

LECTURE 30: INDUCED MAPS BETWEEN CLASSIFYING SPACES, $H^*(BU(n))$

1. INDUCED MAPS BETWEEN CLASSIFYING SPACES

Let G be a topological group.

Definition 1.1. Let X be a right G -space and Y be a left G -space. The *Borel construction* is the quotient

$$X \times_G Y = X \times Y / (xg, y) \sim (x, gy).$$

Observe:

$$\begin{aligned} G \times_G Y &= Y \\ * \times_G Y &= Y/G. \end{aligned}$$

Suppose that $\alpha : G \rightarrow H$ is a homomorphism of topological groups. We can get an induced map

$$\alpha_* : BG \rightarrow BH$$

by providing a natural transformation between the functors that these spaces represent:

$$\alpha_* : \{G\text{-bundles over } X\} \rightarrow \{H\text{-bundles over } X\}.$$

Namely, send a G -bundle $E \rightarrow X$ to the H -bundle

$$H \times_G E \rightarrow X$$

where G acts on H through the homomorphism α .

Thus B may be viewed as a functor

Topological groups \rightarrow homotopy category of CW-complexes.

There is one particular case where the induced map admits a more explicit description. Suppose that H is a sub-Lie group of a Lie group G . On a homework problem you showed that $G \rightarrow G/H$ was an H -bundle. It follows that the quotient

$$EG \rightarrow EG/H$$

is an H -bundle, and is therefore classified by a map ϕ :

$$\begin{array}{ccc} EG & \longrightarrow & EH \\ \downarrow & & \downarrow \\ EG/H & \xrightarrow{\phi} & BH \end{array}$$

Proposition 1.2. The map ϕ is an equivalence, and the map induced by the inclusion

$$i : H \hookrightarrow G$$

is the quotient map

$$i_* : BH \simeq EG/H \rightarrow EG/G = BG.$$

A proof of this proposition is deferred to the next lecture.

Corollary 1.3. There is a fiber sequence

$$G/H \rightarrow BH \xrightarrow{i_*} BG.$$

2. CLASSIFYING VECTOR BUNDLES

Given a paracompact space X , there are 1 – 1 correspondences

$$\begin{array}{c} \{U(n)\text{-bundles over } X\} \\ \updownarrow \\ \{\text{hermitian } n\text{-dimensional complex vector bundles over } X\} \\ \updownarrow \\ \{n\text{-dimensional complex vector bundles over } X\}. \end{array}$$

The first correspondence associates to a principle $U(n)$ -bundle P over X the vector bundle

$$P \times_{U(n)} \mathbb{C}^n \rightarrow X$$

with a fixed hermitian structure on \mathbb{C}^n .

Using partitions of unity, every vector bundle admits a hermitian structure, unique up to isomorphism.

3. CALCULATION OF $H^*(BU(n))$

Theorem 3.1. There is an isomorphism

$$H^*(BU(n); \mathbb{Z}) \cong \mathbb{Z}[c_1, \dots, c_n].$$

The generator c_i lies in degree $2i$, and is called the i th Chern class. The induced map

$$H^*(BU(n)) \rightarrow H^*(BU(n-1))$$

is the quotient by the ideal (c_n) .

Proof. We prove this theorem by induction on n , using the Serre spectral sequence of the fiber sequence

$$S^{2n-1} \rightarrow BU(n-1) \rightarrow BU(n).$$

The E_2 -term takes the form

$$E_2^{*,*} = H^*(BU(n)) \otimes \Lambda[e_{2n-1}]$$

where $H^s(BU(n))$ lies in $E_2^{s,0}$ and $|e_{2n-1}| = (0, 2n-1)$. Because this is a 2-line spectral sequence, the only possible non-trivial differential is d_{2n} , and the spectral sequence degenerates into a long exact sequence (the *Gysin sequence*). Because the inductive hypothesis implies that $H^*(BU(n-1))$ is concentrated in

even dimensions, we may deduce that the differential $d_{2n}(e_{2n-1})$ is non-trivial, and we define $c_n \in H^n(BU(n))$ to be the image of this differential

$$c_n := d_{2n}(e_{2n-1}).$$

The multiplicative structure implies that d_{2n} is given by cupping with c_n . Thus the E_∞ -term is comprised of the elements of $H^*(BU(n))$ annihilated by c_n , and the quotient of $H^*(BU(n))$ by the ideal (c_n) . By inducting over k , using the fact that $H^*(BU(n-1))$ is concentrated in even dimensions, we can argue that $H^k(BU(n))$ must be generated in even dimensions, implying that in fact there can be no elements annihilated by c_n . Thus $H^*(BU(n))$ is generated by c_1, \dots, c_n , with

$$H^*(BU(n))/(c_n) = \mathbb{Z}[c_1, \dots, c_{n-1}].$$

We deduce that there is an isomorphism $H^*(BU(n)) \cong \mathbb{Z}[c_1, \dots, c_n]$. □