

## LECTURE 9: FIBRATIONS

**Definition 0.1.** A map  $f : X \rightarrow Y$  is a *fibration* if it satisfies the covering homotopy property (CHP): for all maps  $f$  and homotopies  $H$  making the square commute

$$\begin{array}{ccc}
 Z \times \{0\} & \xrightarrow{f} & X \\
 \downarrow & \nearrow \tilde{H} & \downarrow f \\
 Z \times I & \xrightarrow{H} & Y
 \end{array}$$

there exists a lift  $\tilde{H}$  making the diagram commute.

We saw that cofibrations  $f : X \rightarrow Y$  had the property that the canonical map

$$C(f) \rightarrow Y/X$$

is a homotopy equivalence. On the homework, you will verify:

**Lemma 0.2.** Suppose that  $f : X \rightarrow Y$  is a fibration, and suppose that  $Y$  is pointed. The canonical map

$$f^{-1}(*) \rightarrow F(f)$$

is a homotopy equivalence.

The infinite fiber sequence therefore yields the following corollary:

**Corollary 0.3.** For a fibration  $f : X \rightarrow Y$  with fiber  $F = f^{-1}(*)$ , there is a long exact sequence of homotopy groups

$$\cdots \rightarrow \pi_n(F) \rightarrow \pi_n(X) \xrightarrow{f_*} \pi_n(Y) \rightarrow \pi_{n-1}(F) \rightarrow \cdots$$

Examples of fibrations:

- (1) Covering spaces: the CHP is easily obtained from the homotopy lifting properties of covering spaces.
- (2) Products: a projection  $X \times F \rightarrow X$  is easily seen to be a fibration.
- (3) Locally trivial bundles: a map  $f : X \rightarrow Y$  is a locally trivial bundle with fiber  $F$  if there is an open cover  $\{U_i\}$  of  $Y$  such that there are homeomorphisms  $f^{-1}(U_i) \approx U_i \times F$ . If  $Y$  is paracompact, then  $f$  is a fibration.