HOMEWORK 6

DUE: MONDAY, 3/20/06

1. (Hatcher) (a) Show that $\mathbb{C}P^\infty$ is a $K(\mathbb{Z},2)$.
(b) Show there is a map $\mathbb{R}P^\infty \to \mathbb{C}P^\infty$ which induces the trivial map on $H_*(-)$ but a nontrivial map on $H^*(-)$. How is this consistent with the universal coefficient theorem?

2. (Hatcher) Given abelian groups G and H and CW complexes $K(G,n)$ and $K(H,n)$, show that the map $[K(G,n),K(H,n)]_* \to \mathrm{Hom}(G,H)$ sending a homotopy class $[f]$ to the induced homomorphism $f_* : \pi_n K(G,n) \to \pi_n K(H,n)$ is a bijection.

3. (This may be useful for the next problem) Let $f : X \to Y$ be a pointed map. Show that the cofiber of

$$f \wedge 1 : X \wedge Z \to Y \wedge Z$$

is given by $C(f) \wedge Z$.

4. Let n be greater than 1. Assuming that there is a natural isomorphism

$$\widetilde{H}_{k+n}(X \wedge M(\pi,k)) \cong \widetilde{H}_n(X,\pi)$$

show that the universal coefficient theorem follows from the long exact sequence of the cofiber seqeuence

$$\bigvee_I S^n \to \bigvee_J S^n \to M(\pi,n).$$

The Snake lemma may be useful in the following two problems:

5. A directed system $\{A_i\}$ of abelian groups is a sequence of homomorphisms

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \cdots .$$

A map of directed systems

$$\{A_i\} \to \{B_i\}$$

is a sequence of homomorphisms $A_i \to B_i$ making the diagrams

$$\begin{array}{ccc} A_i & \longrightarrow & A_{i+1} \\ \downarrow & & \downarrow \\ B_i & \longrightarrow & B_{i+1} \end{array}$$

commute. A short exact sequence of directed systems

$$0 \to \{A_i\} \to \{B_i\} \to \{C_i\} \to 0$$

is a short exact sequence at every level

$$0 \to A_i \to B_i \to C_i \to 0$$

(a) Show that $\varinjlim A_i$ is given by the kernel of the map

$$\phi : \bigoplus A_i \to \bigoplus A_i$$

where $\phi(\sum a_i) = \sum f_i(a_i) + a_i$.
(b) Show that $\varinjlim$ is an exact functor from the category of directed systems of abelian groups to the category of abelian groups. That is to say, the direct limit of a short exact sequence of directed systems is a short exact sequence.

6. In a manner precisely analogous to the previous problem, you can consider the category of inverse systems of abelian groups

$$A_1 \xleftarrow{f_1} A_2 \xleftarrow{f_2} A_3 \xleftarrow{f_3} \cdots .$$

(a) Show that a short exact sequence of inverse systems

$$0 \to \{A_i\} \to \{B_i\} \to \{C_i\} \to 0$$

gives rise to an exact sequence

$$0 \to \varprojlim A_i \to \varprojlim B_i \to \varprojlim C_i \to \varprojlim{}^1 A_i \to \varprojlim{}^1 B_i \to \varprojlim{}^1 C_i \to 0$$

(b) Show that for a prime p, the sequence

$$\mathbb{Z} \xleftarrow{p} \mathbb{Z} \xleftarrow{p} \mathbb{Z} \xleftarrow{p} \cdots$$

has

$$\varprojlim = 0$$
$$\varprojlim{}^1 = \mathbb{Z}_p/\mathbb{Z}$$

Here, $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^i$ are the p-adic integers.