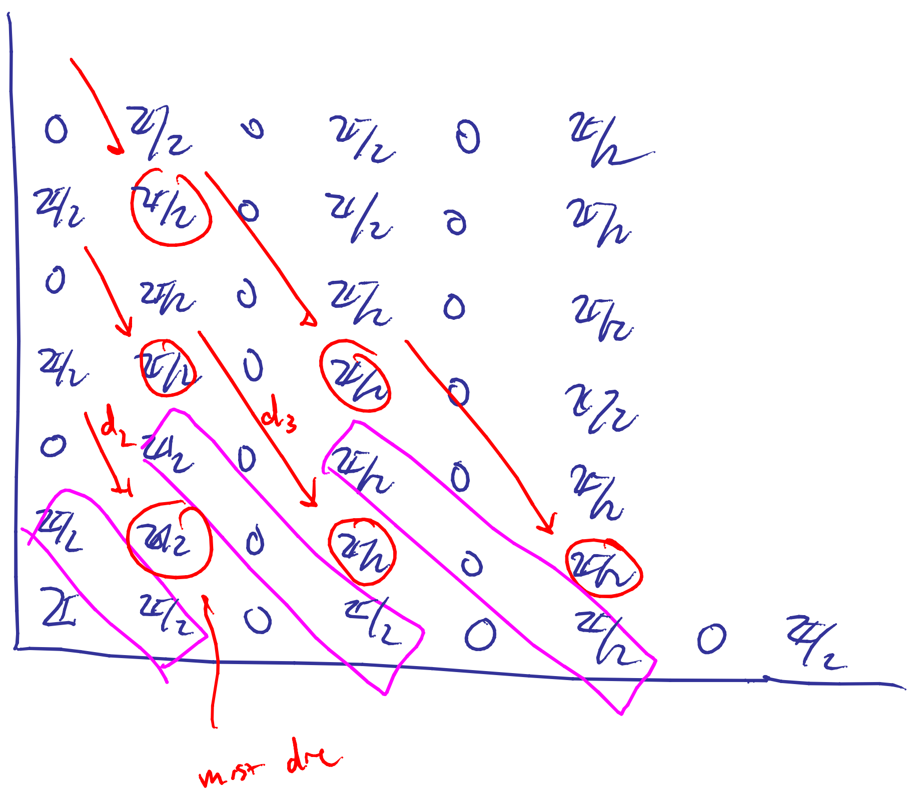
17 More examples, cohomological version

Note Title 4/6/2010

<u>Hidden extension</u>:

$$K(\mathbb{Z}/2,1) \to K(\mathbb{Z}/4,1) \to K(\mathbb{Z}/2,1)$$

$$H_* = \mathbb{Z} \quad \mathbb{Z}/4 \quad 0 \quad \mathbb{Z}/4 \quad 0 \quad \mathbb{Z}/4$$



$E_4 = E_\infty$

$\mathbb{Z}/4$, <u>Not</u> $\mathbb{Z}/2 \oplus \mathbb{Z}/2$

$$0 \to \mathbb{Z}/2 \to H_{2n+1}(K(\mathbb{Z}/4,1)) \to \mathbb{Z}/2 \to 0$$

17 - Cohomological Serre Spectral Sequence

<u>Cohomological</u> <u>Serre</u> <u>spectral</u> <u>sequence</u>

A spectral sequence of <u>cohomological</u> type

$$\{E_r^{s,t}\} \Longrightarrow A_{s+t}$$

(first quadrant if for $r \gg 0$ $E_r^{s,t} = 0$ if s or t neg.)

has:

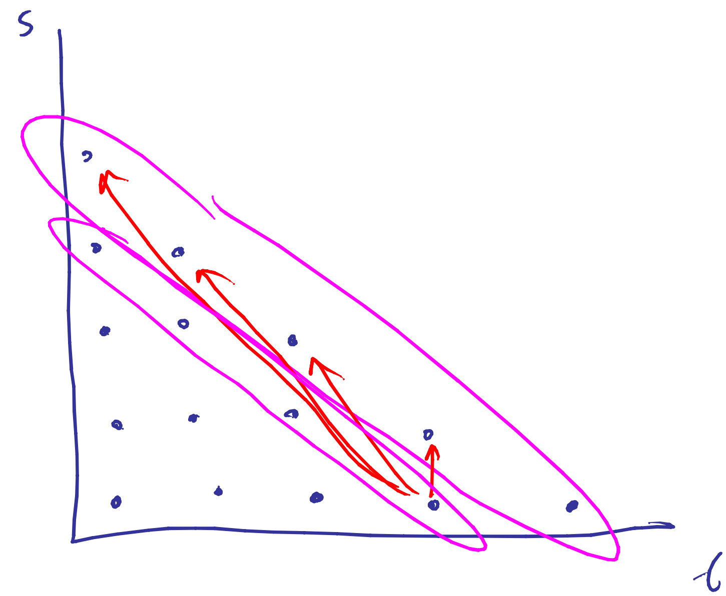
$$d^r : E_r^{s,t} \longrightarrow E_r^{s+r,t-r+1}$$

$$E_{r+1}^{s,t} = H^{s,t}(E_r^{*,*}, d^r)$$

It <u>converges</u> if A has a <u>decreasing</u> <u>filtration</u>

$$A_* = F_0 A_* \supset F_1 A_* \supset F_2 A_* \supset \cdots$$

<u>s.t.</u>
- $\bigcap F_i A_* = 0$
- $E_\infty^{s,t} = \dfrac{F_s A_{s+t}}{F_{s+1} A_{s+t}}$



e.g.

$$C^* = F_0 C^* \supseteq F_1 C^* \supseteq F_2 C^* \supseteq \cdots$$

Cochain cx w/ decreasing filtration

s.t. $\bigcap F_i C^* = 0$

$$E_1^{s,t} = H^{s+t}\left(\frac{F_s C^*}{F_{s+1} C^*}\right) \Rightarrow H^{s+t}(C^*)$$

if finite quantity

(o/w convergence is more complicated
lim' issues)

where $F_s H^n(C^*) = \ker\left(H^n C^* \to H^n C^*/F_s\right)$

r.s.

Serre spectral sequence II

$$F \longrightarrow E \longrightarrow B$$

$$\begin{array}{c} E^{[s]} \\ \downarrow \\ B^{[s]} \end{array} \rightsquigarrow C^*_{sing}(E) = F_0 C^*_{sing} \supset F_1 C^*_{sing} \cdots$$

$$F_s C^*_{sing}(E) = \ker\left(C^*_{sing}(E) \longrightarrow C^*_{sing}(E^{s-1}) \right)$$

Get: $$= C^*_{sing}(E, E^{s-1})$$

$$E_2^{s,t} = H^s(B; H^t(F)) \Longrightarrow H^{s+t}(E)$$

Multiplicative structure

C^* Diff'l graded algebra DGA
w/ decreasing filtration

DGA $\begin{cases} \bullet \text{ cochain cx} \\ \bullet \text{ graded ring} \\ \bullet\ d(xy) = (dx)y + (-1)^{|x|} x (dy) \end{cases}$

$\circ \quad C^* = F_0 C^* \supset F_1 C^* \supset \cdots\cdots$

$$\begin{tikzcd} F_s \otimes F_{s'} \arrow[r, dashed] \arrow[d, hook] & F_{s+s'} \arrow[d, hook] \\ C^* \otimes C^* \arrow[r, "\mu"'] & C^* \end{tikzcd}$$

$\Longrightarrow$ Spectral sequence

$$E_1^{s,t} = H^{s+t}\left(\frac{F_s}{F_{s+1}}\right) \Rightarrow H^*(C^*)$$

is a spectral sequence of algebras

i.e. $(E_r^{s,t}, d_r)$ are DGA's

$$d_r(xy) = d_r(x)\, y + (-1)^{|x|} x\, (d_r y)$$

$$\left[x \in E_r^{s,t} \Rightarrow |x| = s+t \right]$$

$H^* C^*$ is a filtered ring

$$F_s \otimes F_{s'} \longrightarrow F_{s+s'}$$

$$\Rightarrow \mathrm{Gr}_s H^n C^* \otimes \mathrm{Gr}_{s'} H^{n'} C^* \longrightarrow \mathrm{Gr}_{s+s'} H^{n+n'} C^*$$

$\Rightarrow \quad \mathrm{Gr}_* H^* C^*$ (bi)-graded ring

$$E_\infty^{s,t} \cong \mathrm{Gr}_s H^{s+t}(C^*)$$

iso of bi-graded rings.

<u>Thm:</u> The Serre spectral seq:

$$H^s(B; H^t(F)) \Rightarrow H^{s+t}(E)$$

is multiplicative.

Unfortunately: $C^*_{sing}(E, E^{s-1})$ is not a filtered DGA in the sense above, so proof is slightly indirect...

---

filter $B \times B = \varinjlim F_s(B \times B)$

$$F_s(B \times B) = \bigcup_{s_1+s_2=s} B^{s_1} \times B^{s_2}$$

(cw filtration on $B \times B$)

$$\Longrightarrow \quad F_s(E \times E) = \bigcup_{s_1+s_2=s} E^{s_1} \times E^{s_2}$$

$$\begin{tikzcd} E \arrow[r, "\tilde{\Delta}"'] \arrow[d] & E \times E \arrow[d] \\ B \arrow[r, "\tilde{\Delta}"'] & B \times B \end{tikzcd}$$

cellular approx to Δ

$$\tilde{\Delta}(B^s) \subseteq F_s(B \times B)$$

e.g. E CW or $E \to B$ fibration

$$\Rightarrow \quad \begin{tikzcd} & E\times\{0\} \arrow[r, "\Delta"] \arrow[d] & E\times E \arrow[d] \\ E\times\{1\} \arrow[r] \arrow[rru, bend right=60, "\tilde{\tilde{\Delta}}"'] & E\times I \arrow[r] \arrow[ru, dashed] & B\times B \end{tikzcd}$$

In any event

$$\tilde{\tilde{\Delta}}\left(E^s\right) \subseteq F_s\left(E\times E\right)$$

We make use of the following algebraic tool:

Suppose that:

$$C^* = F_0 C^* \supset F_1 C^* \supset \text{-----}$$

$$D^* = F_0 D^* \supset F_1 D^* \supset \text{-----}$$

are filtered cochain complexes.

Give $C^* \otimes D^*$ the filtration

$$F_s(C^* \times D^*) = \sum_{s_1+s_2=s} F_{s_1} C^* \otimes F_{s_2} D^*$$

Then there exists a pairing of spectral sequences:

$$E_r^{s_1,t_1}(C^*) \otimes E_r^{s_2,t_2}(D^*) \xrightarrow[\mu]{} E_r^{s_1+s_2,t_1+t_2}(C^* \otimes D^*)$$

w/

$$d_r(\mu(x \otimes y)) = \mu(d_r(x) \otimes y + (-1)^{|x|} x \otimes d_r(y))$$

__Now:__

Apply this to

$$F_s C^* = C^*_{sing}(E, E^{s-1})$$

$$F_s D^* = D^*_{sing}(E, E^{s-1})$$

to get:

Spectral sequence associated to product filtration

$$\begin{tikzcd} E_r^{s_1,t_1}\left(\begin{smallmatrix} E \\ \downarrow \\ B \end{smallmatrix}\right) \otimes E_r^{s_2,t_2}\left(\begin{smallmatrix} E \\ \downarrow \\ B \end{smallmatrix}\right) \arrow[r, "\mu"'] \arrow[dr] & E_r^{s_1+s_2,t_1+t_2}\left(C^*_{sing}(E)^{\otimes 2}\right) \arrow[d, "\tilde{\Delta}"] \\ & E_r\left(\begin{smallmatrix} E \\ \downarrow \\ B \end{smallmatrix}\right) \end{tikzcd}$$

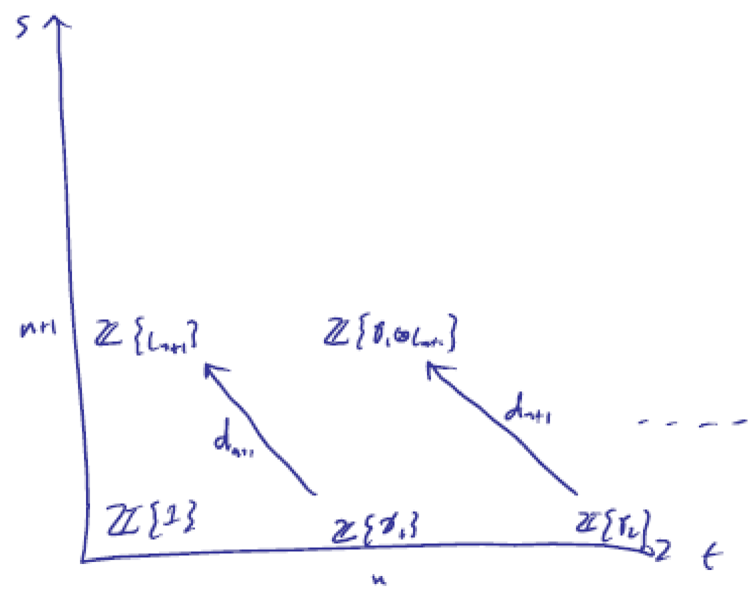
__Here:__ $\tilde{\Delta}$ induces a map of spectral sequences via the cross product

$$\begin{tikzcd} C^*_{sing}(E, E^{s_1-1}) \otimes C^*_{sing}(E, E^{s_2-1}) \arrow[r, "\times"'] \arrow[ddr] & C^*_{sing}(E, E \times E^{s_1-1} \cup E^{s_2-1} \times E) \arrow[d] \\ & C^*_{sing}(E \times E, (E \times E)^{s_1+s_2-1}) \arrow[d, "\tilde{\Delta}^*"] \\ & C^*_{sing}(E, E^{s_1+s_2-1}) \end{tikzcd}$$

//

E.g: compute $H^*(\Omega S^{n+1})$ w/ cup product structure

n even



$$H^*(\Omega S^{n+1}) = \mathbb{Z}\{1, \gamma_1, \gamma_2, \dots\}$$

$$d_{n+1}(\gamma_1^2) = 2(d_{n+1}\gamma_1)\gamma_1$$

$$= 2\iota_1\gamma_1$$

$$\gamma_1^2 = 2\gamma_2$$

$$\vdots$$

$$\gamma_1^n = n!\,\gamma_n$$

$$\gamma_k = \frac{1}{k!}\gamma_1^k$$

$$\gamma_\ell = \frac{1}{\ell!}\gamma_1^\ell$$

$$\Rightarrow \gamma_k \gamma_\ell = \frac{1}{k!\,\ell!}\gamma_1^{k+\ell} = \frac{(k+\ell)!}{k!\,\ell!}\gamma_1^{k+\ell}$$

$$= \binom{k+\ell}{k}\gamma_{k+\ell}$$

$H^*(\Omega S^{n+1})$ is a "divided polynomial ring"

Subring of $\mathbb{Q}[x]$

generated by $\frac{x^n}{n!}$