

## LECTURE 3: POINTED SPACES AND HOMOTOPY GROUPS

### 1. POINTED SPACES

Let  $\text{Top}_*$  denote the category of pointed spaces. The objects are pairs  $(X, x)$  where  $X \in \text{Top}$  and  $x$  is a point of  $X$ . The morphisms are base-point preserving continuous maps.

Let  $X, Y$  be objects of  $\text{Top}_*$ . The space of pointed maps is given as the subspace of the mapping space consisting of the basepoint preserving maps:

$$\underline{\text{Map}}_*(X, Y) \subseteq \underline{\text{Map}}(X, Y).$$

Note that  $\underline{\text{Map}}_*(X, Y)$  is itself pointed, with basepoint given by the constant map.

The topological adjunction between  $-\times-$  and  $\underline{\text{Map}}(-, -)$  has a pointed analog.

**Proposition 1.1.** The natural map gives rise to a homeomorphism

$$\underline{\text{Map}}_*(X \wedge Y, Z) \approx \underline{\text{Map}}_*(X, \underline{\text{Map}}_*(Y, Z)).$$

This homeomorphism is natural in  $X, Y$ , and  $Z$ .

One could view this as motivation for the smash product construction.

Given an unpointed space  $X$ , one can make a pointed space  $X_+$  by adding a disjoint basepoint. We have adjoint functors

$$(-)_+ : \text{Top} \rightleftarrows \text{Top}_* : \text{forget}.$$

We have the following useful identities.

$$S^1 \wedge S^n \approx S^{n+1}$$

$$X \wedge S^0 \approx X$$

$$X_+ \wedge Y_+ \approx (X \times Y)_+$$

Let  $f, g : X \rightarrow Y$  be pointed maps. A pointed homotopy  $H : f \simeq g$  may be conveniently represented by a map

$$H : X \wedge I_+ \rightarrow Y.$$

Let  $[X, Y]_*$  be the set of pointed maps modulo pointed homotopy.

**Lemma 1.2.** There is a bijection

$$\pi_0 \underline{\text{Map}}_*(X, Y) \cong [X, Y]_*.$$

There are two useful adjoint functors

$$\Sigma : \text{Top}_* \rightleftarrows \text{Top}_* : \Omega$$

given by

$$\Sigma X = X \wedge S^1$$

$$\Omega X = \underline{\text{Map}}_*(S^1, X)$$

Note that there are homeomorphisms  $\Sigma S^n \approx S^{n+1}$ .

The disjoint union of unpointed spaces satisfies a universal property:

$$\begin{array}{ccc}
 X & & \\
 \downarrow & \searrow f & \\
 X \amalg Y & \xrightarrow[f \amalg g]{\exists!} & Z \\
 \uparrow & \nearrow g & \\
 Y & & 
 \end{array}$$

For pointed spaces  $X, Y$ , define the wedge to be the union where we have identified the basepoints

$$X \vee Y = X \cup_* Y.$$

The wedge satisfies a similar universal property in the category of pointed spaces. For  $X, Y, Z \in \text{Top}_*$  we have

$$\begin{array}{ccc}
 X & & \\
 \downarrow & \searrow f & \\
 X \vee Y & \xrightarrow[f \vee g]{\exists!} & Z \\
 \uparrow & \nearrow g & \\
 Y & & 
 \end{array}$$

## 2. HOMOTOPY GROUPS

Let  $X$  be a pointed space. Define

$$\pi_n(X) = [S^n, X]_*$$

For  $n \geq 1$  this is a group. Let

$$\text{pinch} : S^n \rightarrow S^n \vee S^n$$

be the pinch map (identify all elements on the equator). For  $\alpha, \beta : S^n \rightarrow X$ , define  $\alpha + \beta$  to be the composite

$$\alpha + \beta : S^n \xrightarrow{\text{pinch}} S^n \vee S^n \xrightarrow{\alpha \vee \beta} X$$

**Lemma 2.1.** This binary operation descends to give a group structure on  $\pi_n(X)$ .

Alternatively, addition may be described on maps  $(I^n, \partial I^n) \rightarrow (X, *)$  by juxtaposition of cubes. See Hatcher or May for this perspective. This perspective gives an easy geometric proof of the following

**Proposition 2.2.** For  $k > 1$ ,  $\pi_k(X)$  is abelian.

Given a pointed map  $f : X \rightarrow Y$ , there is an induced map

$$\begin{aligned}
 f_* : \pi_k(X) &\rightarrow \pi_k(Y) \\
 [\alpha] &\mapsto [f \circ \alpha].
 \end{aligned}$$

Thus the homotopy groups form a functor

$$\pi_k(-) : \text{Top}_* \rightarrow \text{Groups}.$$