$X = \text{variety}/\mathbb{C}$

$X_{\text{an}} = \text{associated complex analytic variety}$

$X(\mathbb{C})$

$H^*(X_{\text{an}}; \mathbb{Z}) = \text{singular cohom}$

S.g. ab gps

... ranks, bett $H^i$, p-torsion for various primes $p$

---

**Key observation:**

These quantities can admit purely algebraic descriptions, and have geometric consequences.

\text{e.g., not our C}

\text{Example: } C = \text{smooth connected curve}/\mathbb{C}

\text{rk } H^i(C_{\text{an}}; \mathbb{Z}) = 2g
\[ g = \text{genres} = \dim H^0(c, \Omega_c) \]
\[ = \dim H^1(c, \Omega_c) \]
\[ = 1 - \chi(\Omega_c) \]

---

**Dream:** Create (purely ablt) \( H^*_c(X) \)

recently \( H^*_c(X^n, \mathbb{Z}) \) for \( X/\mathbb{C} \)

but defined as well behaved as possible if \( X \) not in char 0
\( \{ \text{not smooth} \} \)

---

**Prop:** \( H^*(X^n; \mathbb{Z}) = H^*_c(X^n; \mathbb{Z}) \)

\( \cong \text{constant sheaf} \)

\( \mathbb{Z} \to C^0_{\text{shg}}(X, \mathbb{Z}) \)

\( \uparrow \text{acyclic resolution} \)
What about

\[ H^*_\mathbb{Z}(X; \mathbb{Z})? \text{ Vanishes for } * \geq \text{ dim } X \]


"The Zariski topology is too coarse to compute correct cohomology of local systems."

1st idea: Stable cone

Fine topology

\[ \Rightarrow \text{Some locally constant sheaves get correct cohomology.} \]

2nd idea: Algebraic De Rham cohomology

Replace loc. const. sheaves by things for which Zariski is fine enough."
Def: A (pre-)topology (almost same) is a city \( C \) together with a collection of collections of morphisms \( \{ U_{a} \to U_{b} \} \) which we call "covers".

0) \( \text{Fix} \to \gamma \) iso \( \Rightarrow \{ x \to \gamma \} \) is a cover

1) \( \{ U_{a} \to U \} \) cover

and

\( \{ U_{ab} \to U_{a} \} \) cover

\( \Rightarrow \{ U_{ab} \to U_{a} \to U \} \) cover

2) \( S: V \to U \) \( \text{Uex}_{u} V \) exists

\( \{ U_{a} \to U_{b} \text{ covers} \} \) \( \Rightarrow \{ \exists U_{a} \times_{U} V \to V \} \)

---

(pre-topology is can generalize to sets \( \Gamma \) of pre-Kleisli
A presheaf on $C$ is a functor

$$C^{op} \to \text{Set}$$

A sheaf is a presheaf s.t. for all coverings $\{U_i \to U\}$

$$F(U) \to \prod F(U_i) \to \prod F(U_i \cap U_j)$$

is an equalizer,

A topos is a category of the form $\text{Sh}(C)$

Reason why you want this def:

There are maps of toposes which do NOT arise from maps of sites.
Ex: $X$ zar objects: $U \to X$

morph: $U \to V$

$X_{an}$ objects: $U \to X(\mathfrak{c})$

morph: $U \to X(\mathfrak{c})$

Cover = set theoretically.

\[ X_{et} \text{ objects: } U \to X \]

\[ \text{étale} \]

morph: $U \to V$

\[ X \]
\{ U_n \to U \} \text{ is covering} \quad \text{if} \quad U \cap U_n = U_n.

Assume $X$ is a smooth var $\mathcal{G}$

$X_{an, \acute{e}t}$ objects: $U \to X$

more: contains as before

$X$ smooth $\mathcal{G}$

pretty much same colh $\overset{\text{sam colh}}{\longleftarrow}$ for torsion sheaves

$X_{an}$  $\overset{\text{want to study}}{\longleftarrow}$
Cohomology

\[ c = \text{site}, \ U \in \mathcal{C} \]

\[ \text{Sh}_{\text{Ab}_{qc}}(\mathcal{C}) = \text{a cat w/ enough injectives} \]

\[ \mathcal{F} \]

\[ H^*(U, \mathcal{F}) := R^* I'(U, \mathcal{F}) \]

\[ \text{Then } X \text{ smooth } / S \]

\[ H^*(X_{\text{et}}, \mathbb{Z}/l^n) = H^* (X_{an}, \mathbb{Z}/l^n) \]

So get all ranks, all torsion.

---

Defects: Not very well behaved for non-smooth

but MORE IMPORTANTLY
- $p$-torsion is VERY BAD in char $p$

  Sub-step: Fix this.

  (Crystalline)

Starting Rank 1:

Apply a \text{pro-}$\mathbb{F}_p$ process

$	ext{GAGA} \rightarrow$ Zariski col. sheaves correct answers for coherent sheaves of $\mathcal{O}_X^\text{-}\text{vds}$.

e.g. vector bundles.

Starting Rule 2: (after de Rham thesis)

i) $M$ is a smooth manifold

$TM = \text{ tangent sheaf}$

$\Omega^1_{\text{smooth}} = \text{ dual}$

$\Omega^n_{\text{smooth}} = \bigwedge^n \Omega_M^\text{smooth}$
\[ \Omega^*_M \rightarrow \Omega^1_M \rightarrow \Omega^2_M \rightarrow \cdots \rightarrow \Omega^*_M \]

\[ \Omega^*_M \]

- Poincaré lemma!

\[ \Omega^*_M \] is exact, except at basin:

\[ \ker \, d \big|_{\Omega^*_M} = \mathbb{R} \]

- (Portions of unity) Each

\[ \Omega^*_M \] is "finite" (no higher ch

\[ \Rightarrow H^*(M, \mathbb{R}) = H^*(M, \mathbb{R}) = H^*(TM, \Omega^*_M) \]

\text{De Rham Thm}
(i) $M = \mathbb{R}$ or $\mathbb{C}$ analytic manifold

$\partial M$

- Poincaré lemma is still true!

$K \to \partial M$ quasi-isomorphism

$K = \mathbb{R}$ or $\mathbb{C}$

- Acyclic? \textbf{NO}!

Thus:

$H^*(M, K) = H^*(M, K) = H^4(M, \partial M)$
(iii) $X = \text{smooth proj var.}/\mathbb{C}$

$$\left(\Omega^*_{X/\mathbb{C}}\right)^{\text{an}} = \Omega^*_{X/\mathbb{C}}$$

$\text{GAGA} \Rightarrow H^*(X, \Omega^*_{X/\mathbb{C}}) \not\approx \text{purely analytic}$

$$= H^*(X, \Omega^*_{X/\mathbb{C}})$$

(iv) $X$ smooth $/\mathbb{C}$ but not necessarily proper.

Then (Groth) uses res of singularities

(*) above still true.

\underline{Def.} Suppose $\phi: X \to S$ smooth

$$H_{dR}^*(X/\mathbb{C}) = R\phi_* (\Omega^*_{X/\mathbb{C}})$$

$$H_{dR}^*(x) = H^*(X, \Omega^*_{X/\mathbb{C}})$$
v) \( X \) var/\( C \) not acc smooth.

\[
\begin{align*}
\hat{X} \xrightarrow{\text{check}} \hat{Y} & \quad \text{smooth var/} \ C \\
\check{\iota} = \check{\iota} & \quad \hat{X} \to \hat{Y} \\
\check{\iota} = \check{\iota} & \quad \hat{X} \to \hat{Y} \\
\hat{Y}/\check{\iota} & \quad \text{locally model space} \\
(X, \lim_{\iota} \check{\sigma}_{\check{\iota}}) & \quad \text{topologically like } X \\
\text{geometrically like } \check{\iota} & \quad \text{geometrically like } \check{\iota} \\
H^{\ast}(\hat{Y}/\check{\iota}, \check{\sigma}_{\check{\iota}}) & \quad \text{"good def"}
\end{align*}
\]

---

Problems

- Not intrinsic for non-smooth case
- It would be nice to be able to vary coef's (instead of only \( \check{\omega} \))
\[ f : X \to Y \quad \text{smooth} \]

\[ H^p(Y, R\Omega^i_x \otimes \mathcal{O}_Y) \to H^p(X, \mathcal{O}_X) \]

different coefs

Infinitesimal site solves both!

Def (old) \[ X/S \text{ smooth} \]

An \( S \)-connection on \( M (\mathcal{O}_X \text{-mod}) \) is an \( \pi^{-1} \mathcal{O}_S \)-linear map

\[ \nabla : M \to \Omega^1_{X/S} \otimes_{\mathcal{O}_X} M \]

Satisfies\: \[ \nabla (f m) = df \otimes m + f \nabla m \]
Can extend to

\[ \nabla^c : \Omega^c \otimes \mathcal{O}_x \to \Omega^{c+1} \otimes \mathcal{O}_x \]

\[ \nabla^c (\omega \otimes m) = d\omega \otimes s + (-1)^c \omega \wedge \nabla m \]

\( \nabla \) is integrable if

\[ \nabla' \circ \nabla = 0 \]

Then \[ \nabla^{c+1} \circ \nabla^c = 0 \quad \forall \ c \]

\[ \Omega^c \otimes \mathcal{O}_x \]

Every need must be made sense

\[ E_3 \otimes \mathcal{O}_x \]
In analytic setting:

\[ \{ \text{locally const. sheaves on } X^{\infty} \} \leftrightarrow \{ \text{vector bundles of integrable connection} \} \]

(And C vs. others)

Flat sections
\[ \text{ker } \nabla \]

(Riemann-Hilbert correspondence explains)

This word-folly

equivalence of "bounded derived cuts"

\[ \text{RHom}(\mathcal{O}, \mathcal{O}) = \text{RHom}(\mathcal{O}_d, \mathcal{O}_d) \]

\text{cohomology of D-oddles}
\[ (X/S)^{(x/s)} \]

Start

\[ f(x, T) \rightarrow (u, T) \]

Closed, inwards, the supremal

Def: (New) \((X/S)^{(x/s)}\) if

\[ \psi(x) \rightarrow x \]
What's your connection?

\[ y \xrightarrow{g} x \]

\[ y' \xrightarrow{g_2} \]

\[ \text{Invertible} \]
\[ \text{Connection} \Leftrightarrow g_1^* M = g_2^* M \]

+ something