1. Idea of the de Rham-Witt complex
   (a) It is a complex of sheaves on a scheme over a perfect field of characteristic $p$, or, more
gen般地，它是一个在特征$p$的完美域上的方案上的正则群。更一般地，它可以是一个在特征$p$的完美域上的方案上的正则群。
   (b) Provides a complex which is explicit and computable. Its hypercohomology agrees with
crystalline cohomology.
   (c) It is a pro-system (inverse limit) of differential graded algebras.
      i. In degree zero, it is the Witt vectors.
      ii. The first complex in the inverse limit is the de Rham complex.

2. Witt vectors
   (a) $W(k)$ for $k$ a perfect field of characteristic $p$.
      i. $W(k)$ is a $p$-adically complete dvr with maximal ideal $(p)$ and residue field $k$.
      ii. For each $x \in k$, there exists a distinguished choice of lift $[x] \in W(k)$, called the
Teichmüller lift, whose projection to $k$ is $x$. (The lift is distinguished by the fact
that it possesses $p^n$th roots for every $n$.)
      iii. These Teichmüller lifts are multiplicative: $[x_1 x_2] = [x_1][x_2]$.
      iv. Every element in $W(k)$ may be written as an infinite sum $\sum p^i [x_i]$.
   (b) $W(R)$ in general.
      i. $W(R)$ is a ring.
      ii. Elements of $W(R)$ are infinite sequences $(r_0, r_1, r_2, \ldots)$. Ring operations are defi-
nitely not defined componentwise.
      iii. Ring operations are defined so that the ghost map

$$w : W(R) \rightarrow R^N$$

$$(r_0, r_1, \ldots) \mapsto (r_0, r_0^p + pr_1, r_0^{p^2} + pr_1^p + p^2 r_2, \ldots)$$

is a ring homomorphism (where $R^N$ has componentwise ring operations).
   iv. That determines the ring structure on $W(R)$ if $R$ is $p$-torsion free. The other cases
are forced by the requirement that $R \rightsquigarrow W(R)$ is a functor, where for $f : R \rightarrow
S$, we set $W(f) : W(R) \rightarrow W(S)$ to be the map given by $W(f)(r_0, r_1, \ldots) =
(f(r_0), f(r_1), \ldots)$.
   v. We again have Teichmüller representatives, $[r] = (r, 0, 0, \ldots)$. They are again mul-
tiplicative.
   vi. For the special case $W(\mathbb{F}_p) (= \mathbb{Z}_p)$ the infinite sequences correspond to the sequences
$(x_0, x_1, \ldots)$ with $x_i$ as in 2(a)iv. (This only works because the map $x \mapsto x^p$ is the
identity on $\mathbb{F}_p$. In general the sequence will look like $(x_0, x_1^p, x_2^{p^2}, \ldots)$.)
   vii. Two special maps Frobenius $F$ and Verschiebung $V$.
      vii(1) $F$ is a ring homomorphism. In case $R$ is characteristic $p$, it is induced by the
$p$th power map on $R$. In general, we define $\bar{F}$ on the ring $R^N$ by $\bar{F}(r_0, r_1, \ldots) =
(r_1, r_2, \ldots)$ and we define $F$ to be the map for which $w \circ F = \bar{F} \circ w$.
      vii(2) $V$ is additive but is not a ring homomorphism. It is defined by $V(r_0, r_1, \ldots) =
(0, r_0, r_1, \ldots)$. 

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vii(3) \( xV(y) = V(F(x)y) \). Thus, \( V : F_* W(R) \to W(R) \) is a homomorphism of \( W(R) \)-modules, where the notation \( F_* W(R) \) means the module structure is given by 
\[
w_1 \cdot w_2 := F(w_1)w_2.
\]
vii(4) \( FV = p \) always. If \( \text{char } R = p \), then \( VF = p \).
viii. We can define the truncated or finite length Witt rings as the length \( n \) sequences \((r_0, \ldots, r_{n-1})\), with addition and multiplication as before. In particular, \( W_1(R) \cong R \). We have restriction maps \( r_n : W(R) \to W_n(R) \). Similarly for \( W_{n+1}(R) \to W_n(R) \). We have \( W(R) \cong \lim_{\leftarrow} W_n(R) \).

(c) Two examples
i. If \( R \) is a \( \mathbb{Z}(p) \)-algebra, then so too is \( W(R) \). [Hes05], Lemma 1.9.
ii. If \( R \) is a \( \mathbb{Z}(p) \)-algebra, the ideal \( V(W(R)) \subseteq W(R) \) is equipped with divided powers. [Ill79], p. 510.

3. The de Rham-Witt complex \( W\Omega_A \) for \( A \) a \( \mathbb{Z}(p) \)-algebra.

(a) Two (equivalent) definitions
i. Initial object in the category of \( V \)-pro-complexes over \( A \).
ii. Initial object in the category of Witt complexes over \( A \).
iii. There is a forgetful functor from the category of Witt complexes to the category of \( V \)-pro-complexes where we forget the Frobenius \( F \).
iv. The existence of these initial objects can be proven using the Freyd adjoint functor theorem, but there is also a more constructive proof in the category of \( V \)-pro-complexes. Historically, this constructive proof was given and it was then shown that the resulting (initial) object also had a Frobenius.

(b) First properties
i. The canonical map of pro-complexes \( \pi : \Omega^*_{W(A)} \to W.\Omega^*_A \) is surjective. (Warning: This does not mean there is a surjective map \( \Omega^*_{W(A)} \to W.\Omega^*_A \).)
ii. In degree zero, this induces an isomorphism: \( \Omega^0_{W(A)} = W(A) \sim W.\Omega^0_A \).
iii. In level one, this induces an isomorphism: \( \Omega^*_{W_1(A)} = \Omega^*_A \sim W_1.\Omega^*_A \).
iv. There is a map of complexes \( F \) induced by \( F \) in degree zero. In degree \( i \), we have \( F = p^i F \).

(c) Examples
i. The de Rham-Witt complex over a perfect field. [Ill79], Prop 1.6, p. 545.
ii. The de Rham-Witt complex over \( \mathbb{F}_p[x_1, \ldots, x_n] \). [Ill79], p.550 or [CL98], p. 18.
iii. The de Rham-Witt complex over \( \mathbb{Z}(p) \). [HM03], Example 1.2.4.
iv. The de Rham-Witt complex over \( A[x] \) in terms of the de Rham-Witt complex over \( A \). [HM03], Theorem B.
References


There are very many exercises (over 50) involving Witt vectors beginning on page 42 of chapter 9.


This is a nice overview of the de Rham-Witt complex (in characteristic $p$) and its relation to crystalline cohomology.


This is a good first place to read about Witt vectors. It describes their properties very carefully, although in some more generality than we talked about. (It includes big Witt vectors in addition to $p$-typical vectors, but nothing about the de Rham-Witt complex.)


A good exposition of the de Rham-Witt complex for $\mathbb{Z}(p)$-algebras. This is the source of the definition(s) I gave.


Probably the standard reference for characteristic $p$. It’s 160 pages and in French, but it’s usually the first place I turn when I’m looking for a specific formula, etc.


A relative version of the de Rham-Witt complex in characteristic zero from [HM03]. They compare the cohomology of their complex to crystalline cohomology in chapter 3. The computation is very concrete.