14 - Spectral sequences

Note Title 3/30/2010

Problem: $F \longrightarrow E \longrightarrow B$ fiber sequence

(i.e. $E \to B$ Serre fibration, fiber F)

What is relationship between

$H_*(F)$

$H_*(E)$?

$H_*(B)$

A: Serre Spectral Sequence

$$E_2^{s,t} = H_s(B; H_t(F)) \Longrightarrow H_{s+t}(E)$$

(we will explain)

Idea: $B =$ CW cx

$$E = \varinjlim E^{[s]} \qquad E^{[s]} = p^{-1}(B^{[s]})$$

$$\downarrow p$$

$$B = \varinjlim B^{[s]}$$

Get a filtration: $C_*^{sing}(E) = \varinjlim C_*^{sing}(E^{[s]})$

Problem: $C_* = \varinjlim \left(F_0 C_* \hookrightarrow F_1 C_* \hookrightarrow F_2 C_* \hookrightarrow \cdots \right)$

filtered chain complex

What is relationship between

$$\bigoplus_s H_*\left(\frac{F_s C_*}{F_{s-1} C_*}\right) \quad \text{and} \quad H_* C_* \quad ?$$

A: Spectral sequence of a filtered complex

$$E^1_{s,t} = H_{s+t}\left(\frac{F_s C_*}{F_{s-1} C_*}\right) \Rightarrow H_{s+t} C$$

Def: A spectral sequence (of homological type)

$$\{E^r_{s,t}\}$$

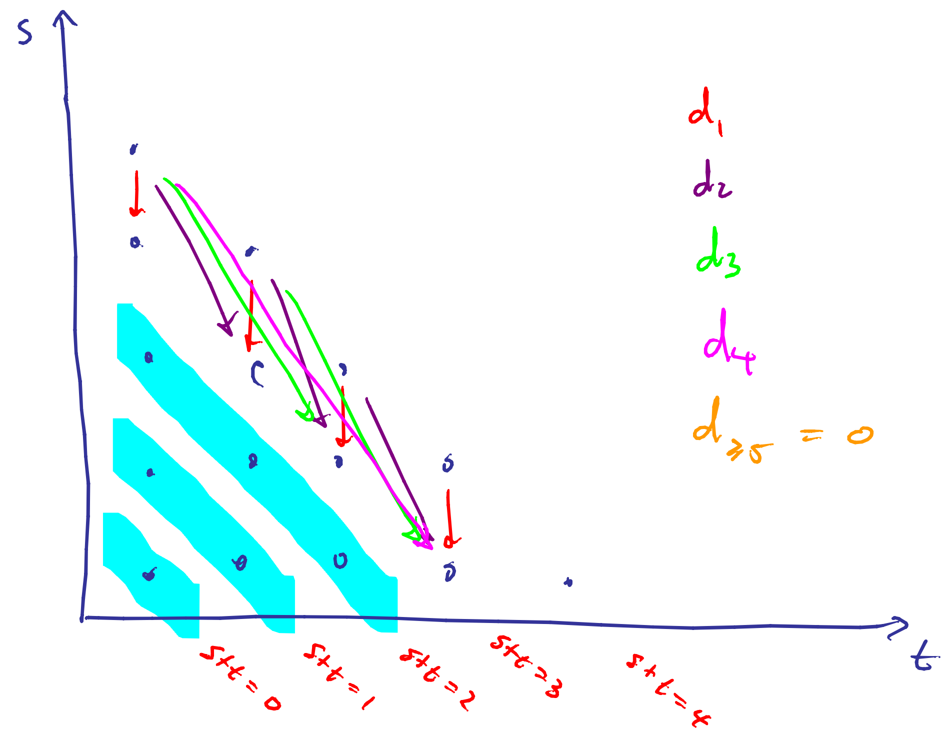
is a sequence of trigraded abelian groups

$$\Big(\text{typically } r \geq 1, \underbrace{s,t \geq 0}_{\text{"First quadrant" (we will always assume)}}\Big)$$

equipped w/ maps

$$d_r : E^r_{s,t} \longrightarrow E^r_{s-r,\, t+r-1}$$

s.t.
- $d_r^2 = 0$
- $E^{r+1}_{s,t} = H_{s,t}(E^r_{*,*}, d_r)$



- $d_r : (s+t) \longrightarrow (s+t)-1$

- In a particular diagonal, only finitely many d_r's are non-zero

For fixed (s,t)

- $E^r_{s,t} = E^{r+1}_{s,t} \quad r \gg 0$

  we call this $E^{\infty}_{s,t}$.

<u>Notation</u>: Let A_n be a graded ab gp

we say: $E^r_{s,t} \Longrightarrow A_{s+t}$

If there exists a filtration

$$F_0 A_n \subseteq F_1 A_n \subseteq F_2 A_n \subseteq \cdots \subseteq A_n$$

S.t. $\quad \bullet \; \varinjlim_s F_s A_n = A_n$

$\bullet \; E^\infty_{s,t} \cong \underbrace{\dfrac{F_s A_{s+t}}{F_{s-1} A_{s+t}}}_{} \; \text{Gr}_s A_{s+t}$

"A_n built out of $\text{Gr}_s A_n$"

In our case: $\text{Gr}_s C_* := \dfrac{F_s C_*}{F_{s-1} C_*}$

$$E^1_{s,t} = H_{s+t}(\text{Gr}_s C_*) \Rightarrow H_{s+t} C_*$$

$$E^\infty_{s,t} = \text{Gr}_s H_{s+t} C_*$$

Slogan: "Spectral sequence commutes Gr_* past H_*"

Construction: "assemble" LES's

$$\cdots \to H_n(F_{s-1} C) \to H_n(F_s C) \to H_n\left(\frac{F_s C}{F_{s-1} C}\right) \xrightarrow[\partial]{} H_{n-1}(F_{s-1} C) \to \cdots$$

Construction:

$$
\begin{tikzcd}
 & & & 0 \arrow[d, dashed] & & \\
 & & & H_{s+t-1}(F_{s-r}) \arrow[r] \arrow[d, dashed] & H_{s+t-1}\left(\frac{F_{s-r}}{F_{s-r-1}}\right) = E^1_{s-r,\,t+r-1} & \\
 & & & \vdots \arrow[d] & & \\
 & H_{s+t}(F_{s-1}) \arrow[d] & & H_{s+t-1}(F_{s-2}) \arrow[r] \arrow[d] & H_{s+t-1}\left(\frac{F_{s-2}}{F_{s-3}}\right) = E^1_{s-2,t+1} & \\
E^1_{s+1,t} = H_{s+t+1}\left(\frac{F_{s+1}}{F_s}\right) \arrow[r] & H_{s+t}(F_s) \arrow[r, "j"'] \arrow[d] & H_{s+t}\left({F_s}/{F_{s-1}}\right) \arrow[r, "\partial"'] \arrow[d, phantom, "\shortparallel"] \arrow[uuur, bend left, "d_r"] & H_{s+t-1}(F_{s-1}) \arrow[r] \arrow[d] & H_{s+t-1}\left(\frac{F_{s-1}}{F_{s-2}}\right) = E^1_{s-1,t} & \\
H_{s+t+1}\left(\frac{F_{s+2}}{F_{s+1}}\right) \arrow[r] \arrow[d, phantom, "\shortparallel"] & H_{s+t}(F_{s+1}) \arrow[d] & E^1_{s,t} & H_{s+t-1}(F_s) & & \\
E^1_{s+2,t-1} & \vdots \arrow[d] & & & & \\
H_{s+t+1}\left(\frac{F_{s+r}}{F_{s+r-1}}\right) \arrow[r] \arrow[d, phantom, "\shortparallel"] & H_{s+t}(F_{s+r-1}) \arrow[d] & & & & \\
E^1_{s+r,\,t-r+1} & H_{s+t}(F_{s+r}) \arrow[d, dashed] & & & & \\
 & H_{s+t}(C) & & & &
\end{tikzcd}
$$

$$E^r_{s,t} = \frac{Z^r_{s,t}}{B^r_{s,t}}$$ Inductively:

$$Z^r_{s,t} = \partial^{-1}\left(\mathrm{Im}\left(H_{s+t-1}(F_{s-r}) \to H_{s+t-1}(F_{s-1})\right)\right)$$

$$B^r_{s,t} = g\left(\ker\left(H_{s+t}(F_s) \to H_{s+t}(F_{s+r-1})\right)\right)$$

Inductively check d_r exists and is well defined.

Note:

$$d_1 : H_{s+t}(F_s/F_{s-1}) \to H_{s+t-1}(F_{s-1}) \to H_{s+t-1}(F_{s-1}/F_{s-2})$$

We think of $E^s_{s,t}$ as a subquotient of $E^1_{s,t}$

$x \in E^1_{s,t}$

$x \in Z^r_{s,t} \iff d_i(x) = 0 \quad i < r$

$x \in B^r_{s,t} \iff x \in \mathrm{im}\, d_i \quad$ some $i < r$

$x \in Z^\infty_{s,t} \iff d_r(x) = 0 \quad \forall\ r$

"x is a permanent cycle"

x P.C. $\implies$ $\partial(x) = 0$

$\implies$ $x = g(x')$

$$H_{s+t}(F_s) \longrightarrow H_{s+t}(C)$$

$$\overset{\varphi}{x'} \longmapsto \tilde{x}$$

Defn a filtration on $H_{s+t}(C)$

$$F_s H_{s+t}(C) := \operatorname{Im}\left(H_{s+t}(F_s) \to H_{s+t}(C) \right)$$

$\implies$ $\tilde{x} \in F_s H_{s+t}(C)$

well defined? suppose x'' is a different choice

$x = g(x'')$

$\implies$ $x' - x'' \in \ker(g)$

$\tilde{x} - \tilde{\tilde{x}} \in F_{s-1} H_{s+t}(C)$

So get well defined map:

$$Z^{\infty}_{s,t} \longrightarrow \frac{F_s H_{s+t}(C)}{F_{s-1} H_{s+t}(C)}$$

kernel $= B^{\infty}_{s,t}$

$$\overset{\varphi}{x} \longmapsto \tilde{x}$$

$\implies E^{\infty}_{s,t} = \dfrac{F_s H_{s+t}(C)}{F_{s-1} H_{s+t}(C)}$