LECTURE 21: SPECTRAL SEQUENCES

In homological algebra, a short exact sequence of chain complexes gives rise to a long exact sequence of homology groups. You could think of a short exact sequence as a 2-stage filtration of a chain complex. A more arbitrary filtered chain complex gives rise to a generalized version of a long exact sequence, called a spectral sequence.

Definition 0.1. A *(increasing) filtered graded abelian group* is a pair $(A_*, \{F_s A_*\}_s)$ where A_* is a graded abelian group, and a sequence of sub-groups

$$F_0 A_n \subseteq F_1 A_n \subseteq F_2 A_n \subseteq \cdots$$

such that $A_n = \varinjlim_s F_s A_n$. By convention, $F_s A_n := 0$ for $s < 0$.

Given a filtered graded abelian group $\{F_s A_*\}$, the successive quotients give rise to a bigraded group $Gr_* A_*$ called the *associated graded group*:

$$Gr_s A_n := F_s A_n / F_{s-1} A_n.$$

Definition 0.2. A filtered chain complex $(C_*, \{F_s C_*\}, d)$ is a chain complex C_* and a sequence of sub-chain complexes $F_s C_*$, such that $(C_*, \{F_s C_*\})$ is a filtered graded abelian group.

Since the differential preserves the filtration, the associated graded of a filtered chain complex $\{F_s C_*\}$ inherits the structure of a chain complex

$$d : Gr_s C_n \to Gr_s C_{n-1}.$$

The homology of C_* also inherits a filtration:

$$F_s H_n(C) := \mathrm{im}(H_n(F_s C) \to H_n(C)).$$

The question is the following: what is the relationship between

(1) the homology of the associated graded $H_*(Gr_* C_*)$, and
(2) the associated graded of the homology $Gr_* H_*(C_*)$?

The answer is that there is a *spectral sequence*

$$E^1_{s,t} = H_{s+t}(Gr_s C_*) \Rightarrow H_{s+t}(C_*).$$

What does this terminology mean?

Definition 0.3. A *spectral sequence (of homological type)* is a sequence of differential bigraded abelian groups

$$\{\{E^r_{s,t}\}_{s,t \in \mathbb{Z}}, d_r\}_r.$$

The index r begins somewhere, typically $r \geq 1$ or $r \geq 2$. The differential d_r is a homomorphism

$$d_r : E^r_{s,t} \to E^r_{s-r,t+r-1}$$

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which satisfies $d_r^2 = 0$. For a fixed r, the bigraded abelian group $E^r_{*,*}$ is called the rth *page* of the spectral sequence. Each page is required to be the homology of the previous page:

$$E^{r+1}_{*,*} = H_*(E^r_{*,*}, d_r).$$

A spectral sequence is called *first quadrant* of there exists an r so that $E^r_{*,*}$ is concentrated in the first quadrant, that is:

$$E^r_{s,t} = 0 \quad \text{if } s < 0 \text{ or } t < 0.$$

All of the spectral sequences we will be dealing with will turn out to be first quadrant, so let's make this assumption for now on. This condition implies that for fixed s, t, the maps

$$d_r : E^r_{s,t} \to E^r_{s-r,t+r-1}$$
$$d_r : E^r_{s+r,t-r+1} \to E^r_{s,t}$$

are zero for $r >> 0$. This implies that for fixed s, t and large r, we have

$$E^r_{s,t} = E^{r+1}_{s,t}.$$

These stable values are denoted $E^\infty_{s,t}$.

Let A_* be a graded abelian group. We use the notation

$$E^r_{s,t} \Rightarrow A_{s+t}$$

and say that the spectral sequence *converges* to A_* provided there exists a filtration $\{F_s A_*\}$ on A_* and an isomorphism

$$E^\infty_{s,t} \cong Gr_s A_{s+t}.$$