

## LECTURE 12: HOMOTOPY EXCISION

Unlike homology, homotopy groups do not satisfy excision. If they did, then the suspension would induce an isomorphism

$$\Sigma : \pi_k(X) \xrightarrow{\cong?} \pi_k(\Sigma X).$$

This cannot happen: we have seen that  $\pi_2(S^1) = 0$  while  $\pi_3(S^2) = \mathbb{Z}$ . However, excision does hold through a range.

**Theorem 0.1** (Homotopy excision). Suppose that  $f : A \rightarrow X$  is an  $m$ -equivalence and  $g : A \rightarrow Y$  is an  $n$ -equivalence. Assume that one of these maps is the inclusion of a relative CW-complex. Let  $Z$  be the pushout

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ g \downarrow & & \downarrow \\ Y & \xrightarrow{f'} & Z \end{array}$$

Then the induced map of homotopy fibers

$$F(f) \rightarrow F(f')$$

is an  $(n + m - 1)$ -equivalence.

Proofs of related theorems may be found in Hatcher or May.

**Definition 0.2.** A map of pairs  $f : (X, A) \rightarrow (Y, B)$  is an  $n$ -equivalence if the induced map on relative homotopy groups

$$f_* : \pi_k(X, A) \rightarrow \pi_k(Y, B)$$

is an isomorphism for  $k < n$  and an epimorphism for  $k = n$ .

Specializing to the case where  $(X, A)$  and  $(Y, A)$  are relative CW-complexes, we get a form of the homotopy excision theorem which bears a more close resemblance to what you might think of by “excision”.

**Corollary 0.3.** Suppose that  $(X, A)$  and  $(Y, A)$  are relative CW complexes, that the inclusion  $A \hookrightarrow X$  is an  $m$ -equivalence, and that the inclusion  $A \hookrightarrow Y$  is an  $n$ -equivalence. Let  $Z = X \cup Y$ . Then the natural map

$$(X, A) \rightarrow (Z, Y)$$

is an  $(m + n)$ -equivalence.

Specializing to the case of a mapping cone, we obtain

**Corollary 0.4.** Suppose that  $A$  is an  $(m - 1)$ -connected CW-complex and that  $f : A \rightarrow X$  is an  $n$ -equivalence. Then the natural map

$$F(f) \rightarrow \Omega C(f)$$

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is an  $(m + n - 1)$ -equivalence. In particular, if  $(X, A)$  is a relative CW-complex, then the map of pairs

$$(X, A) \rightarrow (X/A, *)$$

is an  $(m + n)$ -equivalence.

Specializing the corollary above to the case where  $X = *$ , we recover the Freudenthal suspension theorem.

**Corollary 0.5** (Freudenthal suspension theorem). Suppose that  $A$  is  $(m - 1)$ -connected. Then the suspension

$$A \rightarrow \Omega\Sigma A$$

is a  $(2m - 1)$ -equivalence.

As a result we deduce that the map

$$\pi_k(S^m) \rightarrow \pi_{k+1}(S^{m+1})$$

is an epimorphism if  $k = 2m - 1$  and is an isomorphism if  $k < 2m - 1$ . The Hopf fibration allowed us to deduce that  $\pi_2(S^2) \cong \mathbb{Z}$ . We therefore have

**Corollary 0.6.** The group  $\pi_n(S^n)$  is isomorphic to  $\mathbb{Z}$ , generated by the identity map.