9- Formal groups and BP: a survey

Let $E$ be a homotopy commutative, homotopy associ. Ring spectrum.

TFAE:

1. $R$ complex orientable \( (\sum_{n=0}^{\infty} \mathbb{Q}_n \to \mathbb{Q}) \)

2. There is a factorization:

\[
\begin{array}{ccc}
S^2 & \to & \Sigma^2 E \\
\downarrow & & \downarrow \\
\Sigma^2 \mathbb{Q} & \to & \mathbb{Q}
\end{array}
\]

3. The AHSS

\[
H^*(\mathbb{C}P^\infty, \mathbb{Q}_n) \Rightarrow E^*(\mathbb{C}P^\infty)
\]

Collapses to give

\[
E^2(\mathbb{C}P^\infty) \cong E - [x_1]
\]

\( |x_1| = 2 \)

\( E \) is map of (BP) to spectra

\[
\begin{array}{ccc}
MU & \to & E \\
\phi & & \\
\end{array}
\]
(1) \implies (2) \quad \text{w.r.t.} \quad \text{MUC}(1) = (C\mathbb{P}^\infty)^5 \approx C\mathbb{P}^\infty / C\mathbb{P}^\infty \approx C\mathbb{P}^\infty

\chi = U_3 = \text{then class of } S

(2) \iff (3)

\quad E_2^{2,\alpha} = E_\alpha^{k,\beta} \left[ \chi \right]

(2) \iff \chi \in \text{P.C.}

\text{multiplicity of } \text{AHSS} \Rightarrow d(\chi^k) = 0.

(2) \implies (4)

\Phi \in E^0(\text{MU}) \iff \Phi_n \in E^{2n}(\text{MUC}(n))

\quad \Phi_n \in E^{2n}(\text{MUC}(n)) \quad \xrightarrow{\text{'splitting principle'}} \quad \text{image}

\kappa \ldots \kappa \in E^{2n} \left( \text{MUC}(1) \wedge \ldots \wedge \text{MUC}(1) \right)

(4) \implies (1)

\Phi_n \in E^*(\text{MUC}(n)) = U_{\delta n} \quad \text{Then class}

\text{computation} \iff U_{\delta n} \wedge U_{\delta m} = U_{\delta nm}
Complex orientation

Choice of $\alpha \iff$ choice of $\Theta$

$\iff$ choice of $\nu_{E}$

\[ F_\phi(x, y) \in E_2[[x, y]] \]

Formal group law

Def a (commutative, 1-drill) formal group law is a power series

\[ F(x, y) \in R[[x, y]] \]

S.t.

1. $F(0, x) = F(x, 0) = 0$

2. $F(x, F(z, y)) = F(F(x, y), z)$

3. $F(x, y) = F(y, x)$
\[ \mathbb{C}P^\infty \times \mathbb{C}P^\infty \xrightarrow{\sim} \mathbb{C}P^\infty \]

\text{classifies} \quad \mathbb{S} \times \mathbb{S} \rightarrow \mathbb{C}P^\infty \times \mathbb{C}P^\infty

under \quad \mathbb{C}P^\infty \xrightarrow{\text{H-space}} \text{comm.}

get \quad M^* : E^*(\mathbb{C}P^\infty) \rightarrow E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)

\text{via} \quad E^*([x]) \quad E^*([x,y])

\quad \xrightarrow{\sim} \quad \mathbb{C} \quad \mathbb{C}

\quad \xrightarrow{\sim} \quad \underbrace{F(x,y)} = F_E(x,y)

\text{e.g.} \quad E = \mathbb{H} \mathbb{Z}

\[ \chi = c_1(\mathbb{S}) \in H^2(\mathbb{C}P^\infty) \]

\[ F_{\mathbb{H} \mathbb{Z}}(x,y) = x + y \quad \text{"additive sector"} \]

\text{Con} \quad \text{If} \quad \pi_* E \text{ is concentrated in even degrees} \quad \Rightarrow \quad E \text{ is complex orientable}

\text{(PS) Atiyah for} \quad \mathbb{C}P^\infty \text{ collapses for dual reasons.
\[ \mathfrak{r}_K U = \mathbb{Z}[B, \beta^{-1}] \quad |B| = 2 \]

\[ F_{KU}(x, y) = x + y + \beta xy \]

\[ S^c \rightarrow \Sigma^2 KU = \Sigma^2 \left( \lim_{\rightarrow} \Sigma^2 \Sigma^2_{MU(n)} \right) \]

\[ \xrightarrow{\text{Thm. (Quillen)}} \quad F_{MU}(x, y) \text{ is the universal final group law} \]

\[ L = \text{Lazard ring} \]

\[ \text{Ring}(L, R) \cong \left\{ \text{FGL}_3 / R \right\} \]

\[ \text{Spec}(L)(R) = \text{Map}(\text{Spec}(K), \text{Spec}(L)) \]

\[ \left[ \text{So} \quad \text{Spec}(L) = M_{\text{FGL}} \text{ models spec of } \mathfrak{r}_K U \right] \]

\[ \text{Thm. (Lazard)} \]

\[ L \cong \mathbb{Z}[x_1, x_2, \ldots] \]
\[ F = f^* F_{\text{univ}} \quad F_{\text{univ}} \]

\[ \text{Spec}(\mathbb{R}) \xrightarrow{f} M_{\mathbb{FCL}} \]

\[ f : \mathbb{R} \to T \]

\[ F = \mathbb{FCL}/R \quad F(x,y) = \sum q_{i,j} x^i y^j \]

\[ (f^* F)(x,y) = \sum f(q_{i,j}) x^i y^j \]

\[ f^* F = \mathbb{FCL}/T \]

**Quillen's theorem:**

1. \( x \sim y \)
2. \( \nu \mathbb{M} \mathbb{U} \cong \mathbb{L} \quad \nu \mathbb{R} \mathbb{L} 2i \)
3. under this isomorphism

\[ F_{\mathbb{M} \mathbb{U}} \cong F_{\text{univ}} \]
An isomorphism of formal power series

\[ F_1, F_2 \mathbin{/}/ R \]

\[ f(x) = \sum_{i=1}^{\infty} b_i x^i, \quad b_i \in R^* \]

is \[ f(x) \in R[[x]]^* \]

\[ f : F_1 \to F_2 \]

\[ g_1 : F_2(f(x), f(y)) = f(F_1(x, y)) \]

\[ f \text{ is strict} \quad \text{iff} \quad f(0) = 1 \]

\[ \iff b_0 = 1 \]

Note: given \( f, F_1, F_2 \), \( F_2 \) is determined.
$	extbf{Thm (Quillen - Landweber - Novikov)}$

$M_{\eta} M_{\eta} \cong \mathcal{L}[b_1, b_2, \ldots]$

$\alpha$

$R_{\text{alg}}(M_{\eta} M_{\eta}, R) = \{ (F_1, F_2, f) \mid F_1, F_2 = \text{Fun. S} / R \}$

$f: F_1 \to F_2$

$\text{Spec}(M_{\eta} M_{\eta})(R)$

$\begin{cases} 
  f(x) = \sum \alpha(b_i) x^{c_i} \\
  F_i \text{ is classified by } \alpha \downarrow L \text{ abelian}
\end{cases}$

$\Rightarrow (\text{Spec}(M_{\eta} M_{\eta})(R), \text{Spec}(M_{\eta} M_{\eta})(R))$

is a groupoid.

$\begin{align*}
\text{Cor:} & \quad (\text{Spec}(M_{\eta} M_{\eta})(R), \text{Spec}(M_{\eta} M_{\eta})(R)) \\
& \begin{cases} 
  \text{objects:} & \text{morphisms:} \\
  \text{Groupoid} & (\text{Fun. S} / R, \text{start isos})
\end{cases}
\end{align*}$

From this, can define

$\eta_L, \eta_R, \eta_{\text{NR}}, \text{etc.}$

Recursively (Frames cannot be within)
BP: "p-local vers. of MU"

\[
F(x, y) =: x + y = x + y + \sum_{i+j \geq 2} a_{i,j} x^i y^j
\]

\[\rightsquigarrow \text{any power series } f(x) \in R[[x]] \text{ admits an expression as}
\]

\[
f(x) = a_0 + a_1 x + a_2 x^2 + \ldots
\]

\[
= \sum_{i=0}^{\infty} a_i x^i
\]

\[\forall n \in \mathbb{N}
\]

\[
[n]_F: F \to F \quad \text{endomorphism of } F
\]

\[
[n]_F(x) = x + x_F + \ldots + x_F^{n-1} + x_F^n = nx + \ldots
\]

**Def:** a F-algebra \( F/R \) is \( p \)-typical if

\[\exists v_i \in R
\]

\[
\forall n, [n]_F(x) = px + v_1 x^p + v_2 x^{p^2} + \ldots + v_n x^{p^n} + \ldots
\]
Thus: Suppose $R$ is a $\mathbb{Z}_p$-algebra.

Given $\{v_i\}$, $\exists!$ $p$-typical formal sp $F/R$

s.t.

$$[p]_F(x) = \sum_{i}^F v_i x^i$$

\[\Rightarrow V = \mathbb{Z}_p[\{v_1, v_2, \ldots\}]\]

"Ambi. sexismos."

"Carries a universal $p$-typical formal sp

$R = \mathbb{Z}_p$-alg

$Spec(V)(R) = \{p$-typical $F\text{-algs} \}$

Thus: $R = \mathbb{Z}_p$-alg

"p-weak" $\Gamma = \text{formal sp hom } R$

$\exists$ functorial strict $\text{Hom}_{\mathbb{Z}_p}$

$F \xrightarrow{\pi_p F} F_p$

where $F_p$ is $p$-typical.

\(\text{s.t. If } F \text{ is already } p\text{-typical} \)

$$\pi_p F = \text{Id}$$
\[
\begin{align*}
\text{MU & cx orientable} \\
\downarrow \\
\text{MU} \xrightarrow{\text{Id}} \text{MU} \\
\text{different orientations} \leftrightarrow \text{different chains} \\
\text{of } \kappa \in \text{MU}^* (\mathbb{C} \mathbb{P}^n) \\
\text{MU} \rightarrow [\kappa] \\
\text{Let } \pi_p^\text{uni} : \text{Fun}_p \rightarrow (\text{Fun}_p)_p \\
\kappa_p := \pi_p^\text{uni}(\kappa) \in \text{MU}_p [\kappa] \\
\downarrow \\
\text{new orientation} \\
\text{identitf map } \tau_p : \text{MU} \rightarrow \text{MU} \\
\text{BP} := \text{colim} \left( \text{MU} \xrightarrow{\pi_p} \text{MU} \rightarrow \cdots \right) \\
\text{By construction, BP carries unique } p\text{-torsion} \\
\text{final gp} \\
\Rightarrow \text{BP} = \mathbb{Z}(p) [v_1, v_2, \ldots] \\
|v_i| = 2(p^i - 1)
\end{align*}
\]
\[ \text{Spec}(BPBP)(R) = \{ (F_1, F_2, f) \mid F_1, F_2 \text{ p-typical } \frac{F_1 \otimes F_2}{f} \text{ is } \text{iso} \} \]

Lemma:

\[ f \in R[x_1] \]
\[ f : F_1 \to F_2 \]
\[ F_1 \text{ p-typical} \]

\[ F_2 \text{ is p-typical} \]
\[ \iff f^{-1}(x) = \sum_{t \in F_1} t \cdot x^t \]
\[ \iff f(x) = \sum_{t \in F_2} t \cdot x^t \]

So \( BPBP = BP \langle t_1, t_2, \ldots \rangle \) (6d = 2k-1)

\((\text{Spec}(BP), \text{Spec}(BPBP)(R)) = \text{Hof} \text{f Hurewicz} \)

Object! p-typical \( F_1 \otimes F_2 \otimes R \)
Nonframed strict isos
\[ n_c : B_{c0} \rightarrow B_{c0} B_{c0} \]

\[ n_c : B_{c0} \rightarrow B_{c0} B_{c0} \]

\[ n_c : B_{c0} \rightarrow B_{c0} B_{c0} \]

\[ v_i \rightarrow v_i \]

\[ n_c : B_{c0} \rightarrow B_{c0} B_{c0} \]

\[ v_i \rightarrow n_{c}(v_i) \]

\[ \text{on } B_{c0} B_{c0} \]

\[ \exists \quad F_L \quad F_R \]

\[ \mu_{\mu_P} : \mu_{\mu_P} \rightarrow \mu_{\mu_P} \]

\[ F_{\mu_P} \rightarrow F_{\mu_P} \]

\[ \mu_{\mu_P} (x) = \sum_{i} F_{\mu_P} v_i x^p \]

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\[ \text{Lemma: } \exists \mu_{\mu_P} \text{ s.t. } F_{\mu_P} \]

\[ R = \mathbb{Q} - \text{alg} \]

\[ \exists! \text{ strict iso} \]

\[ \log : F \rightarrow F_{\text{add}} \]

\[ F_{\text{add}}(x, y) = x + y \]
\[ F \text{ p-typical } \iff \log_F(x) = \sum_i m_i x^{p^i} \]

\[ \mathcal{L} !: \text{BP}_c \longrightarrow \text{BP}_c \{c, t_2, \ldots \} \]

\[ \nu_c \longrightarrow \nu_c \]

\[ C(p) \mathcal{L} = \sum_{c}^{F} v_c x^{p^c} \]

Apply \( \log_{(p)} \mathcal{L} \)

\[ p \sum_{i} m_i x^{p^i} = \sum_{i} \log(v_i x^{p^i}) \]

\[ = \sum_{i} m_i v_i x^{p^i + i} \]

Inductively get \( v_c \) in terms of \( m_i \):

\[ m_i \in \text{BP}_c \circ \mathcal{Q} \]

\[ \text{e.g. } m_i p^p + v_i = p^p m_i \]

\[ \Rightarrow m_i = \frac{v_i}{p^{p^p}} \]
\[ [p]_{F_R}^{(z)} = \sum_{i} n_R(v_i) \chi_i \]

\[ F_L \leftarrow f^{-1} \quad \quad F_R \quad \quad \downarrow \log F_R \quad \quad \downarrow \log F_L \quad \quad \downarrow \log F_L \quad \quad \quad \quad \downarrow \log F_L \]

\[ \log F_L^{(z)} = \sum_{i} n_R(v_i) \chi_i \]

\[ \log F_L(f^{-1}(x)) = \log F_R^{(z)} = \sum_{i} n_R(v_i) \chi_i \]

\[ \log F_L(\sum_{i} \chi_i \chi_i) \]

\[ \sum_{i} m_i \chi_i \chi_i \chi_i \chi_i \]

\[ \sum_{i} m_i \chi_i \chi_i \chi_i \chi_i \]

\[ \Rightarrow n_R(v_i) = \sum_{i} m_i \chi_i \chi_i \chi_i \chi_i \chi_i \chi_i \chi_i \]

\[ \Rightarrow n_R(v_i) = \sum_{i} (p - p')m_i = (p - p')m_i + v_i \]

\[ v_i + p' \quad \text{mod} \quad p^2 \]
\[ m_i + t_i + s_i = m_i + \psi(t_i) \]

\[ \Rightarrow \psi(t_i) = t_i \otimes 1 + 1 \otimes t_i \]

\[ \psi(t_2) = t_2 \otimes 1 + 1 \otimes t_2 + t_1 \otimes e_0^p \]

\[ + \frac{v_i}{(p^2 - 1)} \sum_{i=1}^{p^2 - 1} \left( \begin{array}{c} p \end{array} \right)_i t_1 \otimes t_i^{p^2 - i} \]

---

**Note**

\[ I = (p, v_1, v_2, \ldots) \]

\[ \psi(t_i) = \sum_{i' + i'' = i} t_{i'} \otimes t_{i''} \]

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\[ \mathcal{N}_R(v_1) = v_1 \mod (p, v_1, \ldots, v_m) \]

\[ \Rightarrow \mathcal{N}_R(I_n) \subseteq I_n \mathcal{B}_P \mathcal{B}_P \]

\[ \Rightarrow \mathcal{B}_P \mathcal{B}_P / I_n \text{ is } \end{equation} \text{(\mathcal{B}_P, \, \mathcal{B}_P)} \text{ compatible} \]
and the only invariant prime ideals of BP.