

HOMEWORK 5

DUE: MONDAY, 3/13/06

1. (Slightly modified version of Hatcher, Sec. 4.2, problem 30) For a fibration $F \rightarrow E \xrightarrow{p} B$ ($F = p^{-1}(*)$) such that the inclusion $F \rightarrow E$ is homotopic to a constant map, show that the long exact sequence of homotopy groups breaks up into split short exact sequences giving isomorphisms $\pi_n(B) \cong \pi_n(E) \oplus \pi_{n-1}(F)$. In particular, for the Hopf bundles $S^3 \rightarrow S^7 \rightarrow S^4$ and $S^7 \rightarrow S^{15} \rightarrow S^8$ this yields isomorphisms

$$\begin{aligned}\pi_n(S^4) &\cong \pi_n(S^7) \oplus \pi_{n-1}(S^3) \\ \pi_n(S^8) &\cong \pi_n(S^{15}) \oplus \pi_{n-1}(S^7)\end{aligned}$$

Thus $\pi_7(S^4)$ and $\pi_{15}(S^8)$ contain \mathbb{Z} summands.

2. (Hatcher, Sec. 4.2, problem 31) Show that if $S^k \rightarrow S^m \rightarrow S^n$ is a fiber bundle, then $k = n - 1$ and $m = 2n - 1$.

3. Show that there are fiber bundles

$$\begin{aligned}O(n-1) &\rightarrow O(n) \rightarrow S^{n-1} \\ U(n-1) &\rightarrow U(n) \rightarrow S^{2n-1}.\end{aligned}$$

where $O(n)$ is the orthogonal group and $U(n)$ is the unitary group. Deduce that for fixed k the sequences

$$\begin{aligned}\pi_k(U(1)) &\rightarrow \pi_k(U(2)) \rightarrow \pi_k(U(3)) \rightarrow \pi_k(U(4)) \rightarrow \cdots \\ \pi_k(O(1)) &\rightarrow \pi_k(O(2)) \rightarrow \pi_k(O(3)) \rightarrow \pi_k(O(4)) \rightarrow \cdots\end{aligned}$$

eventually stabilize. (The stable values of these homotopy groups is the subject of the celebrated “Bott periodicity theorem”.)

4. (Hatcher, Sec. 4.2, problem 31) Show that a closed simply-connected 3-manifold is homotopy equivalent to S^3 . [Use Poincaré duality, and also the fact that closed manifolds are homotopy equivalent to CW-complexes, from Corollary A.12 in the Appendix of Hatcher. The stronger statement that a closed simply-connected 3 manifold is homeomorphic to S^3 is believed to have been recently proven. This is the Poincaré conjecture.]