

## HOMEWORK 4

DUE: MONDAY, 3/6/06

1. Show that if  $f : X \rightarrow Y$  is a fibration, and  $Y$  is based, then the canonical map

$$f^{-1}(*) \rightarrow F(f)$$

is a homotopy equivalence.

2. A *Serre fibration* is a map  $f : X \rightarrow Y$  satisfying a restricted form of the homotopy lifting property. For all  $n \geq 0$  and all  $g, h$  making the outer square commute

$$\begin{array}{ccc} I^n \times \{0\} & \xrightarrow{g} & X \\ \downarrow & \nearrow \text{dotted} & \downarrow f \\ I^{n+1} & \xrightarrow{h} & Y \end{array}$$

there exists a dotted arrow as above making the diagram commute. The notion of Serre fibration is often times more convenient than the notion of fibration.

Suppose that  $Y$  is pointed. Show that the canonical map  $f^{-1}(*) \rightarrow F(f)$  is a weak equivalence. Deduce that Serre fibrations have long exact sequences of homotopy groups.

3. (Path-loop fibration) Let  $X$  be a pointed space.

(a) Show that the evaluation map

$$ev_1 : \underline{\text{Map}}_*(I, X) \rightarrow X$$

is a Serre fibration, with fiber  $\Omega X$ . (Note: it is actually a fibration.) This fiber sequence is called the path-loop fibration.

(b) Show that if  $p : E \rightarrow X$  is a Serre fibration with contractible total space  $E$ , and fiber  $F$ , then there is a weak equivalence  $F \rightarrow \Omega X$ . (Hint: one approach is to compare with the LES of the path-loop fibration.)

4. Show that all locally trivial bundles are Serre fibrations.

5. Let  $H$  be a closed sub-Lie group of a compact Lie group  $G$ . Show that  $G \rightarrow G/H$  is a locally trivial bundle with fiber  $H$ . (Note: I think that the assumption that  $G$  is compact is not necessary, but might make the problem easier.)