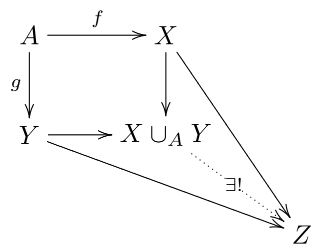
LECTURE 6: PUSHOUTS AND PULLBACKS, THE HOMOTOPY FIBER

1. Pushouts and pullbacks

Let $f : A \to X$ and $g : A \to Y$ be maps of spaces. The *pushout* is the space $X \cup_A Y$ which satisfies the following universal property:



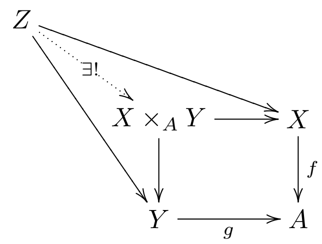
It is defined explicitly as the weak Hausdorfification of the quotient

$$X \cup_A Y = wH((X \amalg Y)/(f(a) \sim g(a) \ : \ a \in A)).$$

Instances of the pushout:

(1) If A is closed in X and Y, $X \cup_A Y$ is the union.
(2) If A is contained in X, $X \cup_A * = X/A$.
(3) Adding an n-cell: $X \cup_{S^{n-1}} D^n$.

Dually, for $f : X \to A$ and $g : Y \to A$, the *pullback* $X \times_A Y$ satisfies the universal property



The pullback is explicitly defined as a subset of the (k-ified) product

$$X \times_A Y = \{(x, y) \in X \times Y \ : \ f(x) = g(y)\}.$$

Instances of the pullback:

(1) $X \times_* Y = X \times Y$.
(2) For $Y = *$, $X \times_A * = f^{-1}(*)$.

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2. The homotopy fiber

Let $f : X \to Y$ be a map of pointed spaces. The homotopy fiber $F(f)$ is defined to be the pullback

$$\begin{array}{ccc} F(f) & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ \underline{\mathrm{Map}}_*(I, Y) & \xrightarrow[ev_1]{} & Y \end{array}$$

where ev_1 is the evaluation at 1 map. Thus $F(f)$ is the space of pairs (x, γ) where $x \in X$ and $\gamma : * \to f(x)$ is a path in Y. The fiber of f is the inverse image $f^{-1}(*)$. The homotopy fiber is an up to homotopy version: it consists of $x \in X$ together with homotopies of $f(x)$ to $*$.

One of the uses of the homotopy fiber is that it completes a long exact sequence of homotopy groups:

$$\begin{aligned} \cdots \to \pi_n(F(f)) \to \pi_n(X) &\xrightarrow{f_*} \pi_n(Y) \\ &\xrightarrow{\partial} \pi_{n-1}(F(f)) \to \cdots \\ \cdots \\ \cdots \to \pi_0(F(f)) \to \pi_0(X) &\xrightarrow{f_*} \pi_0(Y). \end{aligned}$$

In light of the following lemma, this is a generalization of the LES of a pair.

Lemma 2.1. Let $i : A \hookrightarrow X$ be an inclusion. There is an isomorphism $\pi_k(X, A) \cong \pi_{k-1}(F(i))$.

Lemma 2.2. Consider the lifting problem (in Top_*):

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow^{\widetilde{g}} & \downarrow{\scriptstyle g} & & \\ F(f) & \xrightarrow[j]{} & X & \xrightarrow[f]{} & Y \end{array}$$

There is a bijective correspondence:

$$\{\text{lifts } \widetilde{g}\} \\ \updownarrow \\ \{\text{pointed null homotopies } gf \simeq *\}$$

Corollary 2.3. Let Z be a pointed space. The sequence

$$F(f) \to X \xrightarrow{f} Y$$

induces an exact sequence of sets

$$[Z, F(f)]_* \to [Z, X]_* \xrightarrow{f_*} [Z, Y]_*.$$

Letting $Z = S^n$ recovers the exact sequence of homotopy groups at one stage.