LOG ABUNDANCE THEOREM FOR THREEFOLDS.

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§1 Introduction

An important step in the classification of algebraic varieties is to find a good model X in a birational equivalence class. In Mori's minimal model program, we take X to be either a minimal model, or a Mori fibre space.

Now the most important property of a minimal model is that K_X, the class of the canonical divisor, is nef. This means that $K_X \cdot C$ is non-negative, for any curve C in X. (On the other hand, a Mori fibre space is always covered by rational curves.)

However this property is purely numerical. It has been conjectured (the abundance conjecture, 6-1-14 of [10]) that much more is true, that some multiple of K_X defines a base point free linear system.

Moreover an important refinement of this conjecture goes further to the category of log divisors. (Log divisors appear naturally in the classification of open varieties and in many inductive proofs. For an introduction to the minimal model program, see [4], page 28, and for the log minimal model program see the introduction to [10].) The aim of this paper is to prove this conjecture in dimension three:

1.1 Theorem (Log Abundance). *Let the pair (X, Δ) consist of a threefold X and boundary Δ, such that $K_X + \Delta$ is nef, and log canonical.*

Then $|m(K_X + \Delta)|$ is basepoint free for some m.

With the aid of (1.1), we can establish the following birational classification of pairs (X, Δ), where X is a threefold, and $K_X + \Delta$ is log terminal:

The pair (X, Δ) is birational to (X', Δ') (i.e there is a birational map, such that Δ' is the strict transform of Δ) a pair with log terminal singularities and, either

(1) $|m(K_{X'} + \Delta')|$ is base point free for some m, or
(2) there is a morphism $\pi'\colon X' \longrightarrow Y'$, which is a log Mori fibre space.

Indeed we may apply the log minimal model program to (X, Δ). Eventually either $K_X + \Delta$ is nef, in which case (1.1) applies, or (2) holds. A similar result is true if the pair (X, Δ) is log canonical.

Here are some other immediate Corollaries:

Corollary (cf. [2], [6]). *Let (X, Δ) be a log canonical threefold.*

Then the log canonical ring

$$\bigoplus_{m=0}^{\infty} H^0(X, \mathcal{O}_X(\llcorner m(K_X + \Delta)\lrcorner)$$

is finitely generated as a $\mathbb{C}$-algebra.

Corollary. *Let X be a minimal threefold, whose canonical divisor is numerically trivial. (For example a Calabi-Yau manifold, cf. [18].)*

Then any effective nef divisor is semi-ample.

We will prove (1.1) only assuming X is proper, see (7.1). From now on, to simplify the statements of results, we shall however always assume X is projective.

Note that (1.1) generalises the Abundance Theorem of Kawamata ([8]) and Miyaoka ([14], [16], and [15]), and is also considered to be a first step towards a proof of the Abundance Conjecture in dimension four. (See [12] for a good exposition of Abundance in dimension three.)

The Abundance Theorem for threefolds states that the canonical divisor of a minimal model, an endproduct of the minimal model program (or Mori's program), is semi-ample. Recently Shokurov ([21], see also [12]) extended the MMP to the log category. (1.1) is then a natural and important generalisation of the Abundance Theorem.

The log category is a natural setting for inductive proofs. This has been our guiding philosophy, and several Theorems are stated in every dimension, see (5.6), (6.1) and (6.2), and (7.3).

We now give a sketch of the proof of (1.1). First we assume $K_X + \Delta$ is kawamata log terminal.

§§2–4 are devoted to the first step of the proof of (1.1):

1.2 Theorem. *Let the pair (X, Δ) consist of a threefold and boundary Δ such that $K_X + \Delta$ is nef and kawamata log terminal.*

Then $\kappa(K_X + \Delta) \geq 0$.

To prove (1.2), first we run a special minimal model program (see §2), to reduce to the case $\pi \colon X \longrightarrow Y$ is a Mori fibre space and $K_X + \Delta$ is the pull back of $K_Y + B$, for some B. Our strategy is to try to choose B, so that (Y, B) is kawamata log terminal, and thus apply induction. In §3, we prove this for a conic bundle, see (3.8). Roughly speaking, part of B is fixed (any components of Δ which don't surject onto Y, for example) and part we are free to choose in a linear system, without base components, see (3.4). In practice, we need to improve the morphism π using a base change and several applications of the MMP, without changing the linear equivalence class of $K_S + B$. But then we have to deal with non-effective divisors defined only up to linear equivalence, for which we introduce the convenient formalism of compound divisors in (1.4).

(3.10) and (6.3) complete the proof of (1.2), by reducing the del Pezzo case, to a straightforward analysis via classification (§4).

In §5, we apply the minimal model program to improve the pair (X, Δ). Two cases then arise. In one, X is a Mori fibre space (5.4), and we may apply induction (using §§3 and 4).

The defining characteristic of the other case is that X is not covered by rational curves C, that are $K_X + \Delta$ trivial (and this is precisely what is required to apply (6.1)). Indeed this dichotomy is repeated twice more (see (1) and (2) of (2.4), (5.3) and (7.1)). Here (see (5.7)), we try to apply the arguments of Kawamata [8] and Miyaoka [15]. Moreover, using (1.2), we are able to make the technically important reduction to the case where $S = \Delta$ is reduced, and in fact when $\nu = 1$ the result follows immediately. When $\nu = 2$, we need a significant improvement of Miyaoka's semi-positivity result (see §6).

Finally, §7 passes from kawamata log terminal to log canonical.

It should also be pointed out, that (6.1) and it Corollaries ((6.2) and (6.3)) are interesting, independently of the proof of (1.1).

Acknowledgments: We would like to especially thank Lung-Ying Fong and János Kollár for many useful conversations and encouragement. We would like also to thank Lung-Ying Fong, for a previous version of the $\nu = 2$ case, and Professor Kawamata for pointing out that (3.8) should hold. Thanks should also go to Alessio Corti, Bob Gompf, Tie Luo, Shigefumi Mori, Miles Reid and David Saltman for some helpful conversations. The second author was supported in part by a Grant under the Monbusho International Scientific Research Program: 04044081.

1.2 Notation. Let $f\colon X \longrightarrow Y$ be a morphism of normal varieties. If K_Y is $\mathbb{Q}$-Cartier then $K_{X/Y}$ is defined to be $K_X - f^*K_Y$.

We say f is a Mori fibre space, if f has relative Picard number one, $-K_X$ is f-ample, and X has $\mathbb{Q}$-factorial terminal singularities.

In what follows, (X, Δ) will denote a variety with a boundary (resp. subboundary) Δ, i.e. Δ is a $\mathbb{Q}$-Weil divisor and if we write Δ as the sum of its irreducible components $\sum d_i \Delta_i$, then $0 \leq d_i \leq 1$ (resp. $d_i \leq 1$). By convention, we will often write Δ as $S + B$, where S is the part of Δ with coefficient one and B is the rest. We will say Δ is a pure boundary (resp. subboundary), if S is empty. Let $\pi\colon X' \longrightarrow X$ be a generically finite morphism of normal varieties. If π is finite, or $K_X + \Delta$ is $\mathbb{Q}$-Cartier we define the **log pullback** Δ' by the formula

$$K_{X'} + \Delta' = \pi^*(K_X + \Delta).$$

Note that even if $K_X + \Delta$ is log terminal, usually Δ' will only be a subboundary. However $K_X + \Delta$ is semi-ample iff $K_{X'} + \Delta'$ is.

We will also use the following standard definitions:

$\kappa(X, D)$ denotes the Iitaka dimension of the variety X with divisor D. In particular, $\kappa(D) \geq 0$ iff $|mD|$ is non-empty for some positive integer m.

In the case where D is nef, we can define the numerical counterpart

$\nu(X, D) = \max\{\, n \in \mathbb{N} \cup 0 \bigm| (D^n) \text{ not numerically } 0 \,\}$.

(For simplicity, we will often drop the reference to the variety). D_{red} is the boundary, where each component of D is taken with multiplicity one.

D is said to be semi-ample, if the linear system $|mD|$ is base point free, for some positive integer m. D is said to be abundant if $\kappa(D) = \nu(D)$.

A sheaf $\mathcal{F}$ is said to be reflexive, if $\mathcal{F} = \mathcal{F}^{**}$.

We will use the definitions of log terminal and canonical in (2.1.3) of [12]. (We remind the readers that the definition of kawamata log terminal is the same as log terminal in [10].) All these definitions make sense if Δ is only a subboundary, and here we will attach the prefix sub.

1.3 Results.

We collect some standard results, which will be required for the proof of (1.1).

Definition-Lemma: (Log) Terminal Model. *Let the pair (X, Δ) consist of a normal variety of dimension at most three, and a pure boundary (resp. boundary) such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier.*

Then we may find a $\mathbb{Q}$-factorial terminal (resp. log terminal) pair (X', Δ'), a birational morphism $g\colon X' \longrightarrow X$ and a divisor D such that $K_{X'} + \Delta' + D = g^*(K_X + \Delta)$, where D is effective and g-exceptional, and the image of D lies in the locus where $K_X + \Delta$ is not kawamata log terminal (resp. log canonical). Furthermore, g is an isomorphism over any open set where X is $\mathbb{Q}$-factorial and $K_X + \Delta$ is terminal (resp. log terminal).

Proof. The existence of log terminal models is well known (see (17.10) of [12] and (9.1) of [21]).

We construct the terminal model in two stages. Suppose Δ is a pure boundary, and let (X', Δ') be the log terminal model. Since any component of Δ' that has coefficient one is exceptional, shifting components of Δ' with coefficient one into D, we may assume (X', Δ') is kawamata log terminal and apply (6.9.4) of [12]. $\square$

Remark. Note that when the pair (X, Δ) is kawamata log terminal (resp. log canonical), then D is empty, and Δ' is the log pullback.

We will use the non-trivial fact that Abundance holds for semi log canonical surfaces (see Chapters 11 and 12 of [12]). In particular, if $K_X + S + B$ is log canonical and nef, and S is reduced then some multiple of $(K_X + S + B)|_S$ is base point free.

We now recall some of the standard reduction steps already proved in the literature. Suppose $K_X + \Delta$ is kawamata log terminal and nef. Then $K_X + \Delta$ is semi-ample if either

(1) $K_X + \Delta$ is abundant (see (6.1) of [5]), or
(2) $K_X + \Delta$ is big (see 3-1-1 of [10]).

1.4 Preliminaries.

Here we put some non-standard results.

In §3, we will work with pairs $(Y, \mathfrak{B})$, where Y is a normal variety, and $\mathfrak{B}$ is a mixed object: it is the data of a $\mathbb{Q}$-Weil divisor D and the linear equivalence class of a $\mathbb{Q}$-Cartier divisor L. We will call $\mathfrak{B}$ a **compound divisor**. $|\mathfrak{B}|$ denotes the set of $\mathbb{Q}$-Weil divisors B of the form $C + D$, where kC belongs to the linear system $|kL|$, for some positive integer k. We will say two compound divisors $\mathfrak{B}$ and $\mathfrak{B}'$ are equivalent, if $L + D$ and $L' + D'$ are $\mathbb{Q}$-linearly equivalent. We introduce a partial order on equivalent compound divisors, by the rule $\mathfrak{B} \geq \mathfrak{B}'$ if $D \geq D'$ as Weil divisors.

If $K_Y + D$ is $\mathbb{Q}$-Cartier, we define the discrepancies of $K_Y + \mathfrak{B}$, by taking the supremum of the discrepancies of $K_X + B$ over all possible choices of $B \in |\mathfrak{B}|$. This is equivalent to the limit of the discrepancies of $K_Y + B$, for generic choice of kC in $|kL|$, as k goes to infinity. We extend the definitions of log terminal in the obvious way, and we will say $(Y, \mathfrak{B})$ is a pure subboundary, if some $B \in |\mathfrak{B}|$ is a subboundary, with no reduced part. Here B is a pure subboundary for generic choice of kC in $|kL|$ and any sufficiently divisible k.

If $f \colon Y' \longrightarrow Y$ is a generically finite morphism, and either $K_Y + D$ is $\mathbb{Q}$-Cartier or f is finite, we define the log pullback $(Y', \mathfrak{B}')$ of $(Y, \mathfrak{B})$ by the formulae

$$K_{Y'} + D' = f^*(K_Y + D) \qquad \text{and} \qquad L' = f^*L.$$

In words, D' is the log pullback of D, and L' is the ordinary pullback of L.

1.3 Lemma. *Let $\mathfrak{B}$ and $\mathfrak{B}'$ be equivalent compound divisors, where $K_Y + D$ is $\mathbb{Q}$-Cartier and $(Y, \mathfrak{B}')$ is subkawamata log terminal (resp. $\mathfrak{B}'$ is a pure subboundary). Suppose L is pulled back along a birational map g, and $D' - D = A + E$ where A and E are respectively effective and g-exceptional.*

Then the pair $(Y, \mathfrak{B})$ is subkawamata log terminal (resp. $\mathfrak{B}$ is a pure subboundary).

Proof. Clear, since $|\mathfrak{B}'| \subset |\mathfrak{B}|$. $\square$

Note that (1.3) applies when $\mathfrak{B}' \geq \mathfrak{B}$ (just take f to be the identity). However $\mathfrak{B}$ and $\mathfrak{B}'$ may be equivalent and $\mathfrak{B}$ is a subboundary while $\mathfrak{B}'$ is not . For example any divisor is the difference of two ample divisors.

1.4 Lemma. *Suppose $K_Y + D$ is $\mathbb{Q}$-Cartier.*

Then the pair $(Y, \mathfrak{B})$ is subkawamata log terminal (resp. $\mathfrak{B}$ is a pure subboundary), iff either

(1) *the pair (Y, B) is subkawamata log terminal (resp. $\mathfrak{B}$ is a pure subboundary) for some $B \in |\mathfrak{B}|$, or*

(2) *for every birational morphism $f\colon Y' \longrightarrow Y$, the log pullback $(Y', \mathfrak{B}')$ is a pure subboundary.*

Proof. In both cases, one direction is clear.

Otherwise suppose the pair $(Y, \mathfrak{B})$ is subkawamata log terminal, or the log pullback under a birational morphism is always a subboundary. Pick a resolution $f\colon Y' \longrightarrow Y$ such that the union of the support of D, the stable base locus of L and the exceptional locus is a normal crossing divisor. Then, by Bertini, f is a log resolution of the support of B for generic choice of kC in $|kL|$ and k sufficiently divisible, and by assumption all the discrepancies of the exceptional divisors associated to f are strictly less than one. $\square$

1.5 Lemma. *Let $f : Y' \longrightarrow Y$ be a generically finite projective morphism between normal varieties. Suppose $K_Y + D$ is $\mathbb{Q}$-Cartier and the pair $(Y', \mathfrak{B}')$ is the log pullback of the pair $(Y, \mathfrak{B})$.*

Then the pair $(Y, \mathfrak{B})$ is subkawamata log terminal iff the pair $(Y', \mathfrak{B}')$ is. In particular, if $\mathfrak{B}'$ is a pure subboundary, then so is $\mathfrak{B}$ and if f is finite, then $\mathfrak{B}'$ is a pure subboundary iff $\mathfrak{B}$ is.

Proof. Using the Stein factorisation, we may assume f is either birational or finite. If f is birational, the result follows by definition, so we may assume f is finite.

Now if $(Y, \mathfrak{B})$ is subkawamata log terminal, then so is (Y, B) for $B \in |\mathfrak{B}|$. But then (20.3) of [12] implies that (Y', B') is subkawamata log terminal, which implies that $(Y', \mathfrak{B}')$ is subkawamata log terminal.

Now suppose the pair $(Y', \mathfrak{B}')$ is subkawamata log terminal. Passing to a Galois closure (and using the directions we have already proved) we can assume that $f\colon Y' \longrightarrow Y$ is the quotient by a finite group G. By (1) of (1.4), $K_{Y'} + C' + D'$ is subkawamata log terminal, for some choice of C'. But if we set $C'' = (1/|G|)\sum_{g \in G} g^*(C')$, then $K_{Y'} + C'' + D'$ is also subkawamata log terminal and $C'' = f^*C$ for some C on Y.

Now we may again apply (20.3) of [12], and (1) of (1.4). $\square$

1.6 Definition–Lemma. *Let $\pi\colon X \longrightarrow Y$ and $f\colon Y' \longrightarrow Y$ be morphisms of smooth varieties. Suppose f is finite and X has dimension at most three. Let M be a boundary in X, such that every component of M dominates Y, and $K_X + M$ is terminal on the general fibre of π.*

Then we may find a commutative square

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y' & \xrightarrow{f} & Y, \end{array}$$

where X' is $\mathbb{Q}$-factorial, birational to the fibre product, the general fibres of π' and π are the same, and $K_{X'} + M'$ is terminal, where M' is the strict transform of M.

Moreover if we write $K_{X'/Y'} + M' + D = g^*(K_{X/Y} + M)$, then D is effective and no component of D dominates Y'.

Proof. We will construct X' in stages. First let W be the fibre product and Z its normalisation. The graph of π gives a fibre diagram

$$\begin{array}{ccc} W & \longrightarrow & X \times Y' \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\Delta} & Y \times Y. \end{array}$$

Thus W is a local complete intersection in $X \times Y'$, whose normal bundle is the pullback of T_Y. It follows that $K_{W/Y}$ is the pullback of $K_{X/Y}$. If q is the normalisation map, then by (2.3) of [20], we have an equality of Weil divisors on Z: $K_Z + C = q^*(K_W)$, where C is effective and supported on the conductor. Now choose a resolution $X' \longrightarrow Z$ and run the $K_{X'}$-MMP over Z until $K_{X'}$ is relatively nef. It follows that $X' \longrightarrow Z$ is an isomorphism over any open set where Z is $\mathbb{Q}$-factorial and terminal, and in particular along the general fibre of $Z \longrightarrow Y'$. Let $\tilde{C}$ be the strict transform, and h the map to the fibre product. Then, we may write $K_{X'} + \tilde{C} + E = h^*(K_W)$, where E is exceptional. Since $K_{X'}$ is relatively nef, it follows from (2.19) of [12] that E is effective. At this stage, X' is $\mathbb{Q}$-factorial, D is effective and no component dominates Y', and π' and π have the same general fibre. Now pass to a terminal model of the pair (X', M'). □

1.7 Lemma. *Let $f\colon X \longrightarrow Y$ be a projective morphism with connected fibres between normal quasi-projective varieties. Let F be a $\mathbb{Q}$-Cartier numerically f trivial divisor on X, where no component of F dominates Y. Suppose Y is $\mathbb{Q}$-factorial.*

Then F is the pullback of a unique $\mathbb{Q}$-divisor on Y.

Proof. Let A be the image of all the components of F which dominate divisors in Y. Let B be any component of A, and F_1 any divisor which dominates B. If we choose a number r, so that

$F - rf^*B$ does not contain F_1 in its support, then the support of $F - rf^*B$ does not dominate B. Indeed, if we cut Y and X by hyperplanes, we reduce to the well known case of X a surface and Y a curve.

Thus we may assume A is empty, and we want to show F is empty. We drop the hypotheses that f has connected fibres, and proceed by induction on the relative dimension d of f. If $d = 0$, by taking the Stein factorisation, we may assume f is birational and the result follows from (2.10) of [12]. Otherwise, replace X by H a general hyperplane section, F by $F \cap H$, and apply induction. $\square$

§2 A simple variation of the minimal model program

We will need a slight refinement of the Rationality Theorem (see 4-1-1 of [10]).

2.1 Lemma. *Suppose the pair (X, Δ) has $\mathbb{Q}$-factorial kawamata log terminal singularities, D is a nef $\mathbb{Q}$-Cartier divisor, but $K_X + \Delta$ is not nef. Set*

$$\lambda = \sup \{ \mu \mid D + \mu(K_X + \Delta) \text{ is nef} \}.$$

Then λ is rational and moreover there is a $(K_X + \Delta)$-extremal ray R, such that

$$(D + \lambda(K_X + \Delta)) \cdot R = 0.$$

Proof. Let r be a natural number, such that both $r(K_X + \Delta)$ and rD are Cartier divisors. Now by Theorem 1 of [7], for each $(K_X + \Delta)$-extremal ray R_i, there is a rational curve C_i (which generates R_i), such that $-(K_X + \Delta) \cdot C_i \leq 2n$, where n is the dimension of X. In particular

$$\frac{D \cdot C_i}{-(K_X + \Delta) \cdot C_i}$$

is an integer multiple of $1/(2rn)$, for every i. Let

$$\mu = \inf_i \frac{D \cdot C_i}{-(K_X + \Delta) \cdot C_i}.$$

Then certainly there exists an extremal ray R such that $(D + \mu(K_X + \Delta)) \cdot R = 0$, $\mu \in \mathbb{Q}$ and $D + \lambda(K_X + \Delta)$ is not nef for any value of λ greater than μ. But $D + \mu(K_X + \Delta)$ is nef, by the Cone Theorem (4-2-1 of [10]), since it is positive on each R_i. Thus $\lambda = \mu$. $\square$

2.2 Lemma. *Suppose the pair (X, Δ) has $\mathbb{Q}$-factorial kawamata log terminal singularities, and W is an effective $\mathbb{Q}$-divisor such that $K_X + \Delta$ is not nef, but $K_X + \Delta + W$ is nef.*

Then there is a $(K_X + \Delta)$-extremal ray R and rational number $0 < \lambda \leq 1$ such that $K_X + \Delta + \lambda W$ is nef but trivial on R.

Proof. Apply (2.1) to $D = K_X + \Delta + W$. $\square$

2.3 Remarks.

(1) In dimension at most three, we may use (2.2) to define a special minimal model program, which we will refer to as the $(K_X + \Delta)$-MMP with scaling of W.

(2) Suppose we have a morphism $p\colon X \longrightarrow Y$ (or $(K_X + \Delta + \lambda W)$-flop) and divisor D' such that $K_X + \Delta + \lambda W = p^*(D')$. Then $\kappa(D) \geq \kappa(K_X + \Delta + \lambda W) = \kappa(D')$. In particular if p is a divisorial extremal contraction or flop associated to the $(K_X + \Delta)$-MMP with scaling of W, then we may take $D' = K_Y + \Gamma = K_Y + \pi_*(\Delta) + \lambda W$.

2.4 Lemma. *Let the pair (X, Δ) consist of a variety X of dimension at most three, with boundary Δ. Suppose $K_X + \Delta$ is nef and kawamata log terminal.*

Then we may find a pair (X', Δ'), where X' has $\mathbb{Q}$-factorial terminal singularities, $K_{X'} + \Delta'$ is nef and kawamata log terminal, $\kappa(K_X + \Delta) \geq \kappa(K_{X'} + \Delta')$, and either

(1) *X' is a Mori fibre space over a $\mathbb{Q}$-factorial variety Y, and $K_{X'} + \Delta'$ is the pullback of a $\mathbb{Q}$-divisor from Y, or*

(2) *$K_{X'}$ is nef.*

Proof. By passing to a terminal model, we may assume X has $\mathbb{Q}$-factorial terminal singularities. Now run the K_X-MMP with scaling of Δ, at each stage using (2) of (2.3). $\square$

2.5 Lemma. *(1.2) holds in case (2) of (2.4), and in case (1) of (2.4), when Y is either a point, or $\mathbb{P}^1$, or a curve and $K_X + \Delta$ is not numerically trivial.*

Proof. If $K_{X'}$ is nef, then $|nK_X|$ is non-empty for some n, by [16]. The other cases are obvious. $\square$

Thus we may assume $\pi\colon X \longrightarrow Y$ is a Mori fibre space, $K_X + \Delta$ is pulled back from Y, and either π a conic bundle (3.8), or a del Pezzo fibration (4.4).

§3 Conic Bundles

The main results of this section are (3.8) and (3.10).

We fix some notation. Let $I_1, I_2, \dots, I_p$ be a partition of the set of integers between 1 and k. Let $a_1, a_2, \dots, a_k$ be a sequence of rational numbers, such that $0 \leq a_i \leq 1$, which sum to 2. Set

$$\alpha_q = \sum_{i \in I_q} a_i.$$

We are interested in solutions to the linear equations

$$a_i = \sum_j b_{ij} \tag{3.1}$$

where the b_{ij} are non-negative rational numbers, and symmetric in i and j.

We are now ready to state the main combinatorial result.

3.2 Lemma. *Suppose $\alpha_1, \alpha_2, \dots, \alpha_p$ are all at most one.*

Then (3.1) *has a solution, where $b_{ij} = 0$ if i and j belong to the same indexing set I_q.*

Proof. Note that the conditions on the numbers a_i define a convex set. Thus we only need to check the result on the extremal points. They are $a_i = a_j = 1$, all others zero, when the result is immediate. □

We introduce some more notation. Let (X, Δ) be a pair, such that Δ is a $\mathbb{Q}$-Weil divisor and let $\pi\colon X \longrightarrow Y$ be a morphism between normal projective varieties with general fibre $\mathbb{P}^1$. Suppose Y is $\mathbb{Q}$-factorial and $K_X + \Delta$ is pulled back from Y.

We will be interested in $\mathbb{Q}$-divisors B such that

$$(K_{X/Y} + \Delta) = \pi^* B. \tag{3.3}$$

We will decompose Δ as $F + M$, where no component of F dominates Y, and every component of M does, and we will assume that M is a boundary, with no reduced part. We will call a $\mathbb{Q}$-Weil (resp. effective) divisor G an **enzyme** (resp. **effective enzyme**), if no component of G dominates Y, and $K_{X/Y} + M + G$ is $\mathbb{Q}$-Cartier and pulled back from Y. An enzyme G defines a compound divisor $\mathfrak{B}$, by the formulae:

$$K_{X/Y} + M + G = \pi^* L_G \qquad \text{and} \qquad F - G = \pi^* D$$

where D is the Weil divisor given by (1.7). Note that any $B \in |\mathfrak{B}|$ is automatically a solution to (3.3). Note also that if π is a Mori fibre space, we may take G to be the trivial divisor, in which case we will call $\mathfrak{B}$ the standard compound divisor. Note that the standard compound divisor is always bigger than any compound divisor defined by an effective enzyme.

3.4 Definition-Lemma. *Suppose we have a commutative square*

$$\begin{array}{ccc} X' & \xrightarrow[g]{} & X \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y' & \xrightarrow[f]{} & Y, \end{array}$$

of $\mathbb{Q}$-factorial normal varieties. Let G be any enzyme, $\mathfrak{B}$ the corresponding divisor and $\mathfrak{B}'$ the log pullback of $\mathfrak{B}$.

If we define a divisor G' by the formula

$$K_{X'/Y'} + M' + G' = g^*(K_{X/Y} + M + G),$$

then G' is the enzyme that defines $\mathfrak{B}'$.

Proof. Clear. $\square$

3.5 Lemma. *Suppose every component of M maps birationally onto Y, and π is a Mori fibre space.*

Then there is a sufficiently divisible positive integer m, which depends only on the coefficients of M, such that the base locus of the linear system $|mL|$ ($L = L_0$) is supported on those points P, where M restricted to $\pi^{-1}(P)_{\text{red}}$ is not a boundary.

Proof. Suppose $M = \sum_i a_i M_i$, where M has irreducible components $M_1, M_2, \dots, M_k$.

Now, given any partition of the integers from 1 to k, pick solutions to (3.1) (whose existence is guaranteed by (3.2)), and a sufficiently divisible integer m, so that $(m/2)b_{ij}$ is an integer.

Given any solution to (3.1) we have

$$(K_{X/Y} + M) = (1/2)\sum_{ij} b_{ij}(K_{X/Y} + M_i + M_j).$$

But by adjunction

$$\pi_*(K_{X/Y} + M_i + M_j)|_{M_i} = \pi_*(M_j|_{M_i}).$$

Since the relative Picard number is one, $C = (1/2)\sum_{ij} b_{ij}\pi_*(M_i \cdot M_j)$ is then a solution to (3.3).

Fix a point P, where M restricted to $\pi^{-1}(P)_{\text{red}}$ is a boundary. Let $Q_1, Q_2, \dots, Q_p$ be the points of X, over P, through which any components of M pass. Define a partition of the integers from 1 to k as follows:

$$I_q = \{\, i \mid M_i \text{ passes through } Q_q \,\}.$$

Then for the solution to (3.1) already fixed for this partition, and the corresponding solution to (3.3), P does not belong to the support of mC. □

Perhaps a simple example will explain (3.2) and (3.5). Let X be the surface $\mathbb{P}^1 \times \mathbb{P}^1$, and π be a projection. Let $M = \sum_i (1/2)M_i$, consist of four sections of π. Let P be a point of $\mathbb{P}^1$, over which M_1 meets M_2 in Q_1 and M_3 meets M_4 in Q_2. Then $\alpha_1 = \alpha_2 = 1$, and one solution to (3.1), as in (3.2), is $b_{12} = b_{34} = 0$, every other $b_{ij} = 1/4$.

3.5.1 Remark. If $K_X + M$ is canonical then $\pi^{-1}(P)_{red}$ is a boundary for any codimension one point P of Y, and so (3.5) implies that $|mL|$ has no base components. In particular if $K_X + \Delta$ is also subkawamata log terminal, then the standard compound divisor is a pure subboundary.

3.6 Lemma. *Suppose the pair (X, Δ) is subkawamata log terminal, and Y is $\mathbb{Q}$-factorial. Let G be an effective enzyme and $\mathfrak{B}$ the corresponding compound divisor.*

Then $\mathfrak{B}$ is a pure subboundary.

Proof. Passing to a terminal model, we may assume $K_X + M$ is terminal. Taking iterated pullbacks of the form $M_i \longrightarrow Y$, for every component M_i of M, we may find a finite morphism of normal varieties $f\colon Y' \longrightarrow Y$ such that every component of the fibre product of M that dominates Y', maps birationally onto Y'. Since we are free to throw away points of Y, we may assume X, Y and Y' are all smooth. By (1.6), (3.4) and (1.5) and relabeling we may assume from the start that $K_X + M$ is terminal, and every component of M maps birationally onto Y. Now run the $(K_X + M)$-MMP and then the K_X-MMP trivial with respect to $K_X + M$, over Y. The steps of all these programs are automatically $K_X + \Delta$ trivial, since $K_X + \Delta$ is pulled back from Y. Finally we have a Mori fibre space $X \longrightarrow Y'$, where $Y' \longrightarrow Y$ is generically finite, and $K_X + M$ is canonical. Suppose the standard compound divisor is $\mathfrak{B}''$ and the log pullback of $\mathfrak{B}$ is $\mathfrak{B}'$. Now $\mathfrak{B}''$ is a pure subboundary (see (3.5.1)). But $\mathfrak{B}'$ is the compound divisor defined by the effective enzyme G. Thus $\mathfrak{B}'' \geq \mathfrak{B}'$ and (1.3) implies $\mathfrak{B}'$ is a pure subboundary.

Now apply (1.5). □

3.7. Theorem. *Suppose $K_X + \Delta$ is subkawamata log terminal and $\mathfrak{B}$ is the compound divisor defined by an effective enzyme G.*

Then $(Y, \mathfrak{B})$ is subkawamata log terminal.

Proof. By passing to a terminal model, we may assume $K_X + M$ is terminal. Let $f\colon Y' \longrightarrow Y$ be any birational morphism. By (1.4) it suffices to show the log pullback $\mathfrak{B}'$ is a pure subboundary. Blowing up further, we may assume Y' is smooth and that the exceptional locus of $W \longrightarrow X$, for some desingularisation W of $X \underset{Y}{\times} Y'$, lies over the exceptional locus of f. Let D be the strict transform of M, $g\colon X' \longrightarrow X$ a terminal model for $K_W + D$, and G' the enzyme defined in (3.4).

Since the exceptional locus of g lies over the exceptional locus of f, the negative part of G' lies over the exceptional locus of f. It follows that there is an f-exceptional divisor E such that $G'' = G' + \pi'^*(E)$ is an effective enzyme. Since G'' is effective, (3.6) implies the corresponding compound divisor is a pure subboundary.

Since D'' and D' differ by E, we may apply (1.3). $\square$

3.8 Corollary. *Suppose $K_X + \Delta$ is kawamata log terminal.*

Then (3.3) has a solution B, such that the pair (Y, B) is kawamata log terminal.

Proof. Apply (3.7) to the trivial effective enzyme. $\square$

3.9 Corollary. *Let $\pi\colon X \longrightarrow C$ be a projective morphism where X is a threefold and C is curve. Suppose S, the geometric generic fibre, admits a morphism to $\mathbb{P}^1$, and the restriction of $K_X + \Delta$ to S is kawamata log terminal and trivial.*

Then $\kappa(K_{X/C} + \Delta) \geq 0$.

Proof. We can assume that every component of Δ surjects onto C. Note that we are free to take any transform of Δ under a birational morphism, and prove the Corollary for the new Δ. Passing to a terminal model, we can assume $K_X + \Delta$ is terminal on the general fibre. So we may desingularise (this will not affect the general fibre) and assume X is smooth.

Since S is ruled, there is a finite base change $C' \longrightarrow C$, so that the generic fibre is ruled. Let (X', Δ') be the pair defined in (1.6) (with $M = \Delta$). Now resolve the rational map of X' to $C' \times \mathbb{P}^1$, and take Δ' to be the strict transform. Again passing to a terminal model (and renaming) we may assume (X, Δ) is terminal, and X maps to $C \times \mathbb{P}^1$.

Running the $(K_X + \Delta)$-MMP and the K_X-MMP over $C \times \mathbb{P}^1$, we may assume $K_X + \Delta$ is pulled back from a surface S, ruled over C. By (3.8), $K_X + \Delta$ is the pullback of $K_S + B$, which is kawamata log terminal, and nef on the general fibre of S over C.

Now run the $(K_S + B)$-MMP and the K_S-MMP over C. (3.8) then implies that $K_{S/C} + B$ is effective. $\square$

3.10 Corollary. *Let $\pi\colon X \longrightarrow Y$ be a Mori fibre space, where X is a threefold and suppose $K_X + \Delta$ is nef, kawamata log terminal, and pulled back from Y. Suppose the general fibre is not $\mathbb{P}^2$.*

Then $\kappa(K_X + \Delta) \geq 0$.

Proof. If Y is a surface, then use (3.8) and log abundance for surfaces.

Otherwise the geometric generic fibre S of π is a del Pezzo surface. Now if Y is rational, the result follows by (2.5). Thus we may assume Y is not rational, and so it is enough to consider $K_{X/Y} + \Delta$. Since S is not $\mathbb{P}^2$, S admits a ruling. Now apply (3.9). $\square$

§4 del Pezzo Fibrations

Let $\pi\colon X \longrightarrow C$ be a Mori fibre space over a curve.

4.1 Lemma. *Suppose X is factorial and the geometric generic fibre of π is $\mathbb{P}^r$.*

Then π is a smooth $\mathbb{P}^r$-bundle.

Proof. Since the generic fibre of π is a Severi-Brauer variety and the function field of C is a C_1-field, there is birational map of X to $\mathbb{P}^r \times C$, which is an isomorphism on the general fibre. But then there is a line bundle L, such that $K_X + (r+1)L$ is pulled back from C. The result now follows by a standard application of vanishing and Fujita's Δ-genus, cf. [3]. $\square$

4.2 Lemma. *Suppose C is irrational, the pair (X, Δ) is kawamata log terminal, and $K_X + \Delta$ is numerically trivial.*

Then $-K_X$ is nef but not big and C is elliptic. Moreover if the general fibre is $\mathbb{P}^2$ then π is a smooth $\mathbb{P}^2$-bundle.

Proof. Since X has Picard number two, the closure of the cone of curves has two edges. Suppose the edge not corresponding to π is generated by α.

If $-K_X \cdot \alpha$ is not zero, then for some small rational ϵ, $K_X + (1+\epsilon)\Delta$ is kawamata log terminal, but dots α negatively. But then α is generated by a rational curve that dominates C, impossible. Thus $-K_X \cdot \alpha = 0$. Now if $K_C \cdot \alpha = -K_{X/C} \cdot \alpha$ is positive then $-K_{X/C}$ is big and nef and Kawamata-Viehweg vanishing (1-2-5 of [10]) implies $h^1(\omega_C) = h^1(\pi^*\omega_C) = 0$, impossible.

Hence C is elliptic and $-K_X$ is nef, but not big.

Suppose the general fibre of π is $\mathbb{P}^2$. By (6.3) X is Gorenstein ($\chi(\mathcal{O}_X) = \chi(\mathcal{O}_C) = 0$, by relative Kawamata-Viehweg vanishing) and so factorial. Now just apply (4.1). $\square$

4.3 Lemma. *Suppose X is the projectivisation of a vector bundle E of rank $r+1$, $K_X + \Delta$ is kawamata log terminal and $K_X + \Delta$ is numerically trivial.*

Then $\kappa(K_X + \Delta) = 0$.

Proof. By (4.2), C is elliptic, and $-K_X$ is nef. Thus $\lambda_E = -K_{X/C}$ is nef, in the notation of [14]. Thus (3.1) of [14] implies that E is semistable. Taking an étale cover of C, we may assume that E has trivial determinant. But then E is the direct sum of indecomposable vector bundles, each of which has a filtration by isomorphic line bundles, see [1].

Suppose $m\Delta$ is integral. This corresponds to an element of $H^0(C, \mathrm{Sym}^{(r+1)m}(E^*) \otimes M)$, where M is a line bundle of degree zero on C. Moreover we only need to show that M is torsion.

Suppose not. Choose M' so that $M'^{\otimes m(r+1)} = M$ and let $F = E^* \otimes M'$. With this choice of M', $\mathrm{Sym}^{(r+1)m}(E^*) \otimes M = \mathrm{Sym}^{(r+1)m}(F)$. F has a filtration by line bundles $L_0, L_1, L_2, \ldots, L_r$

of degree zero, induced by the direct sum decomposition of E. Let R be the subvector space of $V = \mathbb{Q}^{r+1}$ generated by all $(r+1)$-tuples $(a_0, a_1, a_2, \ldots, a_r)$ such that $L_0^{a_0} \otimes L_1^{a_1} \otimes \cdots \otimes L_r^{a_r}$ is trivial. By assumption R does not contain $(1, 1, \ldots, 1)$. Let W be the dual of V. Pick a non-zero element γ of W, with integer coefficients, which annihilates R, and is non-negative on $(1, 1, \ldots, 1)$. Set $\alpha = (a, a, \ldots, a) - \gamma$, where a is the largest entry of γ. Then every component of α is a non-negative integer, and at least one component of α is zero. We may assume this component is the first.

Choose coordinates $x_0, x_1, x_2, \ldots, x_r$ on the general fibre, which is a copy of $\mathbb{P}^r$, compatible with the direct sum decomposition of F. Then $m\Delta$ restricts to a hypersurface, given by a polynomial f of degree $(r+1)m$. If we write f as a polynomial in the $x_0, x_1, x_2, \ldots, x_r$, then each monomial of f corresponds to an element of R, since the co-ords x_i and the line bundles are compatible with the direct sum decomposition of E. Note that α has the value $am(r+1)$ on each of these monomials (so that in particular α is not zero).

Let $\mathbb{A}^r$ be the affine space where $x_0 \neq 0$, so with coordinates $x_1/x_0, \ldots, x_r/x_0$. Since the first component of α is zero, α gives a weight on monomials in the x_i/x_0. As in Chapter II, §4 of [19], we consider the toric blowup $B \longrightarrow \mathbb{A}^r$ defined by α. By the Proposition on page 373 of [19], the log discrepancy of the exceptional divisor with respect to $K_{\mathbb{P}^r} + \Delta|_{\mathbb{P}^r}$ is

$$\alpha(1, 1, \ldots, 1) - (1/m)\alpha(f(1, x_1/x_0, \ldots, x_r/x_0)) = -\gamma(1, 1 \ldots, 1) \leq 0,$$

a contradiction. □

4.4 Theorem. *Let $\pi\colon X \longrightarrow C$ be a del Pezzo fibration, which is a Mori fibre space. Suppose $K_X + \Delta$ is nef, kawamata log terminal, and pulled back from C.*

Then $\kappa(K_X + \Delta) \geq 0$.

Proof. By (3.10), we may assume π has general fibre $\mathbb{P}^2$. But then we may apply (2.5), (4.2) and (4.3). □

This completes the proof of (1.2).

§5 The minimal model program revisited.

In this section, we will prove (1.1) in the special case that $K_X + \Delta$ is kawamata log terminal (see (5.3), (5.4) and (5.8)).

We first state a simple, but very useful result (cf. §2).

5.1 Lemma. *Suppose the pair (X, Δ) has $\mathbb{Q}$-factorial kawamata log terminal singularities, and W is an effective $\mathbb{Q}$-divisor such that $K_X + \Delta + W$ is nef.*

Then either

(1) *there is a $(K_X+\Delta)$-extremal ray R, which is $K_X+\Delta+W$ trivial, or*
(2) *$K_X+\Delta+(1-\epsilon)W$ is nef, for any small positive rational ϵ.*

Proof. Simple consequence of (2.1). □

5.2 Remarks.

(1) In dimension three or less, (since we need the log MMP, see [21] and [12]), we may use (5.1) to define a special minimal model program, whose steps are $K_X+\Delta+W$ trivial.
(2) Suppose we have a morphism $p\colon X \longrightarrow Y$ (or $(K_X+\Delta+W)$-flop) and divisor D' such that $K_X+\Delta+W=p^*(D')$. Then $K_X+\Delta+W$ is semi-ample iff D' is. In particular if p is a divisorial extremal contraction or flop associated to the $(K_X+\Delta)$-MMP, which is $K_X+\Delta+W$ trivial, then we may take $D'=K_Y+\Gamma=K_Y+\pi_*(\Delta+W)$.
(3) Suppose X is terminal, and covered by rational curves C. Then $K_X\cdot C<0$, as is easily seen by passing to the total space of the family C.

5.3 Lemma. *Let the pair (X,Δ) consist of a threefold X, with boundary Δ. Suppose $K_X+\Delta$ is nef and kawamata log terminal.*

Then we may find a pair (X',Δ'), where X' has $\mathbb{Q}$-factorial terminal singularities, $K_{X'}+\Delta'$ is nef and kawamata log terminal, $K_X+\Delta$ is semi-ample iff $K_{X'}+\Delta'$ is semi-ample and either

(1) *X' is a Mori fibre space over a variety Y, and $K_{X'}+\Delta'$ is the pullback of a $\mathbb{Q}$-divisor from Y, or*
(2) *$K_{X'}+(1-\epsilon)\Delta'$ is nef, where ϵ is any small non-negative rational number. In particular X' is not covered by rational curves C, which are $K_{X'}+\Delta'$ trivial (see remark (3)).*

Proof. By passing to a terminal model, we may assume the pair (X,Δ) has $\mathbb{Q}$-factorial terminal singularities. Now run the K_X-MMP, using extremal rays R which are $K_X+\Delta$ trivial, as in (1) of (5.2), at each stage replacing (X,Δ) with (Y,Γ), as in (2) of (5.2). □

First let us deal with (1).

5.4 Lemma. *Let $\pi\colon X \longrightarrow Y$ be a Mori fibre space, where X is a threefold and suppose $K_X+\Delta$ is nef, kawamata log terminal, and pulled back from Y.*

Then $K_X+\Delta$ is semi-ample.

Proof. Suppose $K_X+\Delta=\pi^*(K_Y+B)$. We want to show that K_Y+B is semi-ample.

If $K_X+\Delta$ is numerically trivial, we may apply (1.2). If $\nu(K_X+\Delta)=1$ and Y is a curve then K_Y+B is ample, and so semi-ample.

Finally, if Y is a surface, then (3.8) implies that the pair (Y, B) is kawamata log terminal, and we may apply log abundance for surfaces. $\square$

(2) is a little more difficult. We start with a reduction step, that is valid in all dimensions. The following is a standard conjecture, that is implied by (1.1) in dimension three. In the proof of (1.1) (see (5.6)) we will only need (5.5) in dimension two, where it is already known.

5.5 Conjecture. Let (X, Δ) be a kawamata log terminal pair of dimension at most n.

Then the minimal model program holds for $K_X + \Delta$ and if $K_X + \Delta$ is nef but not numerically trivial, then $\kappa(K_X + \Delta) \geq 1$.

The following is a simple adaptation of (7.3) of [5] and maybe be proved accordingly (see also Chapter 15 of [12], which establishes a similar result for threefolds).

5.6 Lemma. *Suppose $K_X + \Delta$ is nef, $\kappa(K_X + \Delta) \geq k$ and (5.5) is true for $n - k$-dimensional varieties, where n is the dimension of X.*

Then $K_X + \Delta$ is semi-ample.

5.7 Lemma. *Suppose the pair (X, S) is log terminal, where S is a reduced divisor, X is a threefold, $K_X + S$ is nef but not numerically trivial, and there is a divisor $D \in |m(K_X + S)|$, whose support is exactly S. Suppose further that X is not covered by rational curves C, which are $K_X + S$ trivial.*

Then $\kappa(K_X + S) \geq 1$.

Proof. Let $\nu = \nu(K_X + S)$, and $\kappa = \kappa(K_X + S)$.

We are going to adapt the arguments appearing in Chapters 13 ($\nu = 1$, cf. [15] and [8]) and 14 ($\nu = 2$, cf. [8]) of [12].

If $\nu = 1$ then the argument is almost verbatim (see (13.3.1), (13.3.2) and the proof of (13.1.1) of [12]).

If $\nu = 2$, first we run the K_X-MMP, trivial with respect to $K_X + S$ (cf. (14.2) of [12]). Thus we can assume that $K_X + (1 - \epsilon)S$ is nef, for any small rational ϵ, since in the log Mori fibre case, X is covered by rational curves which are $K_X + S$ trivial. Since the polynomial in ϵ, $(K_X + (1 - \epsilon)S)^3$, vanishes for infinitely many ϵ, its coefficients, K_X^3, $K_X^2 \cdot S$, $K_X \cdot S^2$ and S^3, also vanish.

Clearly the following two conditions are equivalent:

(1) $K_X + (1 - \epsilon)S$ is nef, for small positive rational ϵ,
(2) For every curve C, such that$(K_X + S) \cdot C = 0$, then $K_X \cdot C \geq 0$.

But this is half of property (5) of (14.2) of [12], and the other half, with properties (1)–(4) can be checked as in the proof of (14.2).

However the argument of Chapter 14 of [12] needs to be altered in one important place. (14.3.1) and (14.3.2) go through, but the argument of (14.3.3) fails, since the crucial inequality

$$L' \cdot \hat{c}_2(\hat{\Omega}^1_{X'}(\log B')) \geq 0,$$

deduced from (10.13) does not apply in our case (X may be uniruled). Instead we apply (6.1), with $D_1 = D_2 = D_3 = L = m(K_X + S)$. $\square$

Now we are able to deal with case (2) of (5.3).

5.8 Lemma. *Suppose $K_X + (1-\epsilon)\Delta$ is nef and kawamata log terminal for small, non-negative rational ϵ, where X is a terminal threefold.*

Then $K_X + \Delta$ is semi-ample.

Proof. Let $\nu = \nu(K_X + \Delta)$, and $\kappa = \kappa(K_X + \Delta)$.

If $\nu = 3$, then $K_X + \Delta$ is big, and we may apply 3-1-1 of [10]. If $\nu = 0$, then we may apply (1.2).

Otherwise, using (5.6), we just need to show that $\kappa \geq 1$.

By (1.2), $|m(K_X + (1-\epsilon)\Delta)|$ is non-empty. Thus we may find $D \in |m(K_X + \Delta)|$, such that D contains the support of Δ. Now pick a log resolution $\pi \colon Y \longrightarrow X$ of the pair (X, D_{red}). Let E denote the union of the π-exceptional divisors, with coefficient one, and define a divisor Γ by the formula:

$$K_Y + \Gamma = \pi^*(K_X + \Delta) + E.$$

Since $K_X + \Delta$ is kawamata log terminal, it follows that the support of Γ contains E. Hence $|m(K_Y + \Gamma)|$ contains a divisor G, whose support S contains Γ_{red}. In particular Y is not covered by rational curves C, which are $K_Y + S$ trivial. Running the $(K_Y + S)$-MMP, we may assume $K_Y + S$ is nef, where $\nu(K_Y + S) = \nu$ and $\kappa(K_Y + S) = \kappa$ (cf. the proof of (13.2) of [12]).

Now apply (5.7) to deduce that $\kappa \geq 1$. $\square$

§6 Bogomolov stability and Mori's bending and breaking technique

We fix some notation for this section.

Let X be a normal projective variety of dimension n over an algebraically closed field of characteristic zero. Let $D_1, D_2, \dots, D_n$ be a sequence of nef Cartier divisors, $H_1, H_2, \dots, H_n$ a sequence of $\mathbb{Q}$-ample divisors, and H an ample divisor.

Following [14], we may define the slope, $\mu(\mathcal{F})$ and stability of a reflexive sheaf $\mathcal{F}$ with respect to $D_1, D_2, \dots, D_{n-1}$. See [14] for more details.

In the case where X has quotient singularities in codimension two (for example if X has log terminal singularities), let $\hat{c}_2$ be the second chern class of the sheaf $\hat{\Omega}^1_X$, which is a Q-sheaf in codimension two (see Chapter 10 of [12], for more details on Q-sheaves).

The aim of this section, is to prove the following:

6.1 Theorem. *Suppose $D_1 \cdot D_2 \cdots D_n = 0$, and $-K_X \cdot D_1 \cdot D_2 \cdots D_{n-1}$ is non-negative. Then either*

(1) *X is covered by a family of rational curves C, such that $D_n \cdot C = 0$, or*
(2) *TX is $(D_1, D_2, \ldots, D_{n-1})$-semi-stable, and $-K_X \cdot D_1 \cdot D_2 \cdots D_{n-1}$ is zero.*

Moreover, in case (2), if X has quotient singularities in codimension two and $D_1 \cdot D_2 \cdots D_{n-1}$ is not numerically trivial, then

$$\hat{c}_2 \cdot D_1 \cdot D_2 \cdots D_{n-2} \geq \frac{n-1}{2n} K_X^2 \cdot D_1 \cdot D_2 \cdots D_{n-2}.$$

Proof. For the last statement apply (6.5). We now show that either (1) or (2) holds. Suppose (2) does not hold. Then $c_1(\mathcal{F}) \cdot D_1 \cdot D_2 \cdots D_{n-1} > 0$, where $\mathcal{F}$ is the maximal destabilising subsheaf of TX.

Now for $H_1, H_2, \ldots, H_n$ close enough to $D_1, D_2, \ldots, D_n$, $c_1(\mathcal{F}) \cdot H_1 \cdot H_2 \cdots H_{n-1} > 0$. Thus (9.0.2) of [12] (see also [17]) implies the existence of a family of rational curves C such that

$$H_n \cdot C \leq \frac{2n H_1 \cdot H_2 \cdots H_n}{H_1 \cdot H_2 \cdots H_{n-1} \cdot c_1(\mathcal{F})}.$$

But, as $H_1, H_2, \ldots, H_n$ approach $D_1, D_2, \ldots, D_n$, the right hand side goes to zero and the left hand side approaches the non-negative integer $D_n \cdot C$. □

This result has some interesting Corollaries.

6.2 Corollary. *Suppose $-K_X$ is nef of numerical dimension m, X has canonical singularities Suppose further that $D_1, D_2, \ldots, D_{n-m-1}$ are nef divisors such that $-K_X^m \cdot D_1 \cdot D_2 \cdots D_{n-m-1}$ is not numerically trivial.*

Then $\hat{c}_2 \cdot (-K_X)^{m-1} \cdot D_1 \cdot D_2 \cdots D_{n-m-1}$ is non-negative.

Proof. Apply (6.1), (see (3) of (5.2)). □

6.3 Corollary. *Let X be a projective $\mathbb{Q}$-factorial canonical threefold, such that $-K_X$ is nef, but not big, and $\chi(\mathcal{O}_X) = 0$.*

Then X is Gorenstein.

Proof. First we prove that $c_1 c_2$ is non-negative. If K_X is numerically trivial there is nothing to prove. If K_X^2 is not numerically trivial, then apply (6.2). Otherwise apply (6.2), with D_1 any ample divisor. Thus $c_1 c_2$ is non-negative.

But recall that the Riemann-Roch formula for a threefold with canonical singularities implies

$$\chi(\mathcal{O}_X) = (1/24)c_1c_2 + x,$$

where x is zero iff X is Gorenstein (see (10.3) of [19]). $\square$

We need a little more notation. We may put a total order on $\mathbb{Q}[a]$, the rational polynomials in the variable a, by declaring $f \leq g$, if there exists a positive number ϵ, such that $f(a) \leq g(a)$, for all $0 \leq a \leq \epsilon$.

6.4 Lemma. *Let P be any subset of $\mathbb{Q}[a]$, such that the degree, the coefficients and the denominator of any coefficient are bounded from above.*

Then $(P, \leq)$ satisfies the ascending chain condition, and for every $g \in P$, there is a fixed positive number ϵ, such that $f \leq g$ implies $f(a) \leq g(a)$, for all $0 \leq a \leq \epsilon$, and any f in P.

Proof. Elementary and easy. $\square$

6.5 Lemma. *Let E be a sheaf of rank r, a any small positive number, and H_i the polarisation $D_i + aH$.*

Then the Harder-Narasimhan filtration of E, with respect to $(H_1, H_2, \dots, H_{n-1})$ is independent of a.

Furthermore, if E is $(D_1, D_2, \dots, D_{n-1})$-semi-stable and a Q-sheaf in codimension two, and $D_1 \cdot D_2 \cdots D_{n-1}$ is not numerically trivial then

$$\hat{c}_2(E) \cdot D_1 \cdot D_2 \cdots D_{n-2} \geq \frac{r-1}{2r} c_1^2(E) \cdot D_1 \cdot D_2 \cdots D_{n-2}.$$

Proof. Let $A_0 + A_1 a + \cdots + A_{n-1}a^{n-1}$, be the expansion of $H_1 \cdot H_2 \cdots H_{n-1}$ in powers of a. For any subsheaf $\mathcal{F}$ of E, the numbers $c_1(\mathcal{F}) \cdot A_i$ are universally bounded from above, as A_i is the sum of $(n-1)$-fold products of nef divisors. Thus (6.4) applied to the set of all possible slopes, $\mu(\mathcal{F})$, considered as polynomials in a, implies that the maximal destabilising subsheaf of E is independent of a, and so by induction on the rank of E, the same is true for the Harder-Narasimhan filtration.

Now suppose E is a Q-sheaf in codimension two. Let S be the general intersection of sufficiently high multiples of $H_1, H_2, \dots, H_{n-2}$. It is proved in [13] that the restriction to S, of the Harder-Narasimhan filtration of E w.r.t $(H_1, H_2, \dots, H_{n-1})$ is the Harder-Narasimhan filtration of E w.r.t H_{n-1}. Moreover the quotients of this filtration are Q-sheaves (see the proof of (10.12) of [12]).

Thus for any quotient of the filtration, by applying the Bogomolov inequality on S (see (10.11) of [12]), and then taking the limit as a goes to zero, we may deduce the Bogomolov inequality w.r.t $(D_1, D_2, \dots, D_{n-2})$.

Thus by induction on the length of the filtration, we may suppose we have a saturated subsheaf F of E of rank k, with the same slope as E, such that the Bogomolov inequality for F and E/F holds. Now if we set $D = (1/k)c_1(F) - (1/(r-k))c_1(E/F)$, then $D \cdot D_1 \cdot D_2 \cdots D_{n-1} = 0$, and so by applying the Hodge Index Theorem on S and taking the limit, $D^2 \cdot D_1 \cdot D_2 \cdots D_{n-2} \leq 0$. Using this, it is an easy calculation to deduce the Bogomolov inequality for E. □

6.6 Remark. The authors are extremely grateful to Qihong Xie for pointing out an egregious error in a previous version of this paper, where the hypothesis that $D_1 \cdot D_2 \cdots D_{n-1}$ is not numerically trivial was absent. The problem lies in the last but one line of (6.5). Restricting to S, we have a divisor D and a nef divisor G and we want to conclude that if $D \cdot G \leq 0$ then $D^2 \leq 0$. This is an almost immediate consequence of the HIT, if G is not numerically trivial, but it is surely a problematic implication in the case that G is trivial.

§7 Kawamata log terminal to log canonical

To finish the proof of (1.1), we need to deal with the case that the pair (X, Δ) is log canonical (cf. [9]). We start with a Lemma similar to (2.4) and (5.3).

7.1 Lemma. *Suppose the pair (X, Δ) is nef and log canonical, where X is a complete threefold.*

Then we may find a log canonical pair $(X', \Delta' = S' + B')$, where X' is projective and $\mathbb{Q}$-factorial where $K_{X'} + \Delta'$ is semi-ample iff $K_X + \Delta$ is and either

(1) *X' is a log Mori fibre space over a variety Y, where $K_{X'} + \Delta'$ is the pullback of a $\mathbb{Q}$-divisor from Y, and some component of S' dominates Y', or*
(2) *$K_{X'} + B' + (1-\epsilon)S'$ is nef and kawamata log terminal, where ϵ is a small positive rational number.*

Proof. Passing to a log terminal model, we may assume the pair (X, B) has $\mathbb{Q}$-factorial kawamata log terminal singularities. Now run the $(K_X + B)$-MMP, with extremal rays that are $K_X + \Delta$ trivial. □

We first deal with case (1).

7.2 Lemma. *Suppose the pair (X, Δ) has $\mathbb{Q}$-factorial log canonical singularities, where X is a threefold and $\pi\colon X \longrightarrow Y$ is a Mori fibre space over Y, such that $K_X + \Delta$ is the pullback of a $\mathbb{Q}$-Cartier divisor D and some component S' of S dominates Y.*

Then $K_X + \Delta$ is semi-ample.

Proof. We want to show D is semi-ample. But this is clear, since $(K_X + \Delta)|_{S'}$ is semi-ample, by abundance for semi log canonical surfaces. □

To deal with (2) of (7.1), we need a simple restatement of a vanishing Theorem due to Kawamata ((3.2) of [5]), which in turn is a refinement of a Theorem due to Kollár ((2.2) of [11]). (7.3) may be derived from (3.2) of [5], in the same fashion as 1-2-5 is derived from 1-2-3 of [10].

7.3 Lemma. *Suppose the pair (X, Δ) is kawamata log terminal, and L is an integral $\mathbb{Q}$-Cartier divisor, such that $L-(K_X+\Delta)$ is semi-ample. Suppose D and D' are effective integral $\mathbb{Q}$-Cartier divisors, such that $D + D' \in |m(L - (K_X + \Delta)|$, for some m.*

Then the homomorphisms induced by multiplication by D,

$$\phi^i_D : H^i(X, \mathcal{O}_X(L)) \longrightarrow H^i(X, \mathcal{O}_X(L + D))$$

are all injective.

The following is case (2) of (7.1):

7.4 Corollary. *Suppose $(X, S + B)$ is a log canonical threefold, where S is reduced and $K_X + (1 - \epsilon)S + B$ is nef and kawamata log terminal, where ϵ is any small positive rational number.*

Then $K_X + S + B$ is semi-ample.

Proof. Pick a sufficiently positive integer k, so that $K_X+(1-1/k)S+B$ is nef and kawamata log terminal. let m be a sufficiently positive integer. Set $W = m(K_X + S + B)$. If m is sufficiently divisible then $W|_S$ is base point free by log abundance for surfaces and it is proved in §5 that $m(K_X + S + B) - (m/k)S$ is base point free. Thus the base locus of $W - S$ is supported on S. Looking at the restriction exact sequence

$$0 \longrightarrow \mathcal{O}_X(W - S) \longrightarrow \mathcal{O}_X(W) \longrightarrow \mathcal{O}_S(W) \longrightarrow 0.$$

it is enough to show that we can lift any section of $H^0(S, \mathcal{O}_S(W))$. But the map

$$\phi^1_S : H^1(X, \mathcal{O}_X(W - S)) \longrightarrow H^1(X, \mathcal{O}_X(W))$$

is injective, by (7.3) applied to $L = W - S$. □

This completes the proof of (1.1).

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