

LECTURE 29: LINE BUNDLES

[The first part of this lecture was about the multiplicative structure of the Serre spectral sequence, which followed Hatcher. I also presented Hatcher's example of the cohomological Serre spectral sequence for $S^1 \rightarrow ES^1 \rightarrow \mathbb{C}P^\infty$.]

We have equivalences

$$\mathbb{C}P^\infty \simeq BU(1) \simeq K(\mathbb{Z}, 2).$$

Recall:

$$H^*(\mathbb{C}P^\infty; \mathbb{Z}) = \mathbb{Z}[c_1]$$

where $|c_1| = 2$. This generator is called the "first Chern class".

Given a space X , there are 1 – 1 correspondences

$$\begin{array}{c} \{U(1)\text{-bundles over } X\} \\ \updownarrow \\ \{\text{hermitian complex line bundles over } X\} \\ \updownarrow \\ \{\text{complex line bundles over } X\}. \end{array}$$

The first correspondence associates to a principle $U(1)$ -bundle P over X the line bundle

$$P \times_{U(1)} \mathbb{C} \rightarrow X$$

with a fixed hermitian structure on \mathbb{C} . Since every hermitian structure on \mathbb{C} takes the form

$$(z, w) = az\bar{w}$$

for a a positive real, hermitian structures on a line bundle $L \rightarrow X$ are the same thing as positive valued real functions on X , and any two hermitian structures on L are equivalent.

Given a line bundle L/X , there is a classifying map

$$\begin{array}{ccc} L & \longrightarrow & L_{univ} \\ \downarrow & & \downarrow \\ X & \xrightarrow{f_L} & BU(1) \end{array}$$

which recovers L as the pullback of L_{univ} .

Definition 0.1. The *first Chern class* $c_1(L) \in H^2(X; \mathbb{Z})$ is defined to be the class $f_L^*(c_1)$.

Date: 4/24/06.

Proposition 0.2. The association

$$L \mapsto c_1(L)$$

gives an isomorphism

$$\{\text{complex line bundles over } X\} \cong H^2(X; \mathbb{Z}).$$

Proof. Under the equivalence $BU(1) \simeq K(\mathbb{Z}, 2)$, c_1 gives the fundamental class of $H^2(K(\mathbb{Z}, 2), \mathbb{Z})$. \square

Remark 0.3. A homework problem you were assigned indicates that if the left hand side of the isomorphism in Proposition 0.2 is given the structure of an abelian group by \otimes , then this is an isomorphism of groups.