\textbf{\textbeta{}-FAMILY CONGRUENCES AND THE $f$-INVARINAT}

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\textsc{Abstract.} In previous work, the authors have each introduced methods for studying the 2-line of the $p$-local Adams-Novikov spectral sequence in terms of the arithmetic of modular forms. We give the precise relationship between the congruences of modular forms introduced by the first author with the $\mathbb{Q}$-spectrum and the $f$-invariant of the second author. This relationship enables us to refine the target group of the $f$-invariant in a way which makes it more manageable for computations.

1. Introduction

In [Ada66], J.F. Adams studied the image of the $J$-homomorphism

$$J : \pi_t(SO) \to \pi_t^S$$

by introducing a pair of invariants

$$d = d_t : \pi_t^S \to \pi_t K,$$

$$e = e_t : \ker(d_t) \to \Ext^{1,t+1}_A(K_*, K_*)$$

where $A$ is a certain abelian category of graded abelian groups with Adams operations. (Adams also studied analogs of $d$ and $e$ using real $K$-theory, to more fully detect 2-primary phenomena.) In order to facilitate the study of the $e$-invariant, Adams used the Chern character to provide a monomorphism

$$\theta_S : \Ext^{1,t+1}_A(K_*, K_*) \hookrightarrow \mathbb{Q}/\mathbb{Z}.$$ 

Thus, the $e$-invariant may be regarded as taking values in $\mathbb{Q}/\mathbb{Z}$. Furthermore, he showed that for $t$ odd, and $k = (t + 1)/2$, the image of $\theta_S$ is the cyclic group of order $\text{denom}(B_k/2k)$, where $B_k$ is the $k$th Bernoulli number.

The $d$ and $e$-invariants detect the 0 and 1-lines of the Adams-Novikov spectral sequence (ANSS). In [Lau99], the second author studied an invariant

$$f : \ker(e_t) \to \Ext^{2,t+2}_{\text{TMF}, \text{TMF}[\frac{1}{6}]}(\text{TMF}[\frac{1}{6}]_*, \text{TMF}[\frac{1}{6}]_*),$$

which detects the 2-line of the ANSS for $\pi^S_\ast$ away from the primes 2 and 3. He furthermore used H. Miller’s elliptic character to show that, if $t$ is even and $k = (t + 2)/2$, there is a monomorphism

$$e^2 : \Ext^{2,t+2}_{\text{TMF}, \text{TMF}[\frac{1}{6}]}(\text{TMF}[\frac{1}{6}]_*, \text{TMF}[\frac{1}{6}]_*) \hookrightarrow D_q/(D_{2[\frac{1}{6}]} + (M_0)_q + (M_k)_q).$$

where $D$ is Katz’s ring of divided congruences and $M_k$ is the space of weight $k$ modular forms of level 1 meromorphic at the cusp. It is natural to ask for a description of the image of the map $e^2$ in arithmetic terms.

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Attempting to generalize the \( J \) fiber-sequence

\[
J \rightarrow KO_p \xrightarrow{\psi^{\ell-1}} KO_p
\]

the first author introduced a ring spectrum \( \mathbb{Q}(\ell) \) built from a length two TMF\(_p\)-resolution. In [Beh08], it was shown that for \( p \geq 5 \), the elements \( \beta_{i,j,k} \in (\pi^S_*)^p \) of [MRW77] are detected in the Hurewicz image of \( Q(\ell) \). This gives rise to the association of a modular form \( f_{i,j,k} \) to each element \( \beta_{i,j,k} \). Furthermore, the forms \( f_{i,j,k} \) are characterized by certain arithmetic conditions.

The purpose of this paper is to prove that the \( f \)-invariant of \( \beta_{i,j,k} \) is given by the formula

\[
f(\beta_{i,j,k}) = \frac{f_{i,j,k}}{p^k E_{p-1}^j} \quad \text{(Theorem 4.1)}.
\]

In particular, since the 2-line of the ANSS is generated by the elements \( \beta_{i,j,k} \), the \( p \)-component of the image of the map \( \iota^2 \) is characterized by the arithmetic conditions satisfied by the elements \( f_{i,j,k} \).

J. Hornbostel and N. Naumann [HN07] computed the \( f \)-invariant of the elements \( \beta_{i,j,1} \) in terms of Katz’s Artin-Schreier generators of the ring of \( p \)-adic modular forms. While their result is best suited to describe \( f \)-invariants of infinite families, it is difficult to explicitly get one’s hands on their output. Direct computations with \( q \)-expansions are limited by the computability of \( q \)-expansions of modular forms, hence are generally not well suited for infinite families of computations. In low degrees, however, our formula can directly be used to compute with \( q \)-expansions. We demonstrate this by giving some sample calculations of some \( f \)-invariants at the prime 5.

We outline the organization of this paper. In Section 2, we review the \( f \)-invariant. In Section 3, we review the spectrum \( \mathbb{Q}(\ell) \), and use it to construct an invariant \( f' \) so that

\[
f_{i,j,k} = f'_{i,j,k}.
\]

In Section 4 we show that the \( f \)-invariant is directly expressible in terms of the invariant \( f' \). In Section 5, we give our sample 5-primary calculations.

2. The \( f \)-invariant

This section reviews the \( f \)-invariant and its various aspects in homotopy theory and geometry. Our main sources are [Lau00], [Lau99], and [vB08].

**Theorem 2.1.** Let \( D \) be the ring of divided congruences defined by N. Katz in [Kat73], that is, the ring of all inhomogeneous modular forms for \( SL_2 \mathbb{Z} \) whose \( q \)-expansion is integral. Then for all \( k > 0 \) there is a homomorphism

\[
f : \pi^S_2 \rightarrow D_q/(D_{2(1/q)} \oplus (M_0)_q \oplus (M_{k+1})_q)
\]

whose kernel is the 3rd Adams-Novikov filtration for \( MU[1/6] \).

The construction of \( f \) is closely related to the construction of the classical \( e \)-invariant by F. Adams (see [Ada66]). Let \( T \) be a flat ring spectrum and let

\[
s : X \rightarrow Y
\]

be a stable map from a finite spectrum into an arbitrary one. Suppose further that the \( d \)-invariant of \( s \) vanishes. This simply means that \( s \) vanishes in \( T \) homology.
Then we have a short exact sequence
\[ T_*Y \to T_*C_s \to T_*\Sigma X, \]
where \( C_s \) is the cofiber of \( s \). We can think of the sequence as an extension of \( T_*X \) by \( T_*Y \) as a \( T_*T \)-comodule. This is the classical \( e \)-invariant of \( s \) in \( T \)-theory.

Next, suppose that \( e(s) \in \text{Ext}^1_{T_*T}(T_*X,T_*Y) \) vanishes, that is, the exact sequence of \( T_*T \)-comodules splits and we choose a splitting. We also choose a \( T \)-monomorphism
\[ \iota : Y \to I \]
into a \( T \)-injective spectrum \( I \). For instance, we can take \( I = T \wedge Y \). Then there is a map
\[ t : C_s \to I \]
which is the image of \( \iota_* \) under the induced splitting map
\[ [Y,I] \cong \text{Hom}_{T_*T}(T_*Y,T_*I) \to \text{Hom}_{T_*T}(T_*C_s,T_*I) \cong [C_s,I]. \]
In particular, the map \( t \) coincides with \( \iota \) on \( Y \). Let \( F \) be the fiber of the map \( \iota \). Then \( s \) lifts to a map
\[ \bar{s} : X \to F \]
which makes the diagram
\[
\begin{array}{ccc}
\Sigma^{-1}C_s & \to & X \\
\downarrow & & \downarrow \bar{s} \\
\Sigma^{-1}I & \to & F \\
\end{array}
\]
commute.

**Lemma 2.2.** \( d(\bar{s}) = 0 \).

**Proof.** In the split exact sequence
\[ \text{Hom}_{T_*T}(T_*\Sigma X,T_*\Sigma F) \to \text{Hom}_{T_*T}(T_*C_s,T_*\Sigma F) \to \text{Hom}_{T_*T}(T_*Y,T_*\Sigma F) \]
the map \( \Sigma \bar{s}_* \) restricted to \( C_s \) is in the image of the splitting and hence has to vanish. The claim follows since the map from \( C_s \) to \( \Sigma X \) is surjective in \( T \)-homology. \( \square \)

Lemma 2.2 implies that we again get a short exact sequence
\[ T_*F \to T_*C_{\bar{s}} \to T_*\Sigma X \]
which we can splice together with the short exact sequence
\[ T_*\Sigma^{-1}Y \to T_*\Sigma^{-1}I \to T_*F. \]
This gives an extension of \( T_*\Sigma^{-1}Y \) by \( T_*\Sigma X \) of length 2, that is, an element
\[ f(s) \in \text{Ext}^2_{T_*T}(T_*X,T_*Y). \]
In the case \( X = S^{2k} \), \( Y = S^0 \) and \( T = \text{TMF}(\frac{1}{6}) \), the image of \( f(s) \) under the injection
\[ \iota^2 : \text{Ext}^2 \to D\mathbb{Q}/(D\mathbb{Z}[\frac{1}{6}] \oplus (M_0)\mathbb{Q} \oplus (M_{k+1})\mathbb{Q}) \]
is the second author’s \( f \)-invariant. The map \( \iota^2 \) will be reviewed in Section 4.

We close this section with an alternative description of the \( f \)-invariant. First recall from [Lau00] that a framed manifold \( M \) represents a framed bordism class in
second Adams-Novikov filtration if and only if it is the corner of a \((U, fr)^2\) manifold \(W\). The boundary of \(W\) is decomposed into two manifolds with boundaries \(W^0\) and \(W^1\). The stable tangent bundle of \(W\) comes with a splitting 
\[
TW \cong (TW)^0 \oplus (TW)^1
\]
and the bundles \((TW)^i\) are trivialized on \(W^i\). Therefore, we get associated classes 
\[(TW)^i \in K(W, W^i)\].

Let \(\exp_T\) be the usual parameter for the universal Weierstrass cubic 
\[y^2 = 4x^3 - E_4 x + E_6\]
and let 
\[\exp_K(x) = 1 - e^{-x}\]
be the standard parameter for the multiplicative formal group. Then following theorem was proved in [Lau00].

**Theorem 2.3.** Let \(s\) be represented by \(M\) under the Pontryagin-Thom isomorphism. Then we have 
\[
f(s) = \left\langle \prod_{i,j} \frac{x_i y_j}{\exp_K(x_i) \exp_T(y_j)}, [W, \partial W] \right\rangle.
\]
Here, \((x_i)\) and \((y_j)\) are the formal Chern roots of \((TW)^0\) and \((TW)^1\) respectively.

We remark that there also is a description of the \(f\)-invariant in terms of a spectral invariant which is analogous to the classical relation between the \(e\)-invariant and the \(\eta\)-invariant. We refer the reader to [vB08].

**3. The spectrum \(Q(\ell)\) and the invariant \(f'\)**

For a \(\mathbb{Z}[1/N]\)-algebra \(R\) we shall let \(M_k(\Gamma_0(N))_R\) denote the space of modular forms of weight \(k\) over \(R\) of level \(\Gamma_0(N)\) which are meromorphic at the cusps. For \(N = 1\) we shall simplify the notation by writing 
\[
(M_k)_R := M_k(\Gamma(1))_R.
\]

Let \(\text{TMF}_0(N)\) denote the corresponding spectrum of topological modular forms with \(N\) inverted (see [Beh06, Sec. 1.2.1], [Beh07, Sec. 3]). For primes \(p > 3\), \(\pi_* \text{TMF}_0(N)_p\) is concentrated in even degrees, and we have 
\[
\pi_{2k} \text{TMF}_0(N)_p \cong M_k(\Gamma_0(N))_{\mathbb{Z}_p}.
\]

Fix a pair of distinct primes \(p\) and \(\ell\). In [Beh06], the first author introduced a \(p\)-local spectrum \(Q(\ell)\), defined as the totalization of a certain semi-cosimplicial spectrum 
\[
Q(\ell) = \text{Tot}(Q(\ell)^*)
\]
where \(Q(\ell)^*\) has the form
\[
(3.1) \quad Q(\ell)^* = \left( \begin{array}{c}
\text{TMF}_p \to \text{TMF}_0(\ell)_p \to \\
\times \\
\text{TMF}_p \to \text{TMF}_0(\ell)_p
\end{array} \right).
\]
In [Beh08] the spectrum \( Q(\ell) \) is reinterpreted as the smooth hypercohomology of a certain open subgroup of an adele group acting on a certain spectrum. The semi-cosimplicial spectrum \( Q(\ell)^\bullet \) is actually a semi-cosimplicial \( E_\infty \)-ring spectrum, so the spectrum \( Q(\ell) \) is an \( E_\infty \)-ring spectrum. In particular, there is a unit map

\[
\eta : S \to Q(\ell).
\]

The spectrum \( Q(\ell) \) is designed to be an approximation of the \( K(2) \)-local sphere. More precisely, the spectrum \( Q(\ell)_{K(2)} \) is given as the homotopy fixed points of a dense subgroup of the Morava stabilizer group acting on the Morava \( E \)-theory \( E_2 \) [Beh07], [BL06]. The spectrum \( Q(\ell) \) is \( E(2) \)-local. In [Beh08] it is proven that elements \( \beta_{i/j,k} \in \pi_*(S_{E(2)}) \) of [MRW77] are detected by the map

\[
S_{E(2)} \to Q(\ell).
\]

Applying homotopy to the semi-cosimplicial spectrum \( Q(\ell)^\bullet \) (3.1) gives a semi-cosimplicial abelian group

\[
C(\ell)_{2k} := \left( \begin{array}{c}
(M_k)_{\mathbb{Z}_p} \\
M_k(\Gamma_0(\ell))_{\mathbb{Z}_p}
\end{array} \right) \to \left( \begin{array}{c}
\times \\
M_k(\Gamma_0(\ell))_{\mathbb{Z}_p}
\end{array} \right).
\]

It is shown in [Beh08] that the morphisms

\[
d_0, d_1 : (M_k)_{\mathbb{Z}_p} \to M_k(\Gamma_0(\ell))_{\mathbb{Z}_p} \times (M_k)_{\mathbb{Z}_p},
\]

induced by the initial coface maps of the cosimplicial abelian group \( C(\ell)_{2k}^\bullet \), are given on the level of \( q \)-expansions by

\[
d_0(f(q)) := (\ell^k f(q^\ell), \ell^k f(q)),
\]

\[
d_1(f(q)) := (f(q), f(q)).
\]

The Bousfield-Kan spectral sequence for computing \( \pi_* \text{Tot}(Q(\ell)^\bullet) \) gives a spectral sequence

\[
H^s(C(\ell)^\bullet)_t \Rightarrow \pi_{t-s}Q(\ell).
\]

For \( p > 3 \), this spectral sequence collapses for dimensional reasons [Beh08, Cor. 5.9], giving us the following lemma.

**Lemma 3.7.** The edge homomorphism

\[
H^2(C(\ell)^\bullet)_t \to \pi_{t-2}(Q(\ell))
\]

is an isomorphism for \( t \equiv 0 \mod 4 \).

Since the sequence

\[
* \to Q(\ell) \to \text{TMF}_p \to \text{TMF}_0(\ell)_p \to *
\]

is a \( BP \)-injective resolution of the spectrum \( Q(\ell) \), the spectral sequence (3.6) coincides with the ANSS for \( \pi_*Q(\ell) \). In particular, the map (3.2) induces a map of ANSS’s

\[
\begin{array}{c}
\text{Ext}_{BP,BP}^{s,t}(BP_*, BP_*) \to \pi_{t-s}S_{(p)} \to \pi_{t-s}Q(\ell)
\end{array}
\]
We shall find that this invariant \( \pi \) because

\[
0 \to BP_\ast \to BP_\ast [p^{-1}] \to BP_\ast /p^\infty \to 0,
0 \to BP_\ast /p^\infty \to BP_\ast /p^\infty [v_1^{-1}] \to BP_\ast /(p^\infty, v_1^{-1}) \to 0
\]
give rise to long exact sequences in \( \text{Ext} \), and the connecting homomorphisms give a composite

\[
(3.8) \quad \delta_{v_1, p} : \text{Ext}_{BP, BP}^0(BP_\ast, BP_\ast / (p^\infty, v_1^{-1})) \xrightarrow{\delta_{v_1}} \text{Ext}_{BP, BP}^1(BP_\ast, BP_\ast / p^\infty) \xrightarrow{\delta_p} \text{Ext}_{BP, BP}^2(BP_\ast, BP_\ast).
\]

The computations of [MRW77] imply the following lemma.

**Lemma 3.9.** The homomorphism \( \delta_{v_1, p} \) of (3.8) is an isomorphism for \( t > 0 \).

A choice of \( p \)-typical complex orientation

\[
BP \to \text{TMF}_p \to \text{TMF}_0(\ell)_p
\]
sends \( v_1 \) to a non-zero multiple of the Hasse invariant \( E_{p-1} \mod p \). The complex \( C(\ell)^\bullet / p^k \) is a complex of modules over the ring \( \mathbb{Z}_p[v_1^{p^{-1}}] \). The short exact sequences

\[
0 \to C(\ell)^\bullet \to C(\ell)^\bullet [p^{-1}] \to C(\ell)^\bullet / p^\infty \to 0,
0 \to C(\ell)^\bullet / p^\infty \to C(\ell)^\bullet / p^\infty [v_1^{-1}] \to C(\ell)^\bullet / (p^\infty, v_1^{-1}) \to 0
\]
give rise to long exact sequences in \( H^* \), and the connecting homomorphisms give a composite

\[
\delta_{v_1, p} : H^0(C(\ell)^\bullet / (p^\infty, v_1^{-1})), \xrightarrow{\delta_{v_1}} H^1(C(\ell)^\bullet / p^\infty), \xrightarrow{\delta_p} H^2(C(\ell)^\bullet).
\]

Using Lemmas 3.7 and 3.9, we have the following diagram, for \( t > 0 \).

\[
\begin{array}{ccc}
\pi_{4t-2} S(p) & \xrightarrow{f'} & \pi_{4t-2} Q(\ell) \\
\text{Ext}_{BP, BP}^{2, 4t}(BP_\ast, BP_\ast) & \xrightarrow{\delta_{v_1, p}} & H^2(C(\ell)^\bullet) \\
\text{Ext}_{BP, BP}^{0, 4t}(BP_\ast, BP_\ast / (p^\infty, v_1^{-1})) & \xrightarrow{\delta_{v_1, p}} & H^0(C(\ell)^\bullet / (p^\infty, v_1^{-1})).
\end{array}
\]

Because \( \pi_{4t-2} S(p) \) contains no elements of Adams-Novikov filtration less than 2, the invariant \( f' \) may be regarded as giving a homotopy invariant through the composite

\[
\pi_{4t-2} S(p) \to \text{Ext}_{BP, BP}^{2, 4t}(BP_\ast, BP_\ast) \xrightarrow{f'} H^0(C(\ell)^\bullet / (p^\infty, v_1^{-1})).
\]

We shall find that this invariant \( f' \) is closely related to the \( f \) invariant of the second author.

We end this section by describing some of the salient features of the invariant \( f' \). Namely, we shall show:

(i) the homomorphism \( f' \) is a monomorphism, and if \( \ell \) generates \( \mathbb{Z}_p^\infty \), the homomorphism \( f' \) is almost an isomorphism, and

(ii) the groups \( H^0(C(\ell)^\bullet / (p^\infty, v_1^{-1})). \) admit a precise arithmetic interpretation in terms of congruences of \( q \)-expansions of modular forms.
The injectivity and almost surjectivity of $f'$.

Because $v_2$ is invertible in $C(\ell^\bullet)/(p^\infty, v_1^\infty)$, there is a factorization (3.11)

\[
\begin{array}{ccc}
\Ext_{BP^*}^2(BP_*, BP_*) & \stackrel{f'}{\longrightarrow} & H^0(C(\ell^\bullet)_{4t}) \\
\cong & \delta_{v_1,p} & \cong \\
\Ext_{BP^*_{BP}}^0(BP_*, BP_*/(p^\infty, v_1^\infty)) & \stackrel{\eta'}{\longrightarrow} & H^0(C(\ell^\bullet)/(p^\infty, v_1^\infty)[v_2^{-1}])
\end{array}
\]

where $\bar{\eta}$ represents the change-of-rings isomorphism, and $\bar{\eta}'$ denotes the element $\beta_{i/j}$ for certain combinations of indices $i$, $j$, and $k$. As usual, $\beta_{i/j}$ denotes the element $\beta_{i/j,1}$.

**Proposition 3.12.**

(i) The map $L_{v_2}$ of (3.11) is injective, and the cokernel is an $\mathbb{F}_p$-vector space with basis

$$\{\beta_{p^n/j} : n \geq 2, p^n < j \leq p^n + p^{n-1} - 1\}.$$  

(ii) The map $\bar{\eta}$ of (3.11) is injective, and if $\ell$ generates $\mathbb{Z}_p^\times$, it is an isomorphism.

**Proof.** (i) follows directly from the calculations of [MRW77]. (ii) follows from the fact that the map $\bar{\eta}$ factors as

\[
\begin{array}{ccc}
\Ext_{BP^*_{BP}}^0(BP_*, BP_*/(p^\infty, v_1^\infty)[v_2^{-1}]) & \stackrel{\bar{\eta}}{\longrightarrow} & H^0(C(\ell^\bullet)/(p^\infty, v_1^\infty))_{4t} \\
\cong & \bar{\eta}' & \cong \\
\Ext_{BP^*_{BP}}^0(BP_*, BP_*/(p^\infty, v_1^\infty))_{4t} & \stackrel{\bar{\eta}''}{\longrightarrow} & H^0(\mathbb{S}_2, \mathbb{E}_2/(p^\infty, v_1^\infty))_{4t}^{\Gal}
\end{array}
\]

where $\bar{\eta}'$ is the Morava change-of-rings isomorphism, and $\bar{\eta}''$ is the monomorphism given by combining Corollary 7.12 and Lemma 11.1 of [Beh08], which is an isomorphism if $\ell$ generates $\mathbb{Z}_p^\times$ [Beh08, Lem. 11.1].

We conclude that $f'$ is injective, and if $\ell$ generates $\mathbb{Z}_p^\times$, the only generators of $H^0(C(\ell^\bullet)/(p^\infty, v_1^\infty))$ not in the image of $f'$ are those corresponding to the Greek letter elements $\beta_{p^n/j}$ for $j > p^n$.

**The arithmetic interpretation of the groups $H^0(C(\ell^\bullet))$.**

The groups $H^0(C(\ell^\bullet)/(p^\infty, v_1^\infty))_{4t}$ are computed by the colimit of groups

$$H^0(C(\ell^\bullet)/(p^\infty, v_1^\infty))_{4t} = \colim_{k} \colim_{j=sp^{k-1}} \colim_{s \geq 1} B_{2t/j,k}$$

where

$$B_{2t/j,k} = H^0(C(\ell^\bullet)/(p^k, v_1^j))_{4t+2j(p-1)}.$$
Using the fact that \( v_1 \) corresponds to the Hasse invariant \( E_{p-1} \) in the ring of modular forms, we have

\[
B_{2t/j,k} = \ker \left( \frac{M_{2t+j(p-1)}}{(p^k, E_{p-1}^j)} \oplus \frac{M_{2t+j(p-1)}(\Gamma_0(\ell))}{(p^k, E_{p-1}^j)} \right).
\]

Serre [Kat73, Prop. 4.4.2] showed that two modular forms \( f_1 \) and \( f_2 \) over \( \mathbb{Z}/p^k \) are linked by multiplication by \( E_{j(p-1)} \) (for \( j \equiv 0 \mod p^{k-1} \)) if and only if the corresponding \( q \)-expansions satisfy

\[
f_1(q) \equiv f_2(q) \mod p^k.
\]

Using this, and (3.4)-(3.5), the following theorem is proven in [Beh08].

**Theorem 3.13** ([Beh08, Thm. 11.3]). There is a one-to-one correspondence between the additive generators of order \( p^k \) in \( B_{t/j,k} \) and the modular forms \( f \in M_{t+j(p-1)}(\mathbb{Z}/p^k) \) satisfying

1. We have \( t \equiv 0 \mod (p-1)p^{k-1} \).
2. The \( q \)-expansion \( f(q) \) is not congruent to 0 mod \( p \).
3. We have \( \text{ord}_q f(q) > \frac{t^2}{12} \) or \( \text{ord}_q f(q) = \frac{t^2}{12} \).
4. There does not exist a form \( f' \in M_{t'} \) such that \( f'(q) \equiv f(q) \mod p^k \) for \( t' < t+j(p-1) \).
5. There exists a form \( g \in M_t(\Gamma_0(\ell)) \) satisfying

\[
f(q') - f(q) \equiv g(q) \mod p^k.
\]

**Remark 3.14.** It follows from [Beh08, Cor. 11.8], that a modular form satisfying (1)-(5) corresponding to \( f' \) is independent of the choice of the prime \( \ell \).

4. The relation between \( f \) and \( f' \)

Let \( \ell \) be a generator of \( \mathbb{Z}_p^\times \). We start with a cohomology class

\[
x \in \text{Ext}^{2t}_{BP,BP}(BP_*, BP_*)
\]

with corresponding invariant

\[
f'(x) \in B_{t/j,k} = H^0(C^*(\ell)/(p^k, v_1))_{2t+j(p-1)}.
\]

By Theorem 3.13, a representative of \( f'(x) \) is a \( \mathbb{Z}/p^k \) modular form \( \varphi \) of weight \( t+j(p-1) \) satisfying certain congruences. It is only well defined up to multiples of \( E_{p-1}^j \). We view \( \varphi \) as a divided congruence, more precisely, as an element of \( D \otimes \mathbb{Z}/p^k \).

**Theorem 4.1.** The \( f \)-invariant of the class \( x \) is given by

\[
p^{-k}E_{p-1}^{-j}(\varphi - q^0(\varphi))
\]

where \( q^0 \) is the 0th Fourier coefficient.

**Remark 4.2.** For \( t > 0 \), Theorem 3.13(3) implies that there exists a representative \( \varphi \) of \( f'(x) \) with \( q^0(\varphi) = 0 \).
Corollary 4.3. The class
\[ p^k E^j_{p-1} f(x) \]
is congruent to a \( \mathbb{Z}/p^k \)-modular form \( \varphi \) of weight \( t + j(p-1) \) up to modular forms of weights \( j(p-1) \) and \( t + j(p-1) \). Moreover, \( \varphi \) satisfies the conditions (1)-(5) of 3.13.

Let \( T \) be \( \text{TMF}[\frac{1}{p}] \) and \( M^{(2)} \) the Hopf algebroid of cooperations of \( T \). An element of \( M^{(2)} \) is a modular form in two variables which is meromorphic at \( \infty \) and has (away from 6) an integral Fourier expansion (see [Lau99]). The map
\[ \phi : (M_k^{(2)})_{\mathbb{Z}_p} \rightarrow M_k(\Gamma_0(l))_{\mathbb{Z}_p} \times (M_k)_{\mathbb{Z}_p} \]
given in terms of \( q \)-expansions by
\[ f \mapsto (l^k f(q,q'), l^k f(q,q)) \]
induces a map of resolutions and hence a commutative diagram
\[
\begin{array}{ccc}
(M_k)_{\mathbb{Z}_p} & \xrightarrow{d_0-d_1} & (M_k^{(2)})_{\mathbb{Z}_p} \\
\downarrow & & \downarrow \phi \\
(M_k)_{\mathbb{Z}_p} & \xrightarrow{d_0-d_1} & M_k(\Gamma_0(l))_{\mathbb{Z}_p} \times (M_k)_{\mathbb{Z}_p}
\end{array}
\]

Lemma 4.4. The induced map in cohomology
\[ H^0(M_\ast/(p^\infty, E^\infty_{p-1})) \rightarrow H^0(C^\ast(\ell)/(p^\infty, v_1^\infty)) \]
is an isomorphism.

Proof. By [HS05], there is a change-of-rings isomorphism
\[
H^0(M_\ast/(p^\infty, E^\infty_{p-1})) = \text{Ext}^0_{\text{TMF}_p, \pi_* \text{TMF}_p}(\pi_* \text{TMF}_p, \pi_* \text{TMF}_p/(p^\infty, E^\infty_{p-1})) \\
\cong \text{Ext}^0_{BP, BP}(BP_\ast, BP_\ast/(p^\infty, v_1^\infty)[v_2^{-1}]).
\]
The lemma follows from the isomorphism \( \tilde{\eta} \) of Proposition 3.12. \( \square \)

Next we explain how to get from an element in
\[ H^0(M_\ast/(p^\infty, E^\infty_{p-1})) \cong \text{Ext}^0_{M^{(2)}}(M_\ast, M_\ast/(p^\infty, E^\infty_{p-1})) \]
to a congruence in
\[ D_\mathbb{Q}/(D_{\mathbb{Z}[1/6]} \oplus (M_0)_\mathbb{Q} \oplus (M_{k+1})_\mathbb{Q}). \]
For this, we first describe how a class \( \varphi \) in
\[ \text{Ext}^0_{M^{(2)}}(M_\ast, M_\ast/(p^\infty, E^\infty_{p-1})) \]
gives rise to a class in
\[ \text{Ext}^2_{M^{(2)}}(M_\ast, M_\ast). \]
We use the geometric boundary theorem

Theorem 4.5. [Rav86] Write \( E_\ast(X) \) for the \( E_\ast \)-term of the \( T \)-based Adams Novikov spectral sequence which conditionally converges to the homotopy of the \( T \)-nilpotent completion of \( X \). Let
\[ W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} \Sigma W \]
be a cofiber sequence of finite spectra with $T_*(h) = 0$. Assume further that $[s] \in E^{t,n}_{2+}(Y)$ converges to $s$. Then $\delta[s]$ converges to $h_*(s)$ where $\delta$ is the connecting homomorphism to the short exact sequence of chain complexes

$$0 \to E_1(W) \to E_1(X) \to E_1(Y) \to 0.$$  

For a multi index $I$ let

$$M(I) = M(i_0, \ldots, i_{n-1})$$

be the generalized Moore spectrum with

$$BP_*M(I) = \Sigma^{-||I||-n}BP_*/(p^{i_0}, v^{i_1}_1, \ldots, v^{i_{n-1}}_n)$$

where

$$||I|| = \sum_j 2i_j(p^j - 1)$$

Each $M(I)$ admits a self map

$$\Sigma^{2i_{n-1} - (p^n - 1)}M(I) \to M(I)$$

which induces multiplication by $v^n_i$. Its fiber is $M(I, i_n)$. We apply the geometric boundary theorem to the sequences

$$\Sigma^{2i_1(p-1)}M(i_0) \to M(i_0) \to \Sigma M(i_0, i_1) \to \Sigma^{2i_1(p-1) + 1}M(i_0)$$

and

$$S \xrightarrow{p_{i_0}^*} S \to \Sigma M(i_0) \to S^1.$$  

For

$$\varphi \in E^0_2(M(i_0, i_1)) = \text{Ext}^0_{M(2)}(M_*, M_*/(p^{i_0}, E^{i_1}_{p-1}))$$

we have

$$\delta \varphi = \left[ \frac{d^0 \varphi - d^1 \varphi}{E^{i_1}_{p-1}} \right] \in E^1_2(M(i_0)) = \text{Ext}^1_{M(2)}(M_*, M_*/p^{i_0})$$

and

$$\delta \delta \varphi = \left[ p^{-i_0} \sum_{i=0}^2 (-1)^i d^i \left[ \frac{d^0 \varphi - d^1 \varphi}{E^{i_1}_{p-1}} \right] \right] \in E_2^2(S) = \text{Ext}^2_{M(2)}(M_*, M_*).$$

where $d^i$ denote the differentials of the cobar complex $\Omega_T$. The maps of spectra

$$T \xrightarrow{q^0} K_{\mathbb{Z}[1/6]} \xrightarrow{\rho} H_{\mathbb{Q}}$$

induce maps of complexes

$$\rho : \Omega_T \to \Omega_{T,K,H}$$

with

$$\Omega_{T,K,H} : M_* \xrightarrow{\rho} (D)_{\mathbb{Z}[1/6]} \xrightarrow{\rho} (D)_\mathbb{Q}/((0_M)_\mathbb{Q} \oplus (M_*)_\mathbb{Q}).$$

The first differentials are given by

$$d^0 = \iota, \quad d^1 = q^0$$

and the second ones by

$$d^0 = \iota, \quad d^1 = d^2 = 0$$

where $\iota$ the canonical inclusion. The induced map in cohomology is the inclusion

$$\iota^2 : \text{Ext}^2_{M(2)}(M_*, M_*) \hookrightarrow D_{\mathbb{Q}}/((D)_{\mathbb{Z}[1/6]} \oplus (0_M)_{\mathbb{Q}} \oplus (M_*)_{\mathbb{Q}} \oplus (M_{k+1})_{\mathbb{Q}}).$$
Hence we have
\[ \rho_\ast \delta \varphi = p^{-i_0} E_{p^{-1}i_1}(\varphi - q^0(\varphi)) \]
and the proof of the theorem is completed.

5. EXAMPLES AT \( p = 5 \)

Below are some computations of the \( q \)-expansions of the modular forms \( f_{i/j,k} \) representing \( f'(\beta_{i/j,k}) \) at \( p = 5 \). The \( q \)-expansions of the corresponding \( f \) invariants, by Theorem 4.1, are given by
\[ f(\beta_{i/j,k}) = p^{-k} E_{p^{-1}k} f_{i/j,k}(q). \]
The computations were performed using the MAGMA computer algebra system, with \( \ell = 2 \), as follows.

(i) A basis \( \{ F_\alpha(q) \} \) of \( q \)-expansions of forms in \( M_{24i} \) satisfying Theorem 3.13(3) was generated.

(ii) A basis \( \{ G_\beta(q) \} \) of \( q \)-expansions of holomorphic forms in \( M_{24i-4j}(\Gamma_0(\ell)) \mathbb{Z}/5^k \) was generated.

(iii) Basic linear algebra is used to calculate a basis of linear combinations
\[ \sum_\alpha a_\alpha F_\alpha \] such that
\[ \sum_\alpha a_\alpha (F_\alpha(q^2) - F_\alpha(q)) \equiv \sum_\beta b_\beta G_\beta(q) \mod 5^k. \]

Note 5.1. The following modular forms are normalized so that the leading term has coefficient 1. Therefore, they may differ from the \( f' \)-invariants of \( \beta_{i/j,k} \) by a multiple in \( \mathbb{Z}_p^\times \).

\[ f_{1/1,1} = \Delta^2 = \]
\[ q^2 + 2 q^3 + q^7 + q^{12} + 2 q^{13} + q^{17} + 2 q^{18} + 2 q^{22} + 2 q^{23} + 3 q^{28} + q^{32} + 4 q^{33} + q^{37} + 2 q^{42} + 2 q^{43} + q^{47} + 2 q^{48} + q^{52} + 2 q^{53} + 2 q^{62} + 2 q^{63} + q^{67} + 3 q^{68} + q^{72} + 2 q^{73} + 2 q^{77} + 4 q^{78} + 2 q^{82} + 2 q^{83} + q^{92} + 4 q^{93} + q^{97} + 3 q^{98} + 0(q^{100}) \mod 5 \]

\[ f_{2/1,1} = \Delta^4 = \]
\[ q^4 + 4 q^5 + 4 q^6 + 2 q^9 + 4 q^{10} + 3 q^{14} + 3 q^{15} + 3 q^{16} + 4 q^{19} + 2 q^{20} + 3 q^{21} + 2 q^{24} + 2 q^{26} + q^{29} + 3 q^{30} + 2 q^{34} + 4 q^{35} + 3 q^{36} + 3 q^{39} + 2 q^{44} + 3 q^{45} + q^{51} + 4 q^{54} + 3 q^{55} + q^{56} + 2 q^{59} + 4 q^{60} + 2 q^{70} + q^{79} + 4 q^{80} + 4 q^{81} + 4 q^{84} + 4 q^{85} + q^{86} + 3 q^{89} + 3 q^{90} + q^{91} + 4 q^{94} + 4 q^{96} + 4 q^{99} + 0(q^{100}) \mod 5 \]
\[ f_{3/1,1} = \Delta^6 = \]
\[ q^6 + q^7 + 2q^8 + 3q^9 + 3q^{11} + 2q^{12} + 2q^{13} +
q^{16} + 4q^{17} + q^{18} + 4q^{19} + 2q^{22} + 4q^{24} +
3q^{26} + 3q^{27} + 3q^{28} + 3q^{29} + 4q^{31} + 4q^{32} +
4q^{33} + 4q^{34} + q^{36} + 4q^{44} + 4q^{46} + 4q^{48} + 4q^{49} +
q^{51} + 2q^{53} + 4q^{54} + 3q^{56} + 4q^{58} + q^{62} +
4q^{63} + 3q^{64} + 3q^{66} + 4q^{67} + 3q^{68} + q^{69} +
2q^{72} + 4q^{73} + q^{74} + q^{76} + 4q^{77} + 3q^{78} +
4q^{79} + q^{82} + 3q^{84} + 2q^{86} + q^{87} + 4q^{88} +
4q^{89} + 3q^{91} + q^{92} + 2q^{93} + 4q^{94} + 3q^{96} +
3q^{97} + q^{98} + 2q^{99} + O(q^{100}) \mod 5 \]

\[ f_{4/1,1} = \Delta^8 = \]
\[ q^8 + 3q^9 + 4q^{10} + 2q^{11} + q^{12} + 4q^{13} + 4q^{14} +
3q^{15} + 2q^{16} + q^{19} + 3q^{21} + 4q^{22} + 2q^{24} +
4q^{26} + 4q^{27} + 4q^{28} + 4q^{29} + 3q^{31} +
4q^{33} + q^{34} + 4q^{35} + 3q^{37} + q^{38} + 2q^{39} +
q^{43} + 3q^{44} + 2q^{47} + 4q^{51} + 2q^{52} + q^{53} +
3q^{54} + q^{56} + q^{57} + 3q^{58} + 2q^{59} + q^{60} +
4q^{61} + 2q^{63} + 3q^{65} + 2q^{66} + q^{67} + 4q^{68} +
2q^{69} + 2q^{71} + q^{73} + q^{74} + 2q^{76} + 2q^{78} +
3q^{79} + 2q^{81} + 3q^{82} + 4q^{85} + 4q^{86} + q^{87} +
q^{89} + 3q^{90} + q^{91} + 3q^{92} + 3q^{93} + 3q^{94} +
4q^{97} + 3q^{98} + 4q^{99} + O(q^{100}) \mod 5 \]

\[ f_{5/5,1} = \Delta^{10} = \]
\[ q^{10} + 2q^{15} + q^{35} + q^{60} + 2q^{65} + q^{85} + 2q^{90} +
0(q^{100}) \mod 5 \]

\[ f_{25/29,1} = \Delta^{50} + 4\Delta^{12}E_4^{24} + 3\Delta^{41}E_4^{27} = \]
\[ 3q^{41} + 2q^{42} + 4q^{43} + 4q^{44} + 3q^{47} + 2q^{48} +
3q^{49} + q^{50} + q^{51} + q^{52} + 2q^{54} + q^{56} + 4q^{58} +
q^{59} + 4q^{61} + 4q^{62} + q^{63} + 3q^{64} + q^{66} +
4q^{67} + 3q^{68} + 3q^{69} + q^{71} + q^{74} + 2q^{75} +
2q^{76} + 3q^{78} + 4q^{79} + 2q^{81} + 3q^{82} + 2q^{83} +
4q^{84} + 2q^{88} + 3q^{89} + 4q^{91} + q^{92} + 2q^{94} +
2q^{96} + q^{98} + q^{102} + q^{104} + 4q^{106} + 3q^{107} +
3q^{108} + 2q^{109} + 4q^{111} + 4q^{112} + 4q^{114} +
3q^{116} + 2q^{118} + 2q^{119} + q^{121} + 4q^{122} +
3q^{123} + q^{124} + q^{126} + 2q^{127} + q^{129} + 4q^{132} +
q^{134} + 4q^{136} + 4q^{138} + q^{139} + q^{141} +
3q^{143} + q^{144} + q^{147} + 3q^{149} + O(q^{150}) \mod 5 \]
\[ f_{25/2} = \Delta^{50} = \]
\[ q^{50} + 10q^{55} + 15q^{60} + 5q^{65} + 12q^{70} + 15q^{75} + 20q^{75} + 20q^{80} + 15q^{85} + 10q^{90} + 5q^{95} + 15q^{100} + 10q^{105} + 20q^{110} + 5q^{115} + 20q^{125} + 20q^{135} + 15q^{140} + 20q^{145} + 10q^{150} + O(q^{151}) \mod 25 \]

References

[Beh08] Mark Behrens, Congruences between modular forms given by the divided \(\beta\) family in homotopy theory, Preprint, 2008.