# 7 - Fiber sequences — Whitehead thm

Note Title 2/18/2010

## Homotopy fiber

$X \xrightarrow{f} Y$ any map in $\mathrm{Top}_*$

$$\begin{tikzcd} F(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\ \mathrm{Map}_*(I, Y) \arrow[r, "ev_1"'] & Y \end{tikzcd}$$

$F(f) =$ htpy fiber

**Lem**

$$\begin{tikzcd} & W \arrow[dl, dashed, "\tilde{g}"'] \arrow[d, "g"] & \\ F(f) \arrow[r] & X \arrow[r, "f"'] & Y \end{tikzcd}$$

$\{$ pointed null htpies $fg \simeq * \}$

$\updownarrow$

$\{$ pointed lifts $\tilde{g}\}$

$$
\begin{tikzcd}
W \arrow[dr, dashed, "\tilde{g}"] \arrow[drr, bend left, "g"] \arrow[ddr, dashed, bend right, "H"'] & & \\
& F(f) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
& \mathrm{Map}_*(I, Y) \arrow[r] & Y
\end{tikzcd}
$$

Consequence:

$$[W, F(f)]_{(*)} \longrightarrow [W, X]_{(*)} \longrightarrow [W, Y]_{(*)}$$

is an exact sequence of pointed sets.

## Relationship to fiber:

$$
\begin{tikzcd}
& p^{-1}(*) \arrow[dl, dashed] \arrow[d] & \\
F(p) \arrow[r] & E \arrow[r, "p"'] & B
\end{tikzcd}
$$

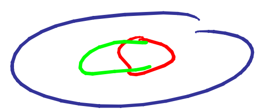
Lem: $p$ fibration $\Rightarrow$ this map is a (pointed) h.e.

(homework)

Lem: $i : A \hookrightarrow X$

$$\pi_k(X,A) \cong \pi_{k-1}(F(i))$$

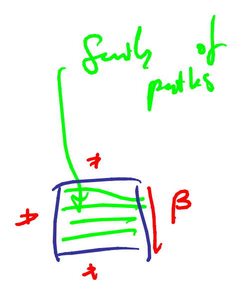
$$F(i) \hookrightarrow \underline{\mathrm{Map}}(I,X) \times A$$



$\partial I^{k-1} \longrightarrow *$

$I^{k-1} \longrightarrow F(i)$

$\updownarrow$

$I^{k-1} \xrightarrow{\beta} A$ s.t. 


$I^{k-1} \times I \xrightarrow{\alpha} X$

# Fiber sequence

$$X \xrightarrow{f} Y$$

get

$$\begin{tikzcd}
F(q) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & F(p) \arrow[r, "q"'] \arrow[d, phantom, "\cong" rotate=90] & F(f) \arrow[r, "p"'] & X \arrow[r, "f"'] & Y \\
\Omega X \arrow[r, "-\Omega f"'] & \Omega Y & & &
\end{tikzcd}$$

$$\begin{tikzcd}
\Omega Y \arrow[r] \arrow[d] & * \arrow[d] \\
F(f) \arrow[r] \arrow[d] & X \arrow[d] \\
\underline{\mathrm{Map}}_*(I, Y) \arrow[r, "ev_1"'] & Y
\end{tikzcd}$$

fibration

fibration

Here we are using $[Z, \Omega W]$ is a gp

ab gp ($[Z, \Omega^2 W]$)

<u>Consequence</u> $\quad X \xrightarrow[f]{} Y$ any map

get LES:

$$\cdots \to \pi_{n+1}(Y) \longrightarrow \pi_n(F(f)) \longrightarrow \pi_n(X) \longrightarrow \pi_n(Y) \to \cdots$$

$$\| $$

$$\begin{cases} \pi_n(f^{-1}(*)), & \text{if } f \text{ is a fibration} \\ \\ \pi_{n+1}(Y, X), & \text{if } f \text{ is an inclusion of a subspace} \end{cases}$$

## Example

$$S^1 \longrightarrow S^3 \xrightarrow[\eta]{} S^2$$

gives

$$\begin{array}{ccccccccccc}
 & & & & & & \mathbb{Z} & & \mathbb{Z} & & \\
 & & & & & & \| & & \| & & \\
\pi_3 S^2 & \longrightarrow & \pi_2 S^1 & \longrightarrow & \pi_2 S^3 & \longrightarrow & \pi_2(S^2) & \longrightarrow & \pi_1(S^1) & \longrightarrow & 0 \\
\uparrow & & & & & & & & & & \\
\pi_3 S^3 & \longleftarrow & \pi_3 S^1 & & & & & & & &
\end{array}$$

$$\pi_n S^3 \xrightarrow[\cong]{} \pi_n S^2$$

for all $n > 2$.

## Hopf invariant

$$f : S^{2n-1} \longrightarrow S^n$$

$$\tilde{H}^*(Cf) : \quad \begin{array}{lcc} y\,\mathbb{Z} & \circ & 2n \\ & \big( f & \\ x\,\mathbb{Z} & \circ & n \end{array}$$

$n \geq 2$

$$x^2 = HI(f) \cdot y$$

$$HI(f) \in \mathbb{Z}$$

e.g.

$$S^3 \longrightarrow S^2 \longrightarrow \mathbb{C}P^2$$

$$HI = 1$$

Similarly, corresponding to $\mathbb{H}$, $\mathbb{O}$

$$S^3 \longrightarrow S^7 \xrightarrow{\nu} S^4 \qquad (\mathbb{H})$$

$$S^7 \longrightarrow S^{15} \xrightarrow[\sigma]{} S^8 \qquad (\mathbb{O})$$

all have HI 1

$\mathbb{D}$ = real division algebra

$\Rightarrow$ get a "Hopf fibration"

$HI = 1$

__Thm__ (Adams) The only maps of $HI$ 1 are $\eta, \nu, \sigma$

__Cor__ the only real division algebras are $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$.