A Two-Phase Free Boundary Problem for Harmonic Measure

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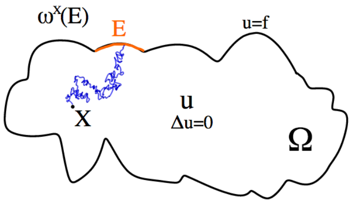
What is Harmonic Measure?

Intuitively, $\omega^X(E)$ is how much a harmonic function "sees" Dirichlet data on E.



Harmonic measure at X. Credit Wikipedia

Harmonic measure is also called "hitting measure" and $\omega^X(E)$ is the probability that a Brownian motion starting at X exits Ω at a point in E.

Ex: For the unit disc $\omega^0 = \frac{\sigma}{2\pi}$. All points of the circle look identical.

Formal Definition of Harmonic Measure

$\Omega \subset \mathbb{R}^n$ simply connected, $f \in C_c(\partial\Omega)$. $\exists U_f \in C^2(\Omega) \cap C(\partial\Omega)$ which satisfies the Dirichlet problem:

$$\Delta U_f(x) = 0, \ x \in \Omega$$
$$U_f(x) = f(x), \ x \in \partial\Omega.$$

Definition

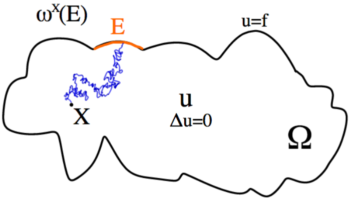
For $X \in \Omega$ the harmonic measure ω^X is the Borel measure such that:

$$\int_{\partial\Omega} f d\omega^X = U_f(X).$$

Note, by the maximum principle, ω^X is a probability measure.

ω VS σ

What is the relationship between harmonic measure ω and surface measure σ for general domains?



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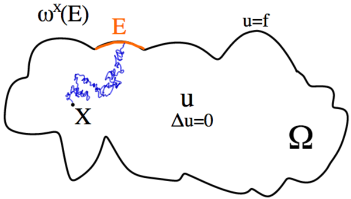

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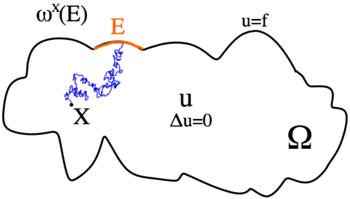

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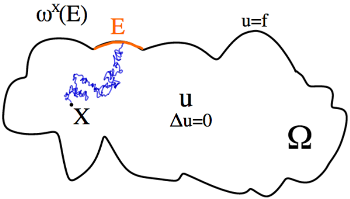

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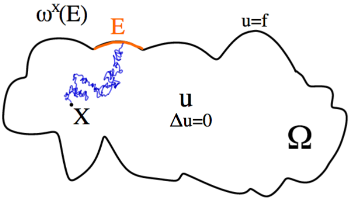

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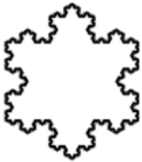
- Integration by parts tells us that $k^X(Q)$ is the normal derivative at Q of the Green function with a pole at X.
- **Ex:** $\Omega = \mathbb{H}, X = x_0 + iy_0 \Rightarrow k^X((x,0)) = \frac{1}{\pi}\frac{y_0}{(x-x_0)^2+y_0^2}$.

ω vs σ. Part II

It is not always true that $\omega << \sigma$.



The von Koch Snowflake. Credit Garnett and Marshall 2005

In the von Koch snowflake there is a set $E \subset \partial\Omega$ such that $\omega(E) = 1$ but $\mathcal{H}^1(E) = 0$. Similarly, there are sets with positive length and 0 harmonic measure.

ω cannot be too "wild" but in general ω may have no relationship with σ.

Motivation: 1-phase problems

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Morally, the oscillation of $\widehat{n}$ should correspond to the oscillation of $\log(k)$.

What about the other direction?

We might ask the converse–can the oscillation of $\log(k)$ tell us about the oscillation of $\widehat{n}$?

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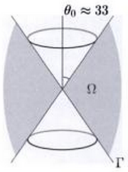

$\log(k) = 0$ but $\partial\Omega$ has a cone point. Figure from Campogna, Kenig and Lanzani 2005

Alt and Caffarelli ('81) found the above counter example.

Note, at the cone point $\partial\Omega$ is not close to any plane.

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- The Alt-Caffarelli cone is not δ-Reifenberg flat for δ small enough.
- This is exactly the additional condition we need for the 1-phase free boundary problem.

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- Kenig and Toro ('03): $\log(k) \in \mathrm{VMO}(\partial\Omega) \Rightarrow \widehat{n} \in \mathrm{VMO}(\partial\Omega)$ (Ω is a vanishing chord arc domain).

- Let $\Omega \subset \mathbb{R}^n$ be a simply connected domain.
- Let ω^+ be the harmonic measure associated to $\Omega^+ := \Omega$
- Let ω^- be the harmonic measure associated to $\Omega^- := \mathbb{R}^n \backslash \overline{\Omega}$.

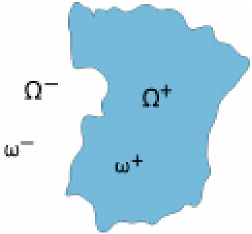

Figure : A typical two-phase setup. Picture by Matthew Badger

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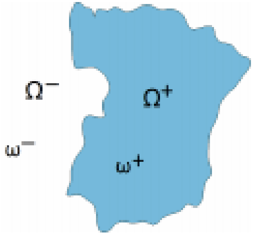

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We will assume that $\omega^+ << \omega^- << \omega^+$ on $\partial\Omega$. Let $h := \frac{d\omega^-}{d\omega^+}$.

Question: What does the regularity of h tells us about the regularity of $\partial\Omega$ (the free boundary)?

That regularity in $\partial\Omega$ implies regularity in h follows immediately from the one-phase analysis: if $\partial\Omega$ is a $C^{k+1,\alpha}$ domain then

$$\log(k^+), \log(k^-) \in C^{k,\alpha} \Rightarrow \log(h) = \log(k^-) - \log(k^+) \in C^{k,\alpha}.$$

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INDEED....

The Main Theorem

Theorem (Main Theorem, E ('14))

Let $\Omega^{\pm} \subset \mathbb{R}^n$ *be NTA domains and assume* $\log(h) \in C^{k,\alpha}(\partial\Omega)$*. Then:*

- *If* $n = 2$ *we conclude* $\partial\Omega$ *is a* $C^{k+1,\alpha}$ *domain.*
- *If* $n \geq 3$ *there exists a* $\delta_n > 0$ *such that if* $\delta < \delta_n$ *and* $\partial\Omega$ *is* δ*-Reifenberg flat then* $\partial\Omega$ *is a* $C^{k+1,\alpha}$ *domain.*

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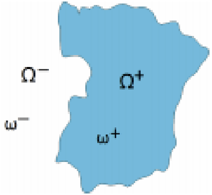
Recall that $\Omega \subset \mathbb{R}^n$
is a simply connected domain, ω^+ is the harmonic measure associated to $\Omega^+ := \Omega$ and ω^- is the harmonic measure associated to $\Omega^- := \mathbb{R}^n \backslash \overline{\Omega}$.
Also $\omega^+ << \omega^- << \omega^+$
on $\partial\Omega$ and $h := \frac{d\omega^-}{d\omega^+}$. We cannot refer to k^+ or k^- as $\partial\Omega$ may be very rough.



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Remarks

- Similar results for C^{∞} or analytic $\log(h)$.
- Some *a priori* flatness assumption is necessary.
- Instead of Reifenberg flat can assume $\partial\Omega$ a graph domain.
- These results are sharp!

δ-Reifenberg flat, NTA domains can be nasty

NTA domains: generalize Lipschitz domains.

Recall: δ-Reifenberg flat $\Rightarrow$ close to a plane at every point and every scale.

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δ-Reifenberg flat, NTA domains can be quite nasty:

- They need not be graph domains or of finite perimeter. See flat snowflakes below.
- In particular it is not necessarily true that $\omega << \sigma$ or $\sigma << \omega$ on a δ-Reifenberg flat NTA domain.

FIGURE : A "flat" snowflake. Photo from generativeart.com

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Note: Every smooth domain is δ-Reifenberg flat (with δ depending on the smoothness) and NTA.

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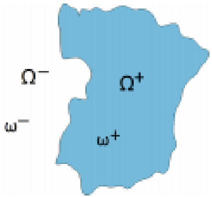
When $n = 2$ there are connections between harmonic measure and the regularity of conformal maps: if $g : \mathbb{D} \to \Omega$ is a conformal isomorphism, $g(0) = A \in \Omega$ then $\frac{d\omega^A}{d\sigma} = \frac{1}{|g'|}$.



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The Situation when $n = 2$

When $n = 2$ there are a plethora of results in both one and two phases. Most relevant to us is the following geometric decomposition: let $\Omega^{\pm}, \omega^{\pm}$ be as above. Then $\partial\Omega^{+} = G^{+} \cup S^{+} \cup N^{+}$ where:

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If $\log(h) \in C^{0,\alpha}$ the above decomposition gives $\partial\Omega = G \cup N$ where G is rectifiable with σ-finite $\mathcal{H}^1$ measure and $\omega^{\pm}(N) = 0$. So even in the case when $n = 2$ our results are new.

Why Flatness is Necessary when $n \geq 2$

Example ($n \geq 4$)

Let $\Omega = \{(x_1, ..., x_n) \mid x_1^2 + x_2^2 > x_3^2 + x_4^2\}$. $\Omega^{\pm}$ are NTA domains. Furthermore, by symmetry $h \equiv 1$. But Ω is not a graph at 0.

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Example ($n = 3$)

$\Omega = \{X \in \mathbb{R}^3 \mid h(X) := x_1^2(x_2 - x_3) + x_2^2(x_3 - x_1) + x_3^2(x_1 - x_2) + x_1x_2x_3 > 0\}$.
The zero set of h separates $\mathbb{R}^3$ into two NTA components (see below).
Symmetry $\Rightarrow \log(h) \equiv c$ a constant. Again Ω is not a graph at 0.

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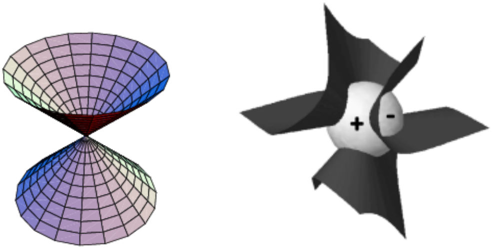
Let $\Omega = \{(x_1, \ldots, x_n) \mid x_1^2 + x_2^2 > x_3^2 + x_4^2\}$. $\Omega^{\pm}$ are NTA domains. Furthermore, by symmetry $h \equiv 1$. But Ω is not a graph at 0.

Example ($n = 3$)

$\Omega = \{X \in \mathbb{R}^3 \mid h(X) := x_1^2(x_2 - x_3) + x_2^2(x_3 - x_1) + x_3^2(x_1 - x_2) + x_1 x_2 x_3 > 0\}$.
The zero set of h separates $\mathbb{R}^3$ into two NTA components (see below).
Symmetry $\Rightarrow \log(h) \equiv c$ a constant. Again Ω is not a graph at 0.



Neither set is δ-Reifenberg flat with δ small. Credit: Mathematica and M. Badger

Arguing Formally

Let $u^{\pm}$ be the Green's functions for $\Omega^{\pm}$. Recall that in a smooth domain $d\omega^{\pm}(Q) = \partial_{\nu^{\pm}(Q)} u^{\pm}(Q) d\sigma$. Arguing formally, we write our problem as:

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$$\begin{gathered} \Delta u^{\pm}(x) = 0, \; x \in \Omega^{\pm} \\ G(Q, \partial_{\nu(Q)} u^{+}(Q)) = -\partial_{\nu(Q)} u^{-}(Q), \; Q \in \partial\Omega \end{gathered} \tag{1}$$

where $\nu(Q)$ is the normal to $\partial\Omega$ pointing into Ω^{+} and $G(Q, a) := h(Q)a$.

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- G **nice** $\Rightarrow$ G **is non-degenerate:** $G(Q, 0) \geq c > 0$**.**
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$G(Q, a) := h(Q)a$ is degenerate: $\nabla u^{+}(Q) = 0 \Leftrightarrow \nabla u^{-}(Q) = 0$.
Note: In an NTA domain $\nabla u^{\pm}(Q)$ may not even exist!

Remarks on Non-Degeneracy

The possibility that $\nabla u^{\pm}(Q) = 0$ (degeneracy) is the biggest obstacle to proving regularity of the free boundary.

Remark

- Proving non-degeneracy is not just convenient but rather "essential".
- Intuitively, $\nabla u^{\pm}(Q)$ should be parallel to ν_Q. If $\nabla u^{\pm}(Q) = 0$ we gain no geometric information. (Analogy: Implicit Function Theorem).
- Non-degeneracy is an important concern in many free boundary problems.

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Issue of degeneracy is most salient in "blowup analysis."

Definition (Blowups)

For $Q \in \partial\Omega$, $r_i \downarrow 0$ define the blowup

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- So if $\frac{u(r_i X+Q)}{r_i} \to u_\infty(X)$ might hope $P = \{u_\infty = 0\}$.
- If $u_\infty \equiv 0 \Leftrightarrow \nabla u(Q) = 0$ then we are lost.

More Blowup Analysis

Figure : Blowing up at a point. Picture courtesy of Matthew Badger.

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In light of these issues we work with,

$$u^{\pm}_{i,Q}(x) := \frac{r_i^{n-2} u^{\pm}(r_i x + Q)}{\omega^{\pm}(B(Q, r_i))}.$$

We abuse notation and sometimes write $u^{\pm}_{r,Q}(x) := \frac{r^{n-2}u^{\pm}(rx+Q)}{\omega^{\pm}(B(Q,r))}$.

- $u^{\pm}_{i,Q}(x)$ is the Green function of $\Omega^{\pm}_i$. Therefore, $u^{\pm}_{i,Q} \to u^{\pm}_{\infty} \neq 0$.

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- $u^{\pm}_{i,Q}(x)$ is the Green function of $\Omega_i^{\pm}$. Therefore, $u^{\pm}_{i,Q} \to u^{\pm}_{\infty} \neq 0$.
- $u^{\pm}_{\infty} \neq 0$ but need to know behavior of $\frac{\omega^{\pm}(B(Q,r_i))}{r_i^{n-1}}$ to get geom. info.

Blowup Analysis Continues

Theorem (Kenig-Toro Crelle's Journal ('06))

Let $\Omega^{\pm} \subset \mathbb{R}^n$ *be NTA. Let* $\log(h) \in \mathrm{VMO}(d\omega^{+})$ *and let* Ω *be* δ*-Reifenberg flat when* $n \geq 3$*. Then* Ω *is vanishing Reifenberg flat.*

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For any $Q \in \partial\Omega$, $r_i \downarrow 0$ *we have* $u^{\pm}_{i,Q}(x) \to p(x)$ *uniformly on compacta (possibly after passing to a subsequence). Here* p *is a 1-homogenous polynomial which may depend on the subsequence.*

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Remarks

- Recall $u^{\pm}_{i,Q}(x) := \frac{r_i^{n-2} u^{\pm}(r_i x + Q)}{\omega^{\pm}(B(Q, r_i))}$.
- Ideally, p is unique and $\{p = 0\}$ is the tangent plane to $\partial\Omega$ at Q.

1. Prove $u^{\pm}$ are locally Lipschitz.
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Outline of The Proof

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2. **Hard Part:** Show the uniform non-degeneracy of $\Theta^{(n-1)}(\omega^{\pm}, Q)$.
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5. Use the Hodograph transform and regularity arguments to finish.
   - Need Schauder-type boundary estimates

Lipschitz Continuity: Monotonicity

Key Tool: monotonicty formula of Alt, Caffarelli and Friedman.

Theorem (ACF Trans. Am. Math. Soc. '84)

Let f be harmonic in $B(x_0, R)\setminus\{f = 0\}$ and vanish at x_0. Then

$$J(f, x, r) := \frac{1}{r^2}\left(\int_{B(x,r)} \frac{|\nabla f^+|^2}{|x-y|^{n-2}} dy\right)^{1/2} \left(\int_{B(x,r)} \frac{|\nabla f^-|^2}{|x-y|^{n-2}} dy\right)^{1/2}$$

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Remarks:

- We can apply this theorem to $u := u^+ - u^-$.
- Monotonicity implies $J(u, Q, 0)$ exists for $Q \in \partial\Omega$.
- Can show $J(u, Q, r) \geq \frac{\omega^+(B(Q,r))}{r^{n-1}}\frac{\omega^-(B(Q,r))}{r^{n-1}}$
- This implies $\limsup_{r\downarrow 0} \sup_{Q\in K\cap\partial\Omega} \frac{\omega^+(B(Q,r))}{r^{n-1}}\frac{\omega^-(B(Q,r))}{r^{n-1}} = C_K < \infty$

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- Clearly $u^{\pm}$ Lipschitz continuous away from $\partial\Omega$.
- If Q is the closest point to x on $\partial\Omega$ and $|x - Q| = r$, NTA estimates yield $\nabla u^{\pm}(x) \lesssim \frac{\omega^{\pm}(B(Q,r))}{r^{n-1}}$.

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- By monotonicty and $\log(h)$ bounded: $\frac{\omega^{\pm}(B(Q,r))}{r^{n-1}} < \infty$.

As a Corollary of the above we get:

Lipschitz continuity of blowups

- $u^{\pm}_{r,Q}(x)$ is uniformly (in r) locally Lipschitz.
- If $r_i \downarrow 0$ and $u^{\pm}_{r_i,Q} \to p$ then $|\nabla u^{\pm}_{r_i,Q}| \overset{*}{\rightharpoonup} \chi_{\{\pm p > 0\}}$.

- We want to show that $\nabla u^{\pm}(Q) \sim \lim_{r \downarrow 0} \frac{\omega^{\pm}(B(Q, r_i)}{r_i^{n-1}}$ does not vanish.

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We use the incongruity between these two observations to show non-degeneracy.

Non-Degeneracy: Frequency Formula

A natural tool to use is Almgren's Frequency Formula:

$$N(r, Q, f) := \frac{r \int_{B(Q,r)} |\nabla f|^2}{\int_{\partial B(Q,r)} f^2}.$$

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Facts about N: Let f be a harmonic function which vanishes at Q.

- $r \mapsto N(r, Q, f)$ is a monotone increasing function.
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For $Q \in \partial\Omega$, define

$$v^{(Q)}(x) := h(Q)u^+(x) - u^-(x).$$

We will study $N(r, Q, v^{(Q)})$.

Non-Degeneracy: Facts about $v^{(Q)}$

We motivate our definition $v^{(Q)}(x) := h(Q)u^+(x) - u^-(x)$:

- As $u^\pm$ are Lipschitz it is clear that $v^{(Q)}$ is Lipschitz.
- For any $r_j \downarrow 0$ define $v_j^{(Q)}(x) := \frac{r_j^{n-2} v^{(Q)}(r_j x + Q)}{\omega^-(B(Q, r_j))}$.

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Finally $v^{(Q)}$ is not harmonic but it is "almost harmonic":

$$\Delta v^{(Q)}(x) = (h(Q)d\omega^+ - d\omega^-)|_{\partial\Omega} = \left(\frac{h(Q)}{h(x)} - 1\right) d\omega^-|_{\partial\Omega} \qquad (2)$$
$$\Rightarrow |\Delta v^{(Q)}(x)| \leq C|x - Q|^\alpha d\omega^-(x)$$

Non-Degeneracy: $N(r, Q, v^{(Q)})$

Note the frequency formula behaves well under blowups. If $r_j \downarrow 0$:

$$N(r_j, Q, v^{(Q)}) = N(1, 0, v_j^{(Q)}) \stackrel{j\to\infty}{\to} N(1, 0, p) \equiv 1.$$

For any $r_j \downarrow 0$ there is a subsequence such that the above holds, therefore

$$\lim_{r\downarrow 0} N(r, Q, v^{(Q)}) = 1.$$

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That $v^{(Q)}$ is "almost harmonic" will imply that $r \mapsto N(r, Q, v^{(Q)})$ is "almost monotonic." In particular:

$$\frac{1}{r}(N(r, Q, v^{(Q)}) - 1) > -Cr^{\alpha/2-1}.$$

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The estimate $\frac{1}{r}(N(r, Q, v^{(Q)}) - 1) > -Cr^{\alpha/2-1}$ is the key quantitative step in proving non-degeneracy. We will outline its proof:

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3. Every term in the numerator includes Δv. On $B_r(Q)$ we have $|\Delta v^{(Q)}| < Cr^{\alpha}$. This gives us the decay we need.

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2. The denominator has terms like $\int_{\partial B(r)}(v^{(Q)})^2$. Standard arguments on NTA domains yield $\int_{\partial B(r)}(v^{(Q)})^2 > c\frac{\omega^-(B(0,r))^2}{r^{n-3}}$
3. Every term in the numerator includes Δv. On $B_r(Q)$ we have $|\Delta v^{(Q)}| < Cr^{\alpha}$. This gives us the decay we need.
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5. **Technical Difficulty:** $\frac{d}{dr}N$ contains terms in which $\nabla v^{(Q)}$ is integrated over sets of dimension $(n-1)$. But $\nabla v^{(Q)}$ only exists a.e. **Fix:** work with $v_\varepsilon^{(Q)}$, a smoothed $v^{(Q)}$.

Non-Degeneracy: Monneau Monotonicty

- Let p be a 1-homogenous polynomial. Monneau ('03) proved that if f is harmonic and vanishes to first order at x_0 then

$$M^{x_0}(r, f, p) := \frac{1}{r^{n+1}} \int_{\partial B_r(0)} (f(x+x_0) - p(x))^2 dx$$

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- Intuitively, M is better for our purposes than N because it detects the derivative of f at x_0.
- Monneau noticed, that for f, p, x_0 as above; $M^{x_0}(r, f, p)$ is monotonic if $N(r, x_0, f)$ is monotonic.
- Similarly, we can show that for any 1-homogenous polynomial p:

$$\frac{1}{r}\left(N(r, Q, v^{(Q)}) - 1\right) \geq -Cr^{\alpha/2-1} \Rightarrow \frac{d}{dr} M^Q(r, v^{(Q)}, p) \geq -Cr^{\alpha/2-1}.$$

So we have

$$\frac{d}{dr}M^Q(r, v^{(Q)}, p) \geq -Cr^{\alpha/2-1}$$

Which allows us to conclude:

THEOREM

For any 1-homogenous polynomial p and any $Q \in \partial\Omega$: **(1)** $M^Q(0, v^{(Q)}, p)$ *exists and* **(2)** $M^Q(r, v^{(Q)}, p) - M^Q(0, v^{(Q)}, p) > -Cr^{\alpha/2}$

Non-Degeneracy: Monneau Part 2

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This theorem immediately implies:

1. $\Theta^{(n-1)}(\omega^-, Q) > 0$
2. Unique blowups.
3. Continuity, in Q, of the blowups $p^Q(x) := \lim u_{i,Q}(x)$.
4. Continuity, in Q, of the density $\Theta^{(n-1)}(\omega^\pm, Q) := \lim_{r\downarrow 0} \frac{\omega^\pm(B(Q,r)}{r^{n-1}}$.

These implications follow from adapting the methods of Garofalo and Petrosyan ('09).

$u^{\pm}$ IS A VISCOSITY SOLUTION

Before we can adapt the work of De Silva-Ferrari-Salsa we need to show that $u := u^+ - u^-$ is a viscosity solution to:

$$\begin{aligned} \Delta u^{\pm}(x) = 0,\ x \in \Omega^{\pm} \\ h(Q)\partial_{\nu(Q)} u^+(Q) = -\partial_{\nu(Q)} u^-(Q),\ Q \in \partial\Omega \end{aligned} \tag{3}$$

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- This follows from a covering argument and the continuity of $Q \mapsto M^Q(r, v^{(Q)}, p^Q)$.

.

$C^{1,s}$ Regularity

Let us recall the work of Caffarelli and De Silva, Ferrari and Salsa:

Theorem (Caf. ('87), DFS ('12))

Let u be viscosity solution to:

$$\begin{aligned} \Delta u^{\pm}(x) &= 0, \; x \in \{u \neq 0\} \\ G(Q, \partial_{\nu(Q)} u^+(Q)) &= -\partial_{\nu(Q)} u^-(Q), \; Q \in \{x = 0\} \end{aligned} \tag{4}$$

If $\{u = 0\}$ is δ-Reifenberg flat and G is "nice" then $\{u = 0\}$ is $C^{1,s}$.

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- In our circumstance $G(Q, a) := h(Q)a$.
- G is "nice" if (1) G is $C^{0,\alpha}$ in Q (2) G grows polynomially in a and (3) $G(Q, 0) \geq c > 0$.
- Our G satisfies the first two conditions but not the third.

$C^{1,s}$ Regularity Part II

Examination of the proof of DFS ('12) shows that the condition $G(Q,0) \geq c > 0$ is used only when ∇u^+ vanishes on $\{u = 0\}$ (it prevents loss of geometric information at that point).

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Let $\Omega^\pm \subset \mathbb{R}^n$ be NTA domains and assume $\log(h) \in C^{0,\alpha}(\partial\Omega)$. Then:

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Now we apply the partial Hodograph transform and use PDE techniques:

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- Follows from an iterative argument similar to the aforementioned article by Jerison ('90)

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Optimal and Higher Regularity

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- Need Schauder estimates for weak solutions. Not explicitly in the literature, but straightforward using the methods of Agmon, Douglis and Nirenberg ('59), ('64).
- Once you have $C^{2,\alpha}$ regularity can apply Schauder estimates to get the full result (Kinderlehrer Nirenberg Spruck '78)

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3. The Parabolic Case
   - Work in progress on the 1-phase case (previous work by Hofmann, Lewis and Nyström).
   - Some work by Nyström on the 2-phase case
   - Recent work of Danielli, Garofalo, Petrosyan and To on parabolic Signorini problem.

Thank You For Listening!

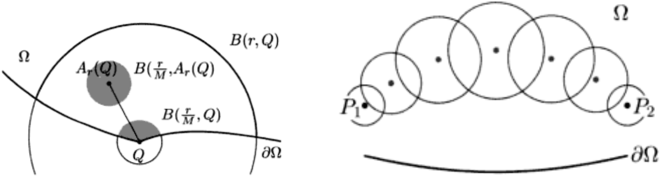

Figure : The Corkscrew and Harnack Chain Conditions. Figures from Campogna, Kenig and Lanzani 2005

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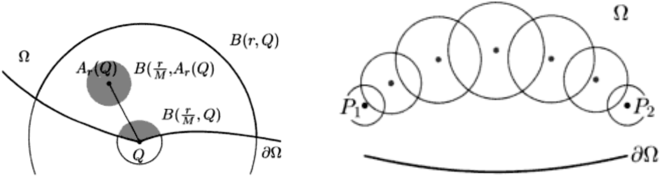

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2. $\overline{\Omega}^c$ satisfies the corkscrew condition.
3. Harnack chain condition:if $\varepsilon > 0$, $P_1, P_2 \in \Omega \cap B(R_0/4, Q)$, for $Q \in \partial\Omega$, and $d(P_i, \partial\Omega) > \varepsilon, |P_1 - P_2| \leq 2^k\varepsilon$ then there exists a chain of balls of length Mk such that $\mathrm{diam} \geq M^{-1} \min_{i=1,2}\{d(P_i, \partial\Omega)\}$.

Definition of Reifenberg flat Domains

Definition

$\Omega \subset \mathbb{R}^n$ is δ-Reifenberg flat if for all $Q \in \partial\Omega$, $r > 0$

$$\theta_r(Q) := \inf_P \frac{1}{r} D[P \cap B(Q,r), \partial\Omega \cap B(Q,r)] \leq \delta.$$

Additionally $\mathbb{R}^n \backslash \partial\Omega$ has two connected components.
We say Ω is **vanishing Reifenberg flat** if $\lim_{r\downarrow 0} \sup_{Q\in K} \theta_r(Q) = 0$.

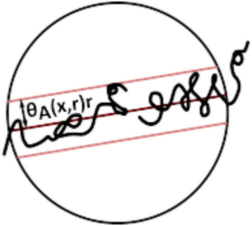

Figure : A slab at scale r. Figure by Matthew Badger

To address issues of higher and optimal regularity we can apply the Hodograph transform to u^+, u^- to obtain the following system in a neighborhood 0 on $\mathbb{H}$.

$$\begin{aligned} L\psi(y) &= 0,\ y \in U \\ L\phi(y) &= 0,\ y \in U \\ \phi(y) &= -\psi(y),\ y \in \{y_n = 0\} \cap \overline{U} \\ \left(\frac{\widetilde{h}(y)}{\psi_n(y)}\right) - \frac{1}{\phi_n(y)} &= 0,\ y \in \{y_n = 0\} \cap \overline{U}. \end{aligned} \tag{5}$$

where $\widetilde{h}(y) = h(y', \psi(y))$ and L is some elliptic operator.

This system is **elliptic and coercive**: imprecisely this means that regularity theory for elliptic boundary value problems should also apply to equation (5).

THEOREM (MAIN THEOREM, E ('14))

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Proof of the Main Theorem

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Comments on the Proof:

- if $\log(h) \in C^{k,\alpha}$ for $k \geq 1$ we know that $\partial\Omega \in C^{2,\alpha}$
- With this *a priori* regularity can apply Schauder-type estimates to get the full theorem.
- Related to work of Kinderlehrer, Nirenberg and Spruck ('78).