Hw 9 Solutions

Note Title 4/14/2009

14-5

$f(x)$ is clearly diff'ble at $x \neq 0$

$$\lim_{x \to 0} \frac{x^2 \sin(1/x) - 0}{x - 0} = \lim_{x \to 0} x \sin(1/x)$$

Since

$$-|x| \leq x \sin(1/x) \leq |x|$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ 0 & & 0 \end{array} \qquad x \to 0$$

We deduce by the squeeze theorem that

$$f'(0) = \lim_{x \to 0} x \sin(1/x) = 0$$

However, for $x \neq 0$,

$$f'(x) = 2x \sin(1/x) + x^2 \cos(1/x)(-x^{-2})$$

$$= 2x \sin(1/x) - \cos(1/x)$$

Now: If $\lim_{x \to 0} f'(x)$ existed,

then, since $\lim_{x \to 0} 2x \sin(1/x) = 0$

we would deduce that

$\lim_{x \to 0} \cos(1/x)$ existed.

But this limit does not exist.

e.g. $x_n = \frac{1}{2\pi n} \to 0$

$y_n = \frac{1}{2\pi n + \pi} \to 0$

but $\cos\left(\frac{1}{x_n}\right) \to 1$, $\cos(1/y_n) \to -1$

violates sequential cont. thm.

15-1

(a) Let $x_1 < x_2 < \cdots < x_n$

be such that $f(x_i) = 0$

Rolle's thm

$\Longrightarrow$ there exists

$y_i \in (x_i, x_{i+1}) \qquad i = 1, 2, \ldots, n-1$

such that

$f'(y_i) = 0$

There are $n-1$ y_i's.

(b) Case $\underline{\underline{n=1}}$

$f(x) = a_1 x + a_0$

$a_1 \neq 0$

f has exactly one distinct zero

(at $x = \frac{-a_0}{a_1}$)

Case $n=2$

$$f(x) = a_2 x^2 + a_1 x + a_0, \quad a_2 \neq 0$$

$$f'(x) = 2a_2 x + a_1 \qquad \text{degree } 1$$

Suppose f has 3 distinct zeros

$\Longrightarrow$ $f'(x)$ has at least 2 distinct zeros

$\rightarrow\leftarrow$ w/ case $n=1$

Thus f has at most 2 distinct zeros.

Keep going

Case 3

Case 4

⋮

Case n $\Longrightarrow$ $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$

$a_n \neq 0$

$$f'(x) = n a_n x^{n-1} + \cdots + a_1$$

$\Rightarrow f'(x)$ has deg $= n-1$

Suppose f has $n+1$ distinct zeros.

(a) $\Longrightarrow$ f' has at least n distinct zeros.

This contradicts case $n-1$

$\Rightarrow\!\Leftarrow$

Thus f has at most n distinct zeros.

<u>15.2(1)</u>

(a) Let $x_0 \in I$, let $L = \text{length } I$

Suppose $x \in I$, $x \neq x_0$

Since f' is bounded, there is an M so that $|f'(y)| \leq M$ for $y \in I$.

Case 1 $x > x_0$

apply MVT to $[x_0, x]$

$\implies \exists\ c \in (x_0, x)$ s.t.

$$f(x) - f(x_0) = f'(c)(x - x_0)$$

$$\implies |f(x) - f(x_0)| = |f'(c)|\,|x - x_0|$$

$$\leq$$

$$M \cdot L$$

Case 2: $x < x_0$

apply MVT to $[x, x_0]$

$\implies \exists\ c \in (x, x_0)$ such that

$$f(x_0) - f(x) = f'(c)(x_0 - x)$$

$$\implies |f(x) - f(x_0)| = |f'(c)|\,|x_0 - x|$$

$$\leq$$

$$M \cdot L$$

Thus:

$$|f(x) - f(x_0)| \leq M \cdot L$$

for $\underline{\underline{\text{all}}}$ $x \in I$

$$\Longrightarrow \quad f(x) \in [f(x_0) - M \cdot L, f(x_0) + M \cdot L]$$

for all $x \in I$

$\Longrightarrow$ $f(x)$ bounded on I

(b) Consider the function

$$f(x) = \sin(1/x) \quad \text{on} \quad (0, 1]$$

f is bounded, but f' is not!

[Indeed, $f'(x) = \frac{-1}{x^2}\cos(1/x)$]

17.4 (1a)

$f(x) = \sin(x)$

$$\left.\begin{aligned} \sin' x &= \cos x \\ \sin^{(2)} x &= -\sin x \\ \sin^{(3)} x &= -\cos x \\ \sin^{(4)} x &= \sin x \\ &\vdots \end{aligned}\right\} \text{4-fold periodic}$$

n	0	1	2	3	4	5	6	7	8 ...
$f^{(n)}(0)$	0	1	0	-1	0	1	0	-1	0 ..

get! Taylor series

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\cdots$$

<u>Taylor's thm</u> $R_n(x_0) = f(x_0) - T_n(x_0)$

there is a c_n between 0 and x_0
such that

$$R_n(x_0) = \frac{f^{(n+1)}(c_n)}{(n+1)!} x_0^{n+1}$$

However, $f^{(n+1)}(x)$ is either $\pm \sin x$
or $\pm \cos x$
depends on n

$$\Longrightarrow \left| f^{(n+1)}(x) \right| \leq 1$$

Thus

$$|R_n(x_0)| = \frac{|f^{(n+1)}(c_n)|}{(n+1)!} |x_0|^{n+1}$$

$$\leq \frac{|x_0|^{n+1}}{(n+1)!}$$

Since $\sum \frac{x_0^{n+1}}{(n+1)!}$ converges absolutely
(Ratio test)

The n^{th} term test implies

$$\frac{|x_0|^{n+1}}{(n+1)!} \longrightarrow 0 \qquad n \to \infty$$

Thus $\quad |R_n(x_0)| \longrightarrow 0 \qquad n \to \infty$

by squeeze thm

$$\Longrightarrow \quad \lim_{n\to\infty} T_n(x_0) = f(x_0)$$

for every $x_0 \in \mathbb{R}$.