LECTURE 31: COMPLETION OF A DEFERRED PROOF, WHITNEY SUM, AND CHERN CLASSES

1. A deferred proof

From the last lecture, I constructed for a sub-Lie group H of a Lie group G a map

$$\phi : EG/H \to BH,$$

and owed you a proof of

Proposition 1.1. The map ϕ is an equivalence, and the map induced by the inclusion

$$i : H \hookrightarrow G$$

is the quotient map

$$i_* : BH \simeq EG/H \to EG/G = BG.$$

We first state some easy lemmas.

Lemma 1.2. The induction functor

$$\mathrm{Ind}_H^G = G \times_H (-) : H\text{-spaces} \to G\text{-spaces}$$

is left adjoint to the restriction functor

$$\mathrm{Res}_H^G : G\text{-spaces} \to H\text{-spaces}$$

which regards a G-space X as an H-space. In particular, there is a natural isomorphism

$$\mathrm{Map}_G(G \times_H X, Y) \cong \mathrm{Map}_H(X, Y)$$

for an H-space X and a G-space Y.

Given G-bundles $E \to B$ and $E' \to B'$, a G-equivariant map

$$f : E \to E'$$

gives rise to a map of bundles

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ \downarrow & & \downarrow \\ B & \xrightarrow[f/G]{} & B' \end{array}$$

Lemma 1.3. There is an equivalence of bundles

$$E \cong (f/G)^* E'.$$

Date: 4/28/06.

Sketch proof of Proposition 1.1. We need to construct a map in the opposite direction. The G-bundle $G \times_H EH \to BH$ is classified by a map

$$\begin{array}{ccc} G \times_H EH & \xrightarrow{f} & EG \\ \downarrow & & \downarrow \\ BH & \longrightarrow & BG. \end{array}$$

Let

$$\widetilde{f} : EH \to EG$$

be the H-equivariant map adjoint to the map f. Let ψ be the induced map of H-orbits:

$$\psi = \widetilde{f}/H : BH = EH/H \to EG/H.$$

The composite $\phi \circ \psi$ is seen to be an equivalence because it is covered by the H-equivariant composite

$$EH \xrightarrow{\widetilde{f}} EG \to EH$$

and thus classifies the universal bundle over BH.

The composite $\psi \circ \phi$ is covered by the H-equivariant composite

$$\widetilde{h} : EG \to EH \xrightarrow{\widetilde{f}} EG$$

whose adjoint h gives a map of G-bundles

$$\begin{array}{ccc} G \times_H EG & \xrightarrow{h} & EG \\ \downarrow & & \downarrow \\ EG/H & \xrightarrow[\bar{h}]{} & BG \end{array}$$

The bundle $G \times_H EG \to EG/H$ is easily seen to be classified by the quotient map $EG/H \to EG/G = BG$. Thus we can conclude that h is G-equivariantly homotopic to the map

$$G \times_H EG \to EG$$

which sends $[g, e]$ to ge. Thus the adjoint $\widetilde{h}$ is H-equivariantly homotopic to the identity map $EG \to EG$. Taking H-orbits, we see that $\psi \circ \phi$ is homotopic to the identity. ☐

2. Whitney sum

Let V be an n-dimensional complex vector bundle over X and W be an m-dimensional complex vector bundle over Y.

Definition 2.1. The *external direct sum* $V \boxplus W$ is the product bundle

$$V \boxplus W = V \times W \to X \times Y$$

where the vector space structure on the fibers is given by the direct sum.

Now assume $X = Y$.

Definition 2.2. The *Whitney sum* $V \oplus W$ is the $n+m$-dimensional complex vector bundle given by the pullback $\Delta^* V \boxplus W$, where

$$\Delta : X \to X \times X$$

is the diagonal.

Let V^n_{univ} be the universal n-dimensional complex vector bundle over $BU(n)$. Let

$$f_{n,m} : BU(n) \times BU(m) \to BU(n+m)$$

be the classifying map of $V^n_{univ} \boxtimes V^m_{univ}$. Then if

$$f_V : X \to BU(n)$$
$$f_W : X \to BU(m)$$

classify V and W, respectively, the composite

$$X \xrightarrow{f_V \times f_W} BU(n) \times BU(m) \xrightarrow{f_{n,m}} BU(n+m)$$

classifies $V \oplus W$.

3. Chern classes

Our computation of $H^*(BU(n))$ allows for the definition of characteristic classes for complex vector bundles.

Definition 3.1. Let $V \to X$ be a complex n-dimensional vector bundle, with classifying map

$$f_V : X \to BU(n).$$

We define the *ith Chern class* $c_i(V) \in H^{2i}(X; \mathbb{Z})$ to be the induced class $f_V^*(c_i)$ for $1 \leq i \leq n$. We use the following conventions:

$$c_0(V) := 1$$
$$c_i(V) := 0 \qquad \text{for } i > n.$$

These classes are *natural*: for a map $f : Y \to X$ we have

$$c_i(f^* V) = f^* c_i(V).$$