HOMEWORK 11

DUE: MONDAY, MAY 8

Most functors on the category of complex vector spaces extend to constructions in the category of complex vector bundles over a space X. For instance, given complex vector bundles V and W, there exist vector bundles $V \otimes W$ and $\mathrm{Hom}(V, W)$. The fibers over a point $x \in X$ are given by

$$(V \otimes W)_x = V_x \otimes_{\mathbb{C}} V_y$$
$$\mathrm{Hom}(V, W)_x = \mathrm{Hom}_{\mathbb{C}}(V_x, W_x).$$

These constructions are easily produced locally using a trivializing cover. Functoriality can than be used to give transition functions.

1. Suppose that L is a complex line bundle on a paracompact space. Let $\overline{L}$ denote the *conjugate bundle*, where the fibers are given the conjugate action of $\mathbb{C}$.

(a) Show that there is a bundle isomorphism $\overline{L} \cong \mathrm{Hom}(L, \mathbb{C})$, where $\mathbb{C}$ denotes the trivial line bundle.
(b) Conclude that there is a bundle isomorphism $L \otimes \overline{L} \cong \mathbb{C}$.
(c) Deduce that $c_1(\overline{L}) = -c_1(L)$.

2. Viewing $\mathbb{C}P^\infty$ as the space of complex lines in $\mathbb{C}^\infty$, consider the line bundle $\mathcal{L}$ over $\mathbb{C}P^\infty$ whose fiber over a line $L \in \mathbb{C}P^\infty$ *is* the line L. Show that this bundle is isomorphic to the universal line bundle. (Hint: use the quotient map

$$S^\infty \to \mathbb{C}P^\infty$$

to give a model for $EU(1) \to BU(1)$. The universal vector bundle was defined to be $EU(1) \times_{U(1)} \mathbb{C} \to BU(1)$.)

3. Let $\mathcal{L}$ be the restriction of the line bundle of problem 2 to $\mathbb{C}P^n$.

(a) Show that the tangent bundle $T\mathbb{C}P^n$ to $\mathbb{C}P^n$ can be identified with the bundle $\mathrm{Hom}(\mathcal{L}, \mathcal{L}^\perp)$. Here, $\mathcal{L}^\perp$ is the perpendicular bundle of dimension n over $\mathbb{C}P^n$, whose fiber over a line L in $\mathbb{C}^{n+1}$ is the perpendicular space $L^\perp$.

(b) Use the axioms of Chern classes to deduce that

$$c_i(T\mathbb{C}P^{n+1}) = (-1)^i \binom{n+1}{i} x^i$$

where $x \in H^2(\mathbb{C}P^1)$ is the generator given by $c_1(\mathcal{L})$. Hint: show that there is an isomorphism

$$T\mathbb{C}P^{n+1} \oplus \mathbb{C} \cong T\mathbb{C}P^{n+1} \oplus (\overline{\mathcal{L}} \otimes \mathcal{L}) \cong \mathrm{Hom}(\mathcal{L}, \mathbb{C}^{n+1}).$$