Solutions for spring 2007 exam.

Note Title 5/14/2009

1.) We must show L satisfies sup-1 and sup-2.

Sup-1: L is an upper bound.

Suppose not. Then there exists n_0 such that $a_{n_0} > L$. But then, for $n > n_0$,

$$L < a_{n_0} \leq a_n$$

However, by sequence location thm

$$a_n < a_{n_0} \quad \text{for} \quad n \gg 0$$

Contradiction!

Sup-2: Suppose M is an upper bound. We must show $L \leq M$.

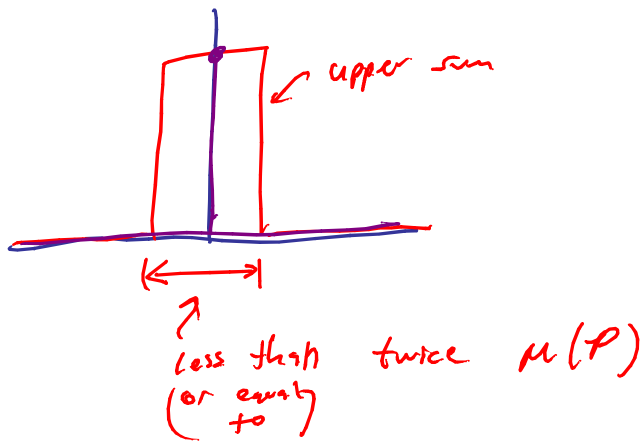
But $a_n \leq M$

$$\underset{\text{Limit location thm}}{\Longrightarrow} L \leq M$$

2.) Pick $\varepsilon > 0$. We must show that there is a $\delta > 0$ so that if $\mu(P) < \delta$,

$$U_f(P) - L_f(P) < \varepsilon.$$

$L_f(P) = 0$.



Choose $\delta = \frac{1}{2}\varepsilon$

$$\Rightarrow U_f(P) - L_f(P) \le 2\mu(P) < \varepsilon$$

3.)

(a) This is uniformly continuous.

Pick $\varepsilon > 0$. Choose $\delta = \varepsilon$

We check, for $|x - x'| < \delta$, $x, x' \in (1, \infty)$

$$\left|\frac{1}{x} - \frac{1}{x'}\right| = \left|\frac{x' - x}{x x'}\right| \underset{x,\, x' > 1}{<} |x' - x| < \delta = \varepsilon$$

(b) This is Not uniformly continuous.

For suppose it were. [I chose to give a full-on proof]

Pick $\varepsilon = 1$. Then there is a δ so that: if $|x - x'| < \delta$

$$\Rightarrow \left|\frac{1}{x} - \frac{1}{x'}\right| < 1$$

$$\shortparallel$$

$$\frac{|x - x'|}{|x||x'|}$$

But set $x' = \frac{\delta}{2}$, $x = \frac{1}{2}$

Then we have, for $0 < x < \frac{\delta}{4}$

$$1 > \frac{|x - x'|}{|x||x'|} = \frac{|x - \frac{\delta}{2}|}{|x|\frac{\delta}{2}} > \frac{\frac{\delta}{4}}{|x|\frac{\delta}{2}} = \frac{2}{x}$$

$$\shortparallel$$

$$4$$

contradiction!

4.) By the 2^{nd} fund. thm of calc,

$g(x)$ is diff'ble (since e^{-t^2} is continuous).

But any diff'ble function is continuous.

5.) $$e^y = \sum_{k \geq 0}' \frac{y^k}{k!} \quad \text{for } \underline{\underline{\text{all}}} \ y \in (-\infty, \infty)$$

$$\Rightarrow e^{-t^2} = \sum_{k \geq 0}' \frac{(-t^2)^k}{k!} = \sum_{k \geq 0}' (-1)^k \frac{t^{2k}}{k!}$$

For all t. (i.e. Radius of convergence $= \infty$)

By uniform convergence of power series,

$\sum_{k \geq 0}' (-1)^k \frac{t^{2k}}{k!}$ converges uniformly on $[0, x]$

$$\underset{\text{Term by term integration}}{\Rightarrow} \int_0^x e^{-t^2} dx = \sum_{k \geq 0}' (-1)^k \frac{x^{2k+1}}{(k!)(2k+1)}$$

this is the taylor series by the Taylor series thm

In particular, this converges,

Since x is arbitrary, $R = \infty$

6 (a) [I really should have stated on which domains we're considering]

$$\left|\frac{\sin(x)}{n^2}\right| \leq \frac{1}{n^2}, \quad \sum_i^\infty \frac{1}{n^2} \text{ converges}$$

$$\underset{\substack{\text{Weierstrass}\\ \text{M-test}}}{\Longrightarrow} \sum_i^\infty \frac{\sin(x)}{n^2} \text{ converges uniformly}$$

(b) on $[0, \infty)$, convergence is <u>NOT</u> uniform.

$$\sqrt{\frac{x}{n}} \longrightarrow 0 \quad (\text{pointwise})$$

but, for $\varepsilon = 1$, for any n

$\sqrt{\frac{x}{n}}$ is <u>unbounded</u>

So $\left|\sqrt{\frac{x}{n}} - 0\right|$ cannot be less than $\varepsilon = 1$ for all x.

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7. Consider $h(x) = |f(x) - g(x)|$.

$h(x)$ is continuous, $[0, 1]$ is compact,

so by the minimum thm, there exists

x_0 s.t. $|f(x_0) - g(x_0)|$ is minimized.

8.)

Apply MVT to $[0, x]$:

There is an $x_0 \in (0, x)$ so that

$$f'(x_0)(x - 0) = f(x) - f(0) = f(x)$$

$\vee$

x