LECTURE 13: THE HUREWICZ HOMOMORPHISM

In 18.905 you saw that there is a Hurewicz homomorphism

$$h : \pi_1(X) \to \widetilde{H}_1(X)$$

which is abelianization if X is path connected. More generally, there is a natural homomorphism

$$h : \pi_k(X) \to \widetilde{H}_n(X)$$

There are two ways to define this homomorphism.

(1) View elements of $\pi_k(X)$ as homotopy classes of maps $(I^k, \partial I^k) \to (X, *)$. By triangulating I^k, you obtain a relative cycle in the relative singular complex $S_*(X, *)$.
(2) Letting $[\iota_k]$ be the fundamental class in $\widetilde{H}_k(S^k)$, send a representative $f : S^k \to X$ to $f_*[\iota_k] \in \widetilde{H}_k(X)$.

The second perspective makes it easier to verify that h is a homomorphism, using the fact that the sum of maps $f, g : S^k \to X$ is represented by the composite

$$S^k \xrightarrow{\text{pinch}} S^k \vee S^k \xrightarrow{f \vee g} X \vee X \xrightarrow{\text{fold}} X.$$

There is a relative Hurewicz homomorphism

$$h : \pi_k(X, A) \to H_k(X, A).$$

Again, there are two perspectives:

(1) View elements of $\pi_k(X, A)$ as homotopy classes of maps $I^k \to X$ with one face constrained to A, and the other faces constrained to $*$. By triangulating I^k, you obtain a relative cycle in the relative singular complex $S_*(X, A)$.
(2) Letting $i : A \to X$ be the inclusion, define h to be the composite

$$\pi_k(X, A) = \pi_{k-1}(F(i)) \to \pi_{k-1}(\Omega C(f)) \cong \pi_k(C(f)) \xrightarrow{h} \widetilde{H}_k(C(f)) \cong H_k(X, A).$$

The proof of the following theorem will be given next time.

Theorem 0.1 (Hurewicz theorem)**.** Suppose that X is an $(m-1)$-connected CW-complex. Then the Hurewicz homomorphism

$$\pi_k(X) \to \widetilde{H}_k(X)$$

is an isomorphism if $k = m$ and is an epimorphism if $k = m + 1$.

We may use this theorem, and homotopy excision, to deduce the following theorem.

Theorem 0.2 (Homology Whitehead theorem)**.** Suppose that $f : X \to Y$ is a homology isomorphism between simply connected CW-complexes. Then it is a weak equivalence, and hence a homotopy equivalence.

Remark 0.3. The simply connected hypothesis is important.

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Remark 0.4. We will prove that weak equivalences are homology isomorphisms. Thus, by using cellular approximation, the CW-complex hypotheses in Theorems 0.1 and 0.2 may be removed. In the latter, you then only get a weak equivalence.