $\bar{M}_{0,n}$ and $\overline{M}_{0,n}$.

We will use (a slight modification of) the notation of, as well as several simple facts from pg. 551–554 of [8]. For the readers convenience we will recall the most important ideas:

A vital divisor is determined by a partition of $\{1, 2, \ldots, n\}$ into disjoint subsets T, T^c, each containing at least two elements. The generic point of the corresponding vital divisor D_{T,T^c} is a curve with two irreducible components, with the labels of T on one component, and the labels of T^c on the other. There is a canonical isomorphism

$$D_{T,T^c} = M_{T \cup \{b\}} \times M_{T^c \cup \{b\}}$$

where e.g. by $M_{T\cup\{b\}}$ we mean a copy of $\overline{M}_{0,|T|+1}$ with the indices labeled by the elements of T, with b an extra index, corresponding to the singular point. We indicate the two projections by π_T and π_{T^c}.

The vital divisors have normal crossings, and each vital codimension k-cycle is uniquely expressible as a complete intersection of vital divisors. Each vital k-cycle has an expression as a product of $\overline{M}_{0,i}$ analogous to that for the vital divisors. In particular, under the above decomposition, any vital curve of D_{T,T^c} is a product of a vital curve on one factor, with a vital point on the second.

Note that the cones W_i are, in this case, simply the cones spanned by curves lying in vital i-cycles.

3.1 Proposition(Kapranov). *For each index $i \in \{1, 2, \dots, n\}$ there is a birational map $q_i : \bar{M}_{0,n} \longrightarrow \mathbb{P}^{n-3}$ with the following properties:*

(1) *q_i is a composition of blow ups along smooth centres, constructed as follows. Fix $n-1$ general points, and blow up successively (from lowest to highest dimensional) the (strict transforms) of every linear subspace spanned by any subset of these points.*
(2) *q_i takes vital cycles to to linear spaces spanned by the chosen points.*
(3) *If $i \in T$ then $q_i|_{D_{T,T^c}} = q_i \circ \pi_T$ for $i \in T$.*
(4) *If F is the general fibre of the map $\bar{M}_{0,n} \longrightarrow \overline{M}_{0,n-1}$ given by dropping the i^{th} point, then $q_i(F)$ is a rational normal curve.*
(5) *q_i is a composition of smooth blow downs, blowing down iteratively the (images of) the divisors D_{T,T^c} with $i \notin T$, and $|T| = 3, 4, \dots, n-2$.*

Proof. See [3]. $\square$

Let $U = \bar{M}_{0,n} \setminus D$, corresponding to the locus of distinct points.

3.2 Lemma. *Let ϕ be an element of $\mathrm{Aut}(\mathbb{P}^1)$ of order p, and let Z be the closure of the points of U fixed by ϕ. Let q be a general point of Z.*

If ϕ fixes j points of q then the dimension of Z is $(n-j)/p - 1 - j$.

Proof. Let $G \subset \mathrm{Aut}(\mathbb{P}^1)$ be the subgroup generated by ϕ. Then G has a non-trivial finite orbit, from it which it follows that G has exactly two fixed points, and after changing coordinates (so the fixed points are 0 and ∞) $\phi : \mathbb{A}^1 \longrightarrow \mathbb{A}^1$ is multiplication by a root a p^{th} root of unity, $p = |G|$. In particular every orbit either consists of the fixed points or has exactly p elements and so $p|(n-j)$. Let $m = (n-j)/p$.

Let Q be any irreducible component of

$$\{ (q, \phi, a, b) \in U \times \mathrm{Aut}(\mathbb{P}^1) \times \mathbb{P}^1 \times \mathbb{P}^1 \mid a \neq b \in \mathbb{P}^1 \text{ and } \phi \text{ permutes } q \text{ and fixes } (a, b) \}.$$

Let ϕ be a general point of the image of $p : Q \longrightarrow \mathrm{Aut}(\mathbb{P}^1)$. Then $p^{-1}(\phi)$ has dimension m, while the fibre of $Q \longrightarrow Z$ has dimension $1 + j$. Thus Z has dimension $(n - j)/p - 1 - j$. $\square$

3.3 Lemma. *S_n acts freely in codimension one on U for $n \geq 7$, and faithfully for $n \geq 5$.*

The action of S_4 on $\overline{M}_{0,4}$ factors through the action on the set of partitions of $\{1, 2, 3, 4\}$ into disjoint subsets of two elements. Nontrivial elements of the kernel are of form $(i, j)(k, l)$ for i, j, k and l distinct.

Proof. The claims about the S_4 action are easily checked, and are left to the reader. The rest follows from (3.2), and the observation that elements σ of S_n which fix points q of U correspond to elements ϕ of $\mathrm{Aut}(\mathbb{P}^1)$ which also fix q. $\square$

Let $B_i = \sum_{|T|=i} D_{T,T^c}$ for $2 \leq i \leq k = [n/2]$. B_i is the orbit under S_n of any D_{T,T^c} with $|T| = i$.

3.4 Lemma. *For $n \geq 7$ the quotient map $q : \bar{M}_{0,n} \longrightarrow \overline{M}_{0,n}$ is unramified in codimension one outside of B_2, and has ramification index two along B_2.*

Proof. Suppose $\sigma \in S_n$ fixes each point of the irreducible divisor $G \subset \bar{M}_{0,n}$. By (3.3), $G = D_{T,T^c}$ for some T preserved by σ. Since the action of σ on $M_{T \cup \{b\}}$ factors through the subgroup of $S_{|T|+1}$ which fixes b, it follows from (3.3) that $T = \{i, j\}$ and $\sigma = (i, j)$. $\square$

We will use the following formulae (which are essentially due to Pandhapripande, [12]):

3.5 Lemma.

$$K_{\bar{M}_{0,n}} + \sum_{j=2}^{k} (2 - \frac{j(n-j)}{n-1}) B_j = 0 = K_{\overline{M}_{0,n}} + (\frac{1}{2} + \frac{1}{(n-1)}) \tilde{B}_2 + \sum_{j=3}^{k} (2 - \frac{j(n-j)}{n-1}) \tilde{B}_j$$

In particular $-K_{\overline{M}_{0,n}}$ (resp. $-K_{\bar{M}_{0,n}}$) is pseudo-effective iff $n \leq 7$.

Proof. The first formula is Proposition 1 of [12], the second follows easily from the first and (3.4) and the last statement then follows from (4.8). In fact we may use (4.3) to prove the first formula in a similar way to the way it is derived in [12].

However it is possible to prove the first formula in an entirely elementary way, using (3.1). Indeed the image D' of D is the union of $\binom{n-1}{2}$ hyperplanes, and the coefficients of B_i are easily identified as the discrepancies of the divisor $K_{\mathbb{P}^{n-3}} + (2/(n-1))D'$. $\square$

3.6 Lemma. *$K_{\bar{M}_{0,n}}+D$ is ample and is linearly equivalent to an effective divisor with the same support as D.*

Proof. We proceed by induction on n. The result is easy for $n = 4$.

By (3.5), $K_{\bar{M}_{0,n}} + D$ is linearly equivalent to an effective divisor Γ with support D.

Note that $(K_{\bar{M}_{0,n}} + D)|D_T$ is the tensor product of the "same expressions" pulled back from the two components in the product description of D_T. Thus by induction $(K_{\bar{M}_{0,n}} + D)|D$ is ample.

It is easy to see that D meets (set theoretically) every curve. Use induction and consider the map $f : \bar{M}_{0,n} \longrightarrow \overline{M}_{0,n-1}$, observe that D meets every fibral curve, and note that $D \supset f^{-1}(D(\overline{M}_{0,n-1}))$.

Thus $(K_{\bar{M}_{0,n}} + D) \cdot C > 0$ for all curves C.

It follows that $K_{\bar{M}_{0,n}} + D$ is nef, and nef and big by induction. Thus by the base point free theorem (applied to the big and nef klt divisor $K_{\bar{M}_{0,n}} + D - \epsilon\Gamma$) $m(K_{\bar{M}_{0,n}} + D)$ is basepoint free for $m \gg 0$. Since it intersects every curve positively, it is thus ample. $\square$

The results above have some interesting geometric consequences:

3.7 Remarks.

(1) By (3.1.1) $\overline{M}_{0,5}$ is isomorphic to $\mathbb{P}^2$ blown up at four points. Thus it is a del Pezzo surface of degree five. It is interesting to note that $K_{\overline{M}_{0,5}} + D = -K_{\overline{M}_{0,5}}$ is very ample and defines the anticanonical embedding of $\overline{M}_{0,5}$ inside $\mathbb{P}^5$. In fact if C is a vital curve,

then $(K_{\bar{M}_{0,n}} + D) \cdot C = 1$. Thus if $K_{\bar{M}_{0,n}} + D$ is very ample, then every vital curve will be embedded as a line.

(2) Note that $\tilde{M}_{0,5}$ is a log del Pezzo of rank one. It is easy to compute, using (3.2) that $\tilde{M}_{0,5}$ has two quotient singularities, one of index two and the other of index five. It is then easy, from the classification of log del Pezzos, to conclude that $\tilde{M}_{0,5}$ has one A_1-singularity and one singularity of type $(2,3)$.

(3) $\overline{M}_{0,n}$ is never a subvariety of $\overline{M}_{0,m}$ for any m. Indeed any subvariety of $\overline{M}_{0,m}$ for $m > n$, of dimension at least two, has Picard number at least $n-1$, by (3.1.1). On the other hand it is not too hard to show that for every n, there is an m such that $\bar{M}_{0,n}$ is a vital subvariety inside $\tilde{M}_{0,m}$.

(4) Note that the map $D_{T,T^c} \longrightarrow \overline{M}_{0,n}$ factors through $M_{|T|+1,|T^c|+1}/S_{|T|} \times S_{|T^c|}$, but not through the quotient by $S_{|T|+1} \times S_{|T^c|+1}$. Thus an inductive study of $\overline{NE}_1(\overline{M}_{0,n})$ seems problematic.

(5) By (3.1), $q_i^*(\mathcal{O}(1))$ is numerically equivalent to an effective divisor with support exactly D. It follows by (3.6) that for any curve $C \subset D$ there is some vital divisor which is negative on C.

3.8 Lemma. *For any projective variety T, $N_1(\bar{M}_{0,n} \times T) = N_1(\bar{M}_{0,n}) \times N_1(T)$ under the map induced by the two projections. The same map induces an isomorphism*

$$NE_1(\bar{M}_{0,n} \times T) = NE_1(\bar{M}_{0,n}) \times NE_1(T)$$

Proof. This follows from Theorem 2 of [8]. $\square$

3.9 Corollary. *Fulton's conjecture for divisors implies the conjecture for curves.*

Proof. Immediate from (2.2) and (3.8). $\square$

Proof of (1.2). As we are going to use induction it is actually more convenient to prove a slightly stronger result. Let M be any product of $\overline{M}_{0,i}$. We will prove (1), (2) and (3) of (1.2) for M.

By a vital cycle on M we mean a product of vital cycles on each component. We will continue to use the same notation, so for example by D_T we mean the inverse image of this divisor from a projection onto one of the components of M.

Let m be the dimension of M, and R an extremal ray of M. We will prove that R is spanned by a vital curve by induction on m.

Suppose f is a contraction $M \longrightarrow Y$. By (2.6) there is a curve C, contracted by f, which also lies in D. By the (3.7.5) there is a vital divisor N, such that $N \cdot C < 0$. (1) now follows immediately from (2.7).

On the other hand note that it is easy to show that R is spanned by a vital curve if $m < 3$. Note also that M is log Fano, for $m \leq 3$ by (3.5) and so R is spanned by a vital curve if $m = 3$ as well, by (2.6.1).

(2) then follows by the strong version of (2.6.2). Similarly R is spanned by a vital curve, in the case $m = 4$ by (2.6.1) and (3.5). In particular (3) holds. $\square$

§4 Intersecting vital curves and divisors.

By a *marked point* of an n-pointed curve, we either mean one of the singular points of the curve, or one of the labeled points $p_1, p_2, \ldots, p_n$.

4.1 Notation: Let C be a vital curve. Let $G = G(C)$ be the n-pointed stable curve corresponding to the generic point of C. G has $n-3$ components, all but one of which contain 3 marked points, and exactly one of which contains 4 *marked points. We call this last component $Q = Q(C)$, the distinguished component of G. Let $s(C)$ be the number of singular points on Q, $l(C)$ be the number of labeled points. C determines a decomposition of* $\{1, 2, \ldots, n\}$ *into* 4 *disjoint subsets: $G \setminus Q$ has exactly $s(C)$ connected components. We decompose* $\{1, 2, \ldots, n\}$ *into those labeled points on each of the components. Additionally we take the singleton sets for each of the $l(C)$ labeled points on G. We call this decomposition P_C.*

There are $n-4$ singular points on G (intersection points of two components). Each singular $p \in G$ defines a decomposition, by letting T_p and T_p^c be the labels on the two connected components *of $G \setminus \{p\}$. C is the complete intersection $\bigcap_{p \in \mathrm{Sing}(G)} D_{T_p,T_p^c}$. Let A_{T_p} and $A_{T_p^c}$ be the connected components of $G \setminus \{p\}$.*

4.2 Lemma. *Let C be a vital curve.*

(1) *For $p \in \mathrm{Sing}(G)$*

$$\pi_{T_p} : D_{T_p,T_p^c} \longrightarrow M_{T_p \cup \{b\}}$$

contracts the vital curve C iff $A_{T_p^c}$ contains the generic point of $Q(C)$.

(2) *q_i contracts C iff i is not one of the labeled points of $Q(C)$ (in particular any C with $l(C) = 0$ is contracted).*

Proof. (1) is immediate and (2) follows from (1) and (3.1.3). □

4.3 Lemma. *P_C uniquely determines the numerical class of C. $K_{\bar{M}_{0,n}} \cdot C = 2 - l(C)$. For any vital divisor D_{T,T^c} we have:*

(1) *$D_{T,T^c} \cdot C = -1$ iff T or T^c is one of the equivalence classes of P_C. Equivalently, iff T or T^c is T_p for some singular point $p \in Q$.*

(2) *$D_{T,T^c} \cdot C = 1$ iff T or T^c is the union of two equivalence classes.*

(3) *Otherwise $D_{T,T^c} \cdot C = 0$.*

Proof. Since the vital divisors generate $\mathrm{Pic}\,(\bar{M}_{0,n})$ the description of $D_{T,T^c} \cdot C$ implies the first statement. The expression for $K_{\bar{M}_{0,n}} \cdot C$ follows from the expression for $D_{T,T^c} \cdot C$ using the adjunction formula, since C is a complete intersection of vital divisors.

Fix $p \in \mathrm{Sing}(G)$ and let S be the intersection of the D_{T_q,T_q^c} for $q \neq p$. Then $C \cdot D_{T_p,T_p^c}$ is the self intersection of C in S. S is a vital surface, and so it is either $\overline{M}_{0,5}$ (which is $\mathbb{P}^2$ blown up in 4 points) or $\overline{M}_{0,4} \times \overline{M}_{0,4}$ (which is $\mathbb{P}^1 \times \mathbb{P}^1$), and C is a vital curve in S. In the first case C is a -1-curve, and in the second a fibre of one of the two projections. Let γ be the pointed stable curve corresponding to a generic point of S. In the first case γ has one component with 5 marked points, and in the second case, two components each with 4 marked points. G is obtained as the limit as two of the marked points (on the same component) come together at

p. It's clear that the first case occurs iff $p \in Q$, whence (1). Note the argument shows that if $C \subset D_{T,T^c}$ then $C \cdot D_{T,T^c}$ is either 0 or 1.

If $D_{T,T^c} \cdot C > 0$ then $D_{T,T^c} \cap C$ is a vital point of $C = \overline{M}_{0,4}$, i.e. a reduced point, thus $D_{T,T^c} \cdot C = 1$. This occurs if T or T^c is a union of two equivalence classes of P_C and every vital divisor of C can be obtained in this way. Since each vital cycle is uniquely a complete intersection of vital divisors, this gives (2).

(3) clearly follows from (1) and (2). □

4.4 Corollary. *The numerical class of $\tilde{C}$ is determined by the cardinalities of the subsets in P_C. If these cardinalities are a, b, c, d then*

$$C \cdot \sum r_i B_i = -r_a - r_b - r_c - r_d + r_{a+b} + r_{a+c} + r_{a+d}$$

where we define $r_1 = 0$ and $r_i = r_{n-i}$ for $i > [n/2]$.

4.5 Lemma.

$$N_{D_{T,T^c}} \bar{M}_{0,n} = (q_b \circ \pi_T)^*(\mathcal{O}(-1)) \otimes (q_b \circ \pi_{T^c})^*(\mathcal{O}(-1)).$$

Proof. Clearly we only need to check how both sides intersect a vital curve $C \subset D_{T,T^c}$. By (3.1.2), and (4.3) the possible values of these intersections are 0 and -1, and it is enough to show show $D_{T,T^c} \cdot C = -1$ iff one of the two maps $q_b \circ \pi_T$ or $q_b \circ \pi_{T^c}$ fails to contract C.

By (4.2.1) we may assume that π_T is finite on C (otherwise switch T and T^c). By (4.2.1) and (4.3.1), $D_{T,T^c} \cdot C = -1$ iff b is a labeled point of $Q(\pi_T(C))$, thus by (4.2.2), iff q_b is finite on $\pi_T(C)$. □

4.6 Lemma. *Let C be a vital curve, and let*

$$D_C = \sum_{p \in \mathrm{Sing}(G) \cap Q} D_{T_p, T_p^C}.$$

$(K_{\bar{M}_{0,n}} + D_C) \cdot C = -2$. $K_{\bar{M}_{0,n}} + D + 1/sD_C$ intersects vital curves non-negatively, and vanishes on exactly those vital curves numerically equivalent to C.

Proof. Immediate from (4.3). $\square$

4.6.1 Remark. (1.1) and the basepoint free theorem imply $K + D + 1/sD_C$ is eventually free, and thus C spans an extremal ray. Presumably this could be checked directly.

The following is immediate:

4.7 Lemma. *Let $T \subset \{1, 2, \ldots, n\}$ with $|T| \geq 3, |T^c| \geq 2$. For $i \in T$,*

$$D_{T\setminus\{i\},T^c\cup\{i\}}|_{D_{T,T^c}} = D_{ib,T\setminus\{i\}} \times M_{T^c\cup\{b\}}$$

under the canonical product decomposition. There is no other vital divisor with the same restriction.

4.8 Lemma. *Suppose there is a numerical equality*

$$\sum_{i=2}^{k} m_i B_i \sim F$$

and either F is nef, or both sides are effective and have no divisor common to their supports. Then

$$rm_{r-1} \geq (r-2)m_r \qquad \text{for } 3 \leq r \leq k$$

$$(n-r)m_{r+1} \geq (n-r-2)m_r \qquad \text{for } 2 \leq r \leq k-1.$$

In particular, when the left hand side is effective, it is either trivial, or has support exactly D. In particular (1.3.1-4) hold.

Proof. We prove the first inequality, the argument for the second is analogous.

Choose T with $|T| = r$. Let Z_r be the general fibre of

$$M_{T\cup\{b\}} \longrightarrow M_T.$$

Let $p \in M_{T^c\cup\{b\}}$ be a general point, and let $D_{T,T^c} \supset C_r = Z_r \times \{p\}$. By (3.1), (4.5) and (4.7) we have

$$C_r \cdot B_i = \begin{cases} r \text{ if } i = r-1 \\ -(r-2) \text{ if } i = r \\ 0 \text{ otherwise} \end{cases}$$

The inequality is obtained by intersecting both sides with C_r.

Note that (1.3.1-3) follow immediately and that (1.3.4) then follows from (2.3). $\square$

§5 An interesting result about $\overline{M}_{0,n}$.

Given (4.8), it is natural to hope that every nef divisor on $\overline{M}_{0,n}$ is eventually free. The obvious approach is to try to use the basepoint free theorem, and thus to realise some positive multiple of a big nef class E (pulled back from $\overline{M}_{0,n}$) as a klt divisor $K_{\bar{M}_{0,n}} + \Delta$.

5.1 Lemma. *If E is a big nef class on a normal $\mathbb{Q}$-factorial variety M, and there is a divisor Δ with $K_M + \Delta$ klt and numerically equivalent to a positive multiple of E, then the extremal subcone of $\overline{NE}_1(M)$ supported by E is rational polyhedral, and is contracted by a log Mori fibre space. If $M = \bar{M}_{0,n}$, the subcone supported by E is spanned by vital curves.*

Proof. By (2.2) we have $E = A + Z$ where A is ample and Z is effective. If $V \subset \overline{NE}_1(M)$ is the extremal subcone supported by E, then $K_M + \Delta + \epsilon Z$ is negative on $V \setminus 0$. Thus the result follows from the cone and contraction theorems, together with (1.2) $\square$

Let E be a nef divisor on $\bar{M}_{0,n}$, pulled back from $\overline{M}_{0,n}$. In general, by (3.5) and (4.8), replacing E by a large multiple one has $E = K_{\bar{M}_{0,n}} + \Delta$ for some Δ supported on D. We can try to make Δ a boundary by subtracting off part of E, thus we are lead to consider:

5.2 Definition-Lemma. *Let E be a non-trivial nef class on $\bar{M}_{0,n}$, pulled back from $\overline{M}_{0,n}$ with $n \geq 8$. Then there is a unique effective class Δ_E with the following properties*

(1) *Δ_E has support a proper subset of D*

(2) *$K_{\bar{M}_{0,n}} + \Delta_E = \lambda E$ for some $\lambda > 0$.*

Proof. For any λ, $-K_{\bar{M}_{0,n}} + \lambda E$ is pulled back from $\overline{M}_{0,n}$, thus by (4.8), (1) is the requirement that Δ_E be on the boundary of NE^1. Since E is in the interior of NE^1, and by (3.5), $-K_{\bar{M}_{0,n}} \notin NE^1$, the result is clear. $\square$

Notation: *For the next corollary, define the integer function $f(a,b,c,d)$ to be 2 minus the number of variables equal to one.*

We will say that P_n holds if for a given integer n the following implication holds:

Let $r_1, r_2, \ldots, r_{n-1}$ be a collection of non-negative real numbers, with $r_1 = 0$, $r_i = r_{n-i}$, and $r_j = 0$ for some $2 \leq j \leq k$. If

$$f(a,b,c,d) + r_{a+b} + r_{a+c} + r_{a+d} \geq r_a + r_b + r_c + r_d$$

for every set of positive integers a, b, c, d with $n = a+b+c+d$, then $r_i < 1$ for all i.

5.3 Corollary. *Δ_E is a pure boundary for every non-trivial nef class pulled back from $\overline{M}_{0,n}$ iff P_n holds.*

Proof. By (4.8) P_n is equivalent to the statement: If $\sum r_i B_i$ has support a proper subset of D and $(K_{\bar{M}_{0,n}} + \sum r_i B_i) \cdot C \geq 0$ for all vital curves C, then $\sum r_i B_i$ is a pure boundary. Thus the only thing to show is that if Δ_E is a pure boundary for every non-trivial nef class, then the images of vital curves generate $NE_1(\overline{M}_{0,n})$. This follows from (5.1). $\square$

For a given n it is straightforward to check whether or not P_n holds:

5.4 Lemma. *P_n holds for $8 \leq n \leq 11$, and fails for $n \geq 12$.*

Proof. We will check P_9. The cases $n = 8$, 10 and 11 are similarly checked.

Let $r_1, r_2, \ldots, r_8$ be a collection of non negative numbers, as in the definition of P_9. From the sums

$$1+2+3+3=9$$

$$1+1+1+6=9$$

$$1+1+2+5=9$$

we obtain the inequalities

$$1 + 2r_4 \geq r_3 + r_2$$

$$2r_3 \geq r_4$$

$$3r_2 \geq r_3 + 1$$

The result follows easily by considering in turn the possibilities $r_4 = 0$, $r_3 = 0$ and $r_2 = 0$.

Now suppose $n \geq 12$, but $n \neq 13$, 14, or 17. These cases can be dealt similarly and it is left as an exercise for the reader. Set, for $2 \leq a \leq k$

$$r_a = \begin{cases} 1, & \text{when } a = 2 \\ 0 & \text{when } a = k-1 \\ 1/2 & \text{otherwise.} \end{cases}$$

I claim this violates P_n. Suppose by way of contradiction that we have a partition $a+b+c+d = n$, where $a \leq b \leq c \leq d$, such that

$$f + r_{a+b} + r_{a+c} + r_{a+d} < r_a + r_b + r_c + r_d$$

where we put $f = f(a,b,c,d)$.

Now if $f = -1$, then $a = b = c = 1$, and $d = n-3$. The resulting inequality is

$$3r_2 - 1 \geq r_3$$

which is clearly satisfied, a contradiction. Thus $f \geq 0$.

Suppose $f = 0$, in which case $a = b = 1$. In particular for the resulting inequality, r_2 appears on the left hand side and only two terms non-zero terms appear on the right hand side. Thus the inequality is trivially satisfied unless r_2 also appears on the right hand side. But then $c = 2$ and $d = n-4$. The inequality becomes,

$$r_2 + 2r_3 \geq r_2 + r_4$$

which is also clearly satisfied, a contradiction. Thus $f \geq 1$.

Suppose $f = 1$. In which case $a = 1$, and $b > 1$. If all three terms r_{a+b}, r_{a+c} and r_{a+d} on the left hand side vanish, then $b = c = d = k - 2$, and $n = 3k - 5$. Thus $n = 10$ or 13, a contradiction. As the right hand side has only three non-zero terms, it follows that at least one of them must be r_2. Thus $b = 2$. Suppose the remaining two terms on the left hand side vanish. Then $c = d = k - 2$. Thus $n = 1 + 2 + 2k - 4 = 2k - 1$, a contradiction. It follows that at least two terms on the right hand side must be r_2. Thus $b = c = 2$, and $d = n - 5$. The inequality becomes

$$1 + 2r_3 + r_4 \geq 2r_2 + r_5$$

which again is satisfied, a contradiction. Thus $f = 2$.

It follows that at least one term on the right hand side is r_2, and so $a = 2$. If all three terms on the left hand side vanish, then $b = c = d = k - 3$, and $n = 3k - 7$. It follows that $n = 14$ or 17. Thus at least two terms on the right hand side must equal r_2. Thus $a = b = 2$. If both of the remaining terms on the left hand side vanish, then $c = d = k - 3$ and $n = 2k - 2$, a contradiction. Hence three terms on the right hand side vanish, and so $a = b = c = 2$, $d = n - 6$. The inequality reduces to

$$2 + 3r_4 \geq 3r_2 + r_6$$

which again is satisfied (note that $r_4 \neq 0$ as $n > 11$), a contradiction. $\square$

Observe that (1.3) follows from (5.1), (5.3) and (5.4).

Δ_E being a pure boundary is equivalent to saying there is a solution to $K_{\bar{M}_{0,n}} < \lambda E < K_{\bar{M}_{0,n}} + D$ for some $\lambda > 0$, where by $>$ we mean the ordering given by the coefficients with basis B_i. In order to apply the basepoint free theorem, much less is actually needed, it is enough that there be a nef class N such that $K_{\bar{M}_{0,n}} + N < \lambda E < K_{\bar{M}_{0,n}} + D + N$ has a solution. However, we do not know how to use this extra flexibility.

[1] V. Aleexev. Moduli space $M_{g,n}$ for surfaces. pages 1–22. Walter de Guyter, 1996. Proceedings of the international conference held in Trento, Italy, June 15-24, 1994.

[2] J. Harris and D. Mumford. On the Kodaira Dimension of the Moduli Space of Curves. *Invent. Math.*, 67:23–88, 1982.

[3] M. Kapranov. Chow Quotients of Grassmannians. I. In S. Gelfand and S. Gindikin, editors, *I. M. Gelfand Seminar*, volume 16, pages 29–110. AMS, 1993. Part 2.

[4] M. Kapranov. Veronese curves and Grothendieck-Knudson moduli space $\overline{M}_{0,n}$. *J. Algebraic Geometry*, 2:239–262, 1993.

[5] Y. Kawamata. Addition formula of logarithmic Kodaira dimensions for morphisms of relative dimension one. In *Algebraic Geometry Kyoto 1977*, pages 207–217. Kinokuniya, 1978.

[6] Y. Kawamata. Subadjunction of log canonical divisors for a a subvariety of codimension 2. Preprint, 1995.

[7] Y. Kawamata, K. Matsuda, and K. Matsuki. Introduction to the minimal model program. In T. Oda, editor, *Algebraic Geometry, Sendai*, pages 283–360. Kinokuniya-North-Holland, 1987. Adv. Stud. Pure Math, vol 10.

[8] S. Keel. Intersection theory of moduli space of stable N-pointed curves of genus zero. *Transactions of the AMS*, 330:545–574, 1992.

[9] J. Kollár. *Rational curves on projective varieties*, volume 32. Springer Ergebnisse der Mathematik, 1996.

[10] M. Kontsevich and Y. Manin. Gromov-witten classes, quantum chomology, enumerative geometry. *Comm. Math. Phys.*, 164:525–562, 1994.

[11] M. Kontsevich and Y. Manin. Qunatum cohomology of a product. *Invent. Math.*, 124:313–339, 1996.

[12] R. Pandharipande. The canonical class of $\overline{M}_{0,n}(\mathbb{P}^d)$ and enumerative geometry. Preprint, 1995.

[13] V. Shokurov. 3-fold log models. Preprint, 1996.

Department of Mathematics, University of Texas, at Austin, Austin TX 78712 Department of Mathematics, University of California at Santa Barbara Santa Barbara, CA 93101