Hesselholt - Madsen, On the K-thy of local fields

Abstract

Let $A$ be a complete DVR

$K = \text{frac}(A)$, $k = \text{Residue field}

\text{mixed char: char } K = 0
\text{ char } k = p \neq 2, k \text{ perfect}

Structure the (five's local fields)

\[
W(k) \xrightarrow{\exists !} A \quad A = \frac{W(k)[x]}{\Phi(x)}
\]

$\Phi(x) = x^e + p \Theta(x)$
e.g., $A = \mathbb{Z}[S_p]$

$$\Phi(x) = \frac{(x-1)^{\rho} - 1}{x}$$

Aim: Compute $K_*(A), K_*(K)$

by Topological methods

Localization sequence:

$$K(k) \xrightarrow{i^*} K(A) \xrightarrow{i} K(K)$$

Problem: don't have localization sequence for $THH(-) = T(-)$

$T(K)$ is an HK-alg

$$\Rightarrow \pi_* (T(k)) = 0$$

$$\Rightarrow \pi_+ TC(K) = 0$$

cannot have localization sequence
**Def:** Let \( C^b_{\geq 2}(P_A) \)

bounded chain complexes of projective \( A \)-modules of finite type

We let \( f: X \to Y \) s.t. \( f \) is a quasiiso in \( C^b(K\text{-mod}) \)

Note: \( K(K) = K(C^b_{\geq 2}(P_A)) \)

**Thm. 1.5.6 + 1.5.7**

We have maps of aff. schemes:

\[
K(k) \to K(A) \to K(K)
\]

\[
\downarrow \quad \downarrow \quad \downarrow
\]

\[
TC(k) \to TC(A) \to TC(A|K)
\]

\[
\downarrow \quad \downarrow \quad \downarrow
\]

\[
TR''(k) \to TR''(A) \to TR''(A|K)
\]
\[ T^r(A) = T(A) \]

was computed \( \overline{\pi} \).

but \( T(A \mid K) \) is \( \text{Nice} \).

\( \text{(has good def.)} \)

The structure of \( \pi \cdot T(A \mid K) \)

(1) Comes' from

\[ d^r : T_n (A \mid K) \rightarrow T_{n+1} (A \mid K) \]

from \( S' \subset T(A \mid K) \)

\[ d^2 = nd = dn \quad n \in \pi \]

at odd primes, \( d^2 = 0 \)
Def: A log ring is a pair $(R, M)$

- $R$ comm ring
- $M$ comm monoid

$\alpha : M \to (R, \times)$ map of monoids

(2) $M = A \cap K^*$

$\alpha : M \to A = T_0(A1K)$

So $(T_0(A1K), M)$ is a log ring.

(3) $d \log : M \to T_1(A1K)$

$M \to K^* = K_1(K) \to T_1(A1K)$

"log" denote $$(T_0(A1K), M) \to T_1(A1K)$$
Set: \[ E = T_1(A1K) \]
\[ d', A \rightarrow E \] is a derivation
\[ dlog', M \rightarrow E \]

Satisfies

**Def:** a log-derivation from a log-ring \((A, M)\) into an \(A\)-mod

is given by a pair \((d, dlog)\)

\[ (A, M) \rightarrow E \]

\[ d = \text{derivation} \]

\[ dlog = \text{map of monoids} \]

\[ d \cdot x(a) = x(a) \cdot dlog(a) \]

\[ \text{for } a \in M \]

A logarithmic diff'd in calculus

\[ d \log (f) = \frac{df}{f} \]

or \[ f \cdot d\log (f) = df \]
We have a universal log-derivation, counting the universal derivation

\[ A \to \Omega^1_A \]

and universal log structure

\[ M \to A \otimes_{\mathbb{Z}} \mathcal{C}_p(M) \]

**Def:** \( w_{(A, M)} = \left[ \Omega^1_A + \left( A \otimes_{\mathbb{Z}} \mathcal{C}_p(M) \right) \right] \frac{(d\omega(a) - a \otimes \omega(a))}{a \in M} \)

\[ d', M \to W'(A, M) \]

\[ a \mapsto (d\alpha, 0) \]

\[ d\log', a \mapsto (0, 1 \otimes a) \]

Computed explicitly.
Definition: A log-differential graded ring is $(E^*, d)$ (graded differential ring) with monoid map

$$\lambda : M \rightarrow E^0$$

a monoid map

$$d \lambda : M \rightarrow E^1$$

such

$$(d, d \lambda) : (E^*, M) \rightarrow E^1$$

is a log-derivation

and such $d \lambda : M \rightarrow E^2$ is zero.
Universal example

\[ \bigwedge_A w'(A, M) = w^*(A, M) \]

"(Kähler differential) \div \log \text{poles}"

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**Rule 2.2.5**

The \( p \)-torsion subring:

\[ (w'(A, M))_p \subset w'(A, M) \]

is a cycle \( A/p \)-module

on \( \mathcal{H} \log (-p) \)

i.e.

\[ \rho(w(A, M)) = A/p \otimes d \log (-p) \]
Rule: A. Understand - Model - Compute

\[ T_\ast(A) \]

\[ \Rightarrow \quad T_\ast(A_{\Lambda K}) \] is a uniquely divisible group.

\[ \Rightarrow \quad \overline{T}_\ast(T(A_{\Lambda K})) \xrightarrow{\delta} \pi_i \overline{T}(A_{\Lambda K}) \]

\[ K \quad \xrightarrow{\text{Log}(-p)} \]

Thm. B The canonical map

\[ \omega^*_{(A_{\Lambda}, M)} \otimes F_p[L] \xrightarrow{\delta} \overline{T}_\ast(T(A_{\Lambda K})) \]

is an iso!

Rule: \( \omega^*_{(A, M)} \) is uniquely divisible for \( i \geq 2 \).
\[
\Rightarrow \omega_1^{(A,M)} \cong D \oplus \bigoplus_p \omega_1^{(A,M)}
\]

\[
\Rightarrow \omega_0^{(A,M)} \otimes \overline{F_p}(K) =
\]

\[
(A \oplus A/p \mathcal{O}(\log(-p))) \otimes \overline{F_p}(K)
\]

Note: This formula is much simpler than Kinderhans-Madsen (but relies on it!)

E.g.,
For \( A = \mathbb{Z}_p[S_p] \)
we obtain
\[
\pi_d \mathcal{T}(\mathbb{Z}_p[S_p] \mid \mathcal{O}_p(S_t)) = P_{\mathbb{P}^1}(\mathbb{C}) \otimes E[d] \otimes P[K]
\]
\[
P_{-1}(x) = \frac{\mathbb{P} \mathcal{L} e}{(x^{-1})}
\]

But
\[
\text{The } T(\mathbb{P} \text{c}(\mathfrak{s}_p)) \text{ is MUCH WORSE!}
\]

Much relation TC is easier to compute than absolute TC.

"\text{Ramified extensions are log-etale}"

\[
\Rightarrow \text{ Descent.}
\]

Thus
\[
\frac{1}{t_0}, \text{ and run in ramified } K \text{ (i.e. } p \neq e_{\text{Lk}})\]
Then we have a map

\[ T(A|K) \cong T(B|L) \]

uses that \( B \otimes_A \frac{w(A,K_A)}{w(A)} \cong \frac{w(B,M_B)}{w(A)} \)

Now for

\[ TR^*(A|K) \]

The \( TR^*(A|K) \) is also a hom-adjunction graded map

for \( M = A \otimes K^{e} \)

\[ \lambda : M \xrightarrow{\mathcal{L}} A \xrightarrow{\mathcal{W}} w(A) \]

\[ T^*(A|K) \]
$\mathcal{N} : \text{TR}_m^n \to \text{TR}_{m+1}^n$ comes from $S'$-action

$\mathcal{M}_{\text{log}} : M \to K_i(K) \to \text{TR}_i^n(A|K)$

Thus makes $(\text{TR}_x^n, M)$ into a log-diff' graded ring.

\[\begin{array}{ccc}
M & \xrightarrow{\alpha_3} & W_3(A) \xrightarrow{\lambda} \text{TR}_0^3(A|K) \\
\downarrow & & \downarrow \\
F & \downarrow R & V
\end{array}\]

\[\begin{array}{ccc}
M & \xrightarrow{\alpha_2} & W_1(A) \xrightarrow{\lambda} \text{TR}_0^2(A|K) \xrightarrow{d} \text{TR}_1^2(A|K) \\
\downarrow & & \downarrow & \downarrow & \downarrow & \downarrow
\end{array}\]

\[\begin{array}{ccc}
M & \xrightarrow{\alpha_1} & W_1(A) \xrightarrow{\lambda} \text{TR}_0^1(A|K) \xrightarrow{d} \text{TR}_1^1(A|K) \\
\downarrow & & \downarrow & \downarrow & \downarrow & \downarrow
\end{array}\]
These satisfy:

\[ E^n_m = TR^n_{m} (A | K) \]

(1) \((E^*_a, M)\) is a pro-log
diff'ed graded \(R\)
(wrt. \(R\))
i.e. \(R\) coincides w/ empty

(2) \(\lambda : (W_a(A), M) \rightarrow (E^*_0, M)\)
is a map of pro-log rings

(3) \(F : E^{n+1}_{a} \rightarrow E^n_{a}\)
is a map of pro-log rings.

\[ \lambda F = F \lambda \]

\[ F \text{dlog}_n = \text{dlog}_{n-1}, \]
\[ Fd [a]_n = [a]_{n-1}^p d([a]_{n-1}) \]

\[(4) \quad V : E^n_x \to E^{n+1}_x \text{ is a map of pro-graded } E^{n+1}_x \text{-modules} \]

\[(\text{via} \quad E^{n+1}_x \to E^n_x) \]

\[\begin{cases}
\lambda F = F \\
FV = p \\
FdV = d
\end{cases} \quad (\text{see 3.2.1})
\]

This structure is a big Witt complex over \((A, M)\)

There is a city of such
This is an initial object, the de-Rham-Witt co of log poles:

\[ W_{\ast} (A, M) \]

\[ W_{\ast} w_{\ast} (A, M) \rightarrow TR_{\ast} \]

Curs names + all elts in spectral sequences.

Assum: \[ \text{Map } C \rightarrow K \]

\[ \Rightarrow K(K) \text{ is 2-period} \]
\[ M_p = \pi_1 B M_p \xrightarrow{\sim} \pi_2 B M_p \rightarrow K_2(K) \xrightarrow{\text{TR}_2(A,1K)} \]

Take:
\[ \text{Sym}(M_p), \quad F, V, R = 1 \]
\[ d = 0 \]

Theorem 6.14

The composite map
\[ \Phi_p[\mathbb{S}] \]
\[ W_0 \otimes \text{Sym}(M_p) \rightarrow \text{TR}^*_p(A,1K) \]

is a pro-isomorphism!

\[ \text{iso} \] of \( \text{po-object} \)

Remark: \( W_0^i, A, M \) is uniquely determined for \( n \geq 1 \) and \( i > 2 \).
[so will not contribute]

\[ \Rightarrow \overline{\text{TR}}_d(A_{1k}) \text{ is 2-pure}\]

\[ \overline{TC} \rightarrow \overline{TR} \rightarrow \overline{TR}_{F^{-1}} \]

\[ \Rightarrow \overline{TC}_*(A_{1k}) \text{ is 2-pure}\]

and also have

\[ \overline{K}_*(k) \text{ 2-pure for } \Sigma_{m+1} \text{ for } m \geq 1 \]

\[ \overline{K}_0(k) \rightarrow \overline{K}_0(A) \rightarrow \overline{K}_0(k) \rightarrow 0 \]

\[ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \]

\[ \overline{TC}_0(k) \rightarrow \overline{TC}_0(A) \rightarrow \overline{TC}_0(A_{1k}) \rightarrow \overline{TC}_0(k)^{10} \]

\[ \text{known } K_0, TC_0 \]

\[ \Rightarrow \overline{K}_1(k) = k^*/(k^*)^p \]
\[ \text{TC}_0(A|k) = \mathbb{Z}/p \oplus \mathbb{Z}/p \]

If you don't have root of unity, adjoin it, and use log-étale descent.

\[ \text{TC}(A|k) \cong \text{TC}(A[\mathbb{F}_p]|K(\mathbb{F}_p)) \]

Identify up to very étale K-thy.

\[ \text{TC}(A|k) \cong \text{TC}(A[\mathbb{F}_p]|K(\mathbb{F}_p)) \]

Example:

\[ K(Q(\mathbb{F}_p)) = \mathbb{B}F \psi_{p-1}^{\text{ss}} \times \Phi \psi_{p-1}^{\text{ss}} \times U^{p-1} \]

\[ \text{TC}(Z_p|Q_p) = E(1) \oplus P(K) \]