8. CW approximation, whitehead thm

Note Title 3/2/2010

Whitehead Thm

$f : X \to Y$ is an

n-equivalence if

(n-1)-connected

$\pi_k(X, x) \longrightarrow \pi_k(Y, f(x))$

is $\begin{cases} \text{iso} & k < n \\ \text{epi} & k = n \end{cases}$

Rmk f is n connected

$\Longleftrightarrow \forall\ x \in X$

$F(f)_x \longrightarrow X \longrightarrow Y$

$x \in X$, $f(x) \in Y$

$F(f)_x$ is n connected

n-connected $= \pi_{\le n} = 0$

0-connected = path connected

1-connected = simply connected.

Thm (Whitehead thm)

$f: X \to Y$ map of CW cx's

Suppose f is a weak equivalence

$\Rightarrow$ f is a htpy equivalence

Lemma (Compression lemma)

Suppose (X, A) is a CW pair

(Y, B) is any pair, $B \neq \emptyset$

and $\left(\begin{array}{l} X - A \text{ has cells of dim } n \\ \Rightarrow \pi_n(Y, B, y) = 0 \ \forall y \in B \end{array} \right)$

$\Longrightarrow$ $f : (X, A) \to (Y, B)$

is homotopic rel A to a map $X \to B$.

(Htpy rel A means htpy is **CONSTANT** on A)

Rmk $$\left[\left(I^n, \partial I^n, \partial I^n - I^{n-1}\times\{0\}\right), (Y,B,*)\right]$$

$\cong$ ← collapse $\partial I^n - I^{n-1}\times\{0\}$

$$\left[\left(D^n, \partial D^n, *\right), (Y,B,*)\right]$$

Lem

$$(D^n, \partial D^n, *) \xrightarrow[\alpha]{} (Y, B, *) \quad \text{null}$$

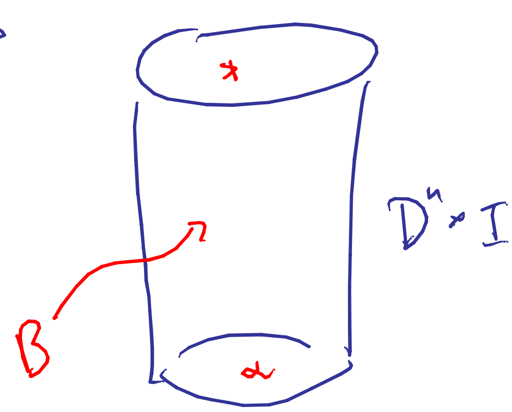
in $\pi_n(Y, B)$

$\Longleftrightarrow$ $\alpha \simeq \alpha'$ rel ∂D^n

s.t. α' maps D^n into B

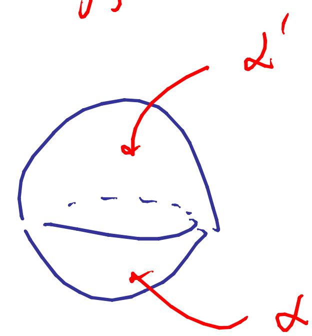
(Pf) $\Leftarrow$ obvious

$\Rightarrow$



$\wr\wr$

$$\frac{D^n \times I}{S^{n-1} \times I} \approx$$



$\square$

Pf of compression

k-skeleton of X

Let $X^{(k)} = A \cup X^{[k]}$

Suppose

$f \simeq_{\text{rel } A} f_k : (X, A) \longrightarrow (Y, B)$

in such a way that

$f_k(X^{(k)}) \subseteq B$

$$\begin{tikzcd} \coprod (D^{k+1}, \partial D^{k+1}) \arrow[r] \arrow[rr, bend right, "\coprod \alpha_i"] & (X^{(k+1)}, X^{(k)}) \arrow[r, "f_k|_{X^{k+1}}"] & (Y, B) \end{tikzcd}$$

hypothesis $\Rightarrow$ each α_i is htpc rel ∂ to a map

$\alpha_i' : D^{k+1} \longrightarrow B$

$h_i : \alpha_i \simeq \alpha_i'$

$$
\begin{tikzcd}
\coprod \partial D^{k+1} \times I \arrow[r] \arrow[dd] \arrow[ddr, phantom, "\ulcorner"] & X^{(k)} \times I \arrow[dd] \arrow[dr, bend left] & \\
& & X^{(n)} \arrow[d] \\
\coprod D^{k+1} \times I \arrow[r] \arrow[rr, bend right, "\coprod h_i"'] & X^{(k+1)} \times I \arrow[r, dashed, "H"] & Y
\end{tikzcd}
$$

$$
\begin{tikzcd}
X^{(k+1)} \arrow[r, "H'"] \arrow[d] & \underline{\mathrm{Map}}(I, Y) \arrow[d] \\
X \arrow[r, "f_k"'] \arrow[ur, dashed, "\tilde{H}'"] & Y
\end{tikzcd}
$$

$$\tilde{H}_{k+1} : X \times I \to Y$$

Let $f_{k+1} = \tilde{H}\big|_{X \times \{1\}}$

$$f_{k+1}\left(X^{(k+1)}\right) \subseteq B$$

$X^{(\infty)} = X$??

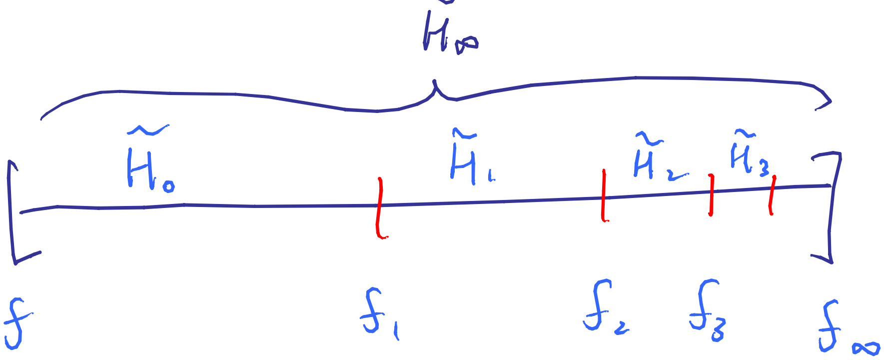
Note $$f_{k+1}\big|_{X^{(k)}} = f_k$$

So

$$\begin{tikzcd}
X^{(0)} \arrow[rr, "f_0"] \arrow[d] & & Y \\
X^{(1)} \arrow[urr, "f_1"'] \arrow[d] & & \\
X^{(2)} \arrow[uurr, "f_2"'] \arrow[d] & & \\
\vdots \arrow[d] & & \\
X^{(\infty)} \arrow[uuuurr, dashed, "f_\infty"'] & &
\end{tikzcd}$$

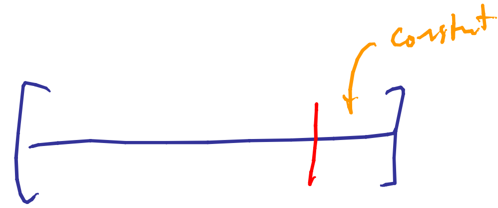
$\lim X^{(k)} = X^{(\infty)}$

is $f_\infty \simeq f$ rel A ?



continuous ??

Note! $\left.\tilde{\tilde{H}}_\infty\right|_{X^{(k)}\times I} = \tilde{\tilde{H}}_k$



$$X\times I = \varinjlim X^{(k)}\times I$$

$$\begin{tikzcd}
X^{(0)}\times I \arrow[r, "\tilde{\tilde{H}}_0"] \arrow[d] & Y \\
X^{(1)}\times I \arrow[ur, "\tilde{\tilde{H}}_1"'] \arrow[d] & \\
\vdots & \\
X\times I \arrow[uuur, dashed, "\tilde{\tilde{H}}_\infty"'] &
\end{tikzcd}$$

$\square$

Cellular approximation thm

Suppose $f : X \to Y$ is a map between CW cx's. Then $f \simeq f'$, where f' is cellular ($f'(X^{[k]}) \subseteq Y^{[k]}$ $\forall k$)

Requires

Lemma

$$\pi_k(Y, Y^{[k]}) = 0$$

pf of lemma similar to pf of $\pi_{<k} S^k = 0$

Given $(D^k, \partial D^k) \xrightarrow[\alpha]{} (Y, Y^{[k]})$

Need to show $\alpha \simeq_{\text{rel } \partial D^k} \alpha'$

$\alpha'(D^k) \subset Y^{[k]}$

D^k cpt $\Rightarrow$ WLOG $Y - Y^{[k]}$ has finite # cells

induction $\Rightarrow$ WLOG $Y - Y^{[k]}$ has one cell.

Use Sard's thm + smooth approximation to argue that $\alpha \simeq_{\text{rel } \partial D^k} \alpha'$ s.t.

$\exists\ y \in Y - Y^{[k]}$, s.t. $y \notin \alpha'(D^k)$

push-off cell away from y $\square$

(pf of cellular approx)

Inductively construct $f_k : X \to Y$ as follows

$$f_k(X^{[i]}) \subseteq Y^{[i]} \quad i \le k$$

$$f_k\big|_{X^{[k+1]}} : (X^{[k+1]}, X^{[k]}) \to (Y, Y^{[k+1]})$$

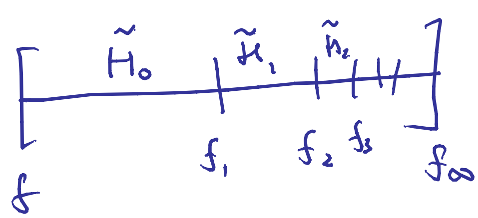
Compression lemma $\Longrightarrow$ $f_k\big|_{X^{[k+1]}} \underset{H_k}{\simeq} g_{k+1}$ rel $X^{[k]}$

$$g_{k+1}(X^{[k+1]}) \subseteq Y^{[k+1]}$$

<u>HEP!</u>

$$\begin{tikzcd}
X^{[k+1]} \arrow[r, "H_k'"] \arrow[d] & \mathrm{Map}(I, Y) \arrow[d] \\
X \arrow[r, "f_k"'] \arrow[ur, dashed, "\tilde{H}_k'"] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
\vdots \arrow[d, dashed] & \\
X^{[k]} \arrow[r, "f_k"] \arrow[d] & Y \\
X^{[k+1]} \arrow[ur, "g_{k+1}"'] \arrow[d, dashed] & \\
X \arrow[uur, dashed, "f_\infty"'] &
\end{tikzcd}$$



∎

(pf of whitehead)

Suppose $X \hookrightarrow Y$ is a CW pair.

w.e. $\Rightarrow$ $\pi_n(Y,X) = 0$

$\Rightarrow$ $\mathrm{Id} : (Y,X) \longrightarrow (Y,X)$

is homotopic, rel X, to a map $\varphi : Y \to X$

i.e. X is a deformation retract of Y

$\Rightarrow$ h.e.

More generally

Assume f is cellular (cellular approx)

$$\begin{tikzcd} X \arrow[rr, "\text{h.e}"] \arrow[dr, hook, "\text{h.e.}"'] & & Y \\ & \mathrm{Cyl}(f) \arrow[ur, "\text{h.e.}"'] & \end{tikzcd}$$

inclusion of a CW cx

□

Thm (CW approximation)

$X \in \mathrm{Top}$

$\exists$ CW cx $\check{X}$

w.e. $\check{X} \xrightarrow[\simeq]{} X$

(pf) Suppose X is path connected, pick bspt

$\Rightarrow$ suffices to find based CW cx $\check{X}$

$\check{X} \longrightarrow X$ iso on $\pi_*(-,*)$
$* \geq 1$

$\tilde{X}_0 = \bigvee_{\alpha_i} S^{\alpha_i}$ α_i runs over all elts of $\pi_{\geq 1} X$

$\tilde{X}_0 \longrightarrow X$ epi on π_*

Inductively

(homotop this to factor thru k-skeleton)

$$\begin{tikzcd}
\coprod S^k \wedge \partial I_+ \arrow[r, "{(\alpha,\beta)}"] \arrow[d] & \tilde{X}_{k-1} \arrow[d, "\text{attached } k+1 \text{ cells}"] \arrow[dr, bend left, "f_{k-1}"] & \\
\coprod S^k \wedge I_+ \arrow[r, "hep"] \arrow[rr, bend right] & \tilde{X}_k \arrow[r, dashed, "f_k"] & X
\end{tikzcd}$$

$\alpha,\beta \in \pi_k \tilde{X}_{k-1}$

$f_k(\alpha) = f_k(\beta)$

iso on $\pi_{\leq k-1}$
epi on π_k

$\square$

<u>Note</u>: All f_k automatically epi:

$$\begin{tikzcd} \pi_i \tilde{X}_0 \arrow[rr, two heads] \arrow[dr] & & \pi_i X \\ & \pi_i \tilde{X}_k \arrow[ur, "f_k"'] & \end{tikzcd}$$

$\tilde{X}_k$ obtained from $\tilde{X}_{k-1}$ by attaching $(k+1)$-cells

$\Longrightarrow$ $\pi_i \tilde{X}_{k-1} \xrightarrow{\cong} \pi_i \tilde{X}_k \quad i \leq k-1$

$\Longrightarrow$ $f_k : \pi_i \tilde{X}_k \to \pi_i X$ iso $i \leq k-1$

<u>mono for $i = k$?</u>

$$\begin{tikzcd} (\tilde{\alpha}, \tilde{\beta}) \in \pi_k \tilde{X}_{k-1} \arrow[r, "f_{k-1}"] \arrow[d] & \pi_k X \\ \pi_k \tilde{X}_k \arrow[ur, "f_k"'] & \end{tikzcd}$$

$\rotatebox{90}{$\in$}$
(α, β)

suppose $f_k(\alpha) \simeq f_k(\beta)$. Choose $\tilde{\alpha}, \tilde{\beta}$, lifts

$f_{k-1}(\tilde{\alpha}) \simeq f_{k-1}(\tilde{\beta}) \Longrightarrow \alpha \simeq \beta$

$\square$