NSF PROJECT DESCRIPTION: MAX ENGELSTEIN

Introduction: My research is focused on the study of free boundary problems (FBPs), a class of partial differential equations in which an unknown function, u, satisfies a PDE on a domain, Ω, which itself depends (often in a highly non-linear way) on u. Originally introduced by physicists to understand liquid-solid phase transition (the classic Stefan problem), recently FBPs have been used to model a broad range of real-world phenomena from option pricing [SP07] to tumor growth [QPV14]. Free boundary problems are also ubiquitous in pure mathematics, e.g., in the study of geometric flows [Das14].

For an open, unbounded $\Omega \subset \mathbb{R}^n$ and a Radon measure ω, consider the boundary value problem,

$$\begin{aligned} \Delta u(x) &= 0,\ x \in \Omega \\ u(x) &> 0,\ x \in \Omega \\ u(x) &= 0,\ x \in \partial\Omega \\ \int_\Omega u \Delta \varphi\, dx &= \int_{\partial\Omega} \varphi d\omega,\ \forall \varphi \in C_c^\infty(\mathbb{R}^n). \end{aligned} \tag{1}$$

The function, u, (uniquely determined up to a constant multiple for sufficiently regular Ω) is the **Green function** of Ω with a pole at infinity and ω, is the corresponding **harmonic measure**. Harmonic measures arise in many areas of math, e.g. in probability as the exit distribution of a Brownian motion in Ω.

Equation (1) is overdetermined and does not have a solution for every combination of Ω and ω (as we are prescribing both its Neumann and Dirichlet data). Therefore, *a priori* assumptions on ω should impose additional conditions on the "free boundary", $\partial\Omega$. In particular, we ask the question:

(2) If we know ω is "regular" what can we say about $\partial\Omega$?

The converse question to (2) is a fundamental one in the study of analysis on rough domains; given the regularity of $\partial\Omega$ what can we conclude about the regularity of the harmonic measure it supports? There is a vast literature devoted to this question (see, e.g., [Dah77], [JKe82] and [HM14]) and its subsequent applications (e.g. to the solvability of the Dirichlet problem, [Dah79]).

Question (2) has been studied under various assumptions; in particular, smoothness of $\frac{d\omega}{d\sigma}$ has been shown to imply smoothness of $\partial\Omega$ (see, e.g., [AC81], [Jer90], [KT03]). Other conditions, involving, e.g., the integrability of $\frac{d\omega}{d\sigma}$, [HMU14], and the doubling rate of ω, [KT97], imply weaker notions of regularity on Ω.

One reason these problems are of interest is because they are non-variational, i.e. do not arise as the minimizer of some energy functional. Thus many "geometric" tools and techniques, which were developed to study minimal surfaces and now are being used by the FBP community (e.g. epiperimetric inequalities, see [FS16] and [GSP16]), are not easily adapted to problems concerning harmonic measure. Instead, researchers in this area use a blend of ideas from harmonic analysis and geometric measure theory.

The focus of my proposed project is to use harmonic analysis and geometric measure theory in combination with tools from geometric analysis to study free boundary problems related to harmonic measure.

A Two-Phase Free Boundary Problem for Harmonic Measure In addition to being mathematically natural, two-phase analogues of (2) are often used to model the separation of two different mediums (e.g. plasma and a vacuum in [KNS78]). Given a domain Ω, let ω^+ be the associated harmonic measure and ω^- the harmonic measure of $\overline{\Omega}^c$. We can then ask,

(3) How is the structure of $\partial\Omega$ determined by the relationship between ω^+ and ω^-?

When $\Omega \subset \mathbb{R}^2$ is simply connected and bounded by a Jordan curve, the combined work of Makarov, McMillan and Pommerenke (see [GM05], Chapter 6) tells us that,

$$\omega^+ << \omega^- << \omega^+ \Rightarrow \omega^+ << \mathcal{H}^1|_G << \omega^- << \omega^+, \tag{4}$$

for some 1-rectifiable set $G \subset \partial\Omega$, with the additional property that $\omega^\pm(\partial\Omega \backslash G) = 0$. That is, the free boundary is comprised of a "good part" which is the union of Lipschitz curves on which harmonic measure and Hausdorff measure are comparable, and a "bad part" which is not seen by harmonic measure.

The analogous decomposition in n-dimensions follows from the work of Kenig, Preiss and Toro [KPT09] and then Azzam, Mourgoglou and Tolsa [AMT16]. In essence, their combined work states that if $\Omega, \overline{\Omega}^c$ are non-tangentially accessible domains, then $\omega^+ << \omega^- << \omega^+$ implies that the support of $\omega^\pm$ is $(n-1)$-rectifiable. Furthermore, $\partial\Omega \backslash \sup\, \omega^\pm$ is purely $(n-1)$-unrectifiable (i.e. its intersection with any $(n-1)$-dimensional Lipschitz graph has null Hausdorff measure). Recall that *non-tangentially accessible (NTA)* domains (introduced in [JKe82]), are a generalization of Lipschitz domains in which the boundary behavior of harmonic functions is well understood. These results are remarkable because they use powerful tools from geometric measure theory and harmonic analysis ([Pre87] and [GT16], respectively).

In other words, the free boundary, $\partial\Omega$, is split into two disjoint pieces, the regular set, Γ_r, where there is a tangent at each point, and the singular set, Γ_s, which is not seen by the harmonic measures. From minimal surfaces theory we expect to be able to conclude higher regularity on Γ_r (given smoothness of $\frac{d\omega^-}{d\omega^+}$) and to be able to control the size and structure of the singular set, Γ_s. In this vein there was work by Kenig and Toro [KT06] and Badger [Bad11], [Bad13], classifying the blowups (see below) of the free boundary.

Analysis of Γ_r, the regular points in $\partial\Omega$: In [Eng16a], we addressed the issue of higher regularity on the part of Γ_r, assuming that $\log(\frac{d\omega^-}{d\omega^+})$ has additional smoothness. Our main theorem is the following;

Theorem 1. *[Theorem 1.1 and Theorem 1.2 in* [Eng16a]*] Let $\Omega, \overline{\Omega}^c$ be NTA domains such that $h \equiv \frac{d\omega^-}{d\omega^+}$ satisfies* $\log(h) \in C^{k,\alpha}(\partial\Omega)$ *where $k \geq 0$ and $\alpha \in (0,1)$.*

- *When $n = 2$: $\partial\Omega$ is locally given by the graph of a $C^{k+1,\alpha}$ function.*
- *When $n \geq 3$: there is some $\delta_n > 0$ such that if Ω is a δ-Reifenberg flat domain and $\delta < \delta_n$, then $\partial\Omega$ is locally given by the graph of a $C^{k+1,\alpha}$ function.*

The same result is true if we assume that Ω is a Lipschitz domain and that $\log(h) \in C^{k,\alpha}(\partial\Omega)$.

The δ-Reifenberg flat assumption above is included to ensure that the set of singular points, Γ_s, is empty. If we omit the flatness assumption, then an analogous result holds true locally in Γ_r. The result is also sharp in the sense that there are examples in $\mathbb{R}^3$ showing flatness is necessary.

The key step in the proof of Theorem 1, is to show quantitative non-degeneracy: that there exists a $c > 0$ such that for every subset $F \subset \partial\Omega$, we have $\omega^\pm(F)/\mathcal{H}^{n-1}(F) \geq c > 0$ (this was unknown even assuming that $\partial\Omega$ is a Lipschitz graph). Degeneracy is a fundamental obstacle in two phase free boundary problems; imprecisely, if both phases vanish at a point of the free boundary, then there is no hope of recovering any geometric information there. To prevent both phases from vanishing simultaneously, many two-phase problems have non-degeneracy "baked in" to the free boundary condition (see, e.g., [Caf87], [DFS14]). However, in this problem, $\omega^+(F) = 0 \Leftrightarrow \omega^-(F) = 0$, so *a priori* both phases could disappear simultaneously.

To show non-degeneracy we constructed a function, v, such that Almgren's frequency formula (see [Alm79]) applied to v is "almost monotonic" (i.e. its derivative is bounded from below by a function integrable at zero). Monotonicity formulas, like Almgren's frequency function, are a traditional tool to prove non-degeneracy. However, these formulas usually arise in a variational context (where they come from appropriate modifications of the energy). Since our problem is non-variational, we had to use estimates from harmonic analysis and geometric measure theory to bound the growth of the function from below. In particular, we played two estimates off one another: the first is that a harmonic function which vanishes at a point to first order must grow at least as fast as $|x|^2$ near that point. The second is that the quotient, $\omega^\pm(B(Q,r))/r^{n-1}$, cannot grow or decay faster than any power of r as long as $\partial\Omega$ is flat around $Q \in \partial\Omega$.

Analysis of Γ_s, the singular points in $\partial\Omega$: In [BET15], I, together with Matthew Badger and Tatiana Toro, studied the structure of the singular set, Γ_s. To state our result, recall that a set, S, is a *blowup* of $\partial\Omega$ at Q if there are $r_i \downarrow 0$ such that $(\partial\Omega - Q)/r_i \to S$. We proved:

Theorem 2. *[see Theorem 7.4 in* [BET15]*] Assume that $\Omega, \overline{\Omega}^c$ are NTA domains equipped with harmonic measure ω^+, ω^- as above. Further assume that $h \equiv \frac{d\omega^-}{d\omega^+}$ satisfies* $\log(h) \in \mathrm{VMO}(d\omega^+)$. *Then we can write the free boundary as $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \ldots \cup \Gamma_d$ such that*

- *For $Q \in \Gamma_k$, every blowup of $\partial\Omega$ at Q is the zero set of a k-homogenous harmonic polynomial.*
- *The set $U_k \equiv \Gamma_1 \cup \ldots \cup \Gamma_k$ is relatively open in $\partial\Omega$ for all $k \leq d$.*
- *$\Gamma_1 = \Gamma_r$. Equivalently, $\Gamma_s \equiv \Gamma_2 \cup \ldots \cup \Gamma_d$.*

- *The singular set, $\Gamma_s \equiv \partial\Omega \backslash \Gamma_1$ has upper Minkowski dimension at most $n-3$.*
- *The "even" singular set $\Gamma_2 \cup \Gamma_4 \cup \ldots \cup \Gamma_{2k}$ has Hausdorff dimension at most $n-4$.*

Stratification and dimension estimates of this kind have been obtained for many problems such as minimal surfaces ([Sim93], [Sim95]) and, more recently, the singular set of an FBP (e.g. for the thin obstacle problem in [GP09]). Generally, these arguments require three key ingredients; a monotonicity formula, the knowledge that there is a unique blowup at every point and some uniformity in the rate of blowup.

To prove Theorem 2 we needed new tools as no monotonicity formula has been found or uniqueness of blowups result proven, when $\log(h) \in \text{VMO}$. However, Kenig and Toro [KT06], showed that at every point, $x_0 \in \partial\Omega$, and every scale, $r > 0$, there exists a harmonic polynomial p, such that $\partial\Omega \cap B(x_0, r)$ is well approximated by $\{p = 0\} \cap B(x_0, r)$. This fact, combined with a framework developed by Badger and Lewis [BL15] (which was inspired by the aforementioned paper of Preiss [Pre87]), meant that to prove Theorem 2 it sufficed to study the structure of zero sets of harmonic polynomials.

More precisely, we wanted to understand when a portion of the zero set of a degree, d, harmonic polynomial can be well approximated by the zero set of a harmonic polynomial of lower degree. The central step is an "improvement type" lemma which roughly states the following: if p, h are harmonic polynomials with $\deg(h) = k \leq n = \deg(p)$ and $\{p = 0\}$ is close to $\{h = 0\}$ inside of $B(x_0, r_0)$, then, as $s \downarrow 0$, $\{p = 0\}$ is increasingly well approximated inside of $B(x_0, sr_0)$ by the zero sets of other harmonic polynomials of degree $\leq k$. This is analogous to Alt and Caffarelli's "flat at one scale implies smooth" result in [AC81]. The improvement lemma was obtained by proving Łojasiewicz-type inequalities for harmonic polynomials of bounded degree. These inequalities then allowed us to control the geometry of $\{p = 0\}$ at x_0 by analyzing the first k terms in the Taylor expansion of p around x_0.

Future work related to the two-phase problem: In work in progress with Badger and Toro, we are combining the methods of [Eng16a] and [BET15], to study the structure of the singular set, when $\log(h) \in C^{0,\alpha}$. Essentially, we want to emulate the work of Simon [Sim83], [Sim95] and prove rectifiability of the singular set and parametrization of the free boundary near isolated singular points. To do so, we first extend the monotonicity formula of [Eng16a] to singular points by refining our estimates on $\omega^{\pm}(B(Q, r))/r^{n-1}$. Then we establish an epiperimetric inequality for this almost-monotone quantity. This result should be of particular interest, as epiperimetric inequalities at singular points are often hard to prove.

In the long term, we hope that our work provides a blueprint for using monotonicity formulas and Łojasiewicz-type inequalities in other non-variational FBPs. Łojasiewicz inequalities in particular have been used with great success in the geometric analysis community (see, e.g., [Sim83], [CM15]), and we believe that they should have important applications to free boundary theory as well.

Parabolic Free Boundary Problems: We now discuss a one-phase problem for caloric measure. Caloric measure is the time dependent version of (2),

$$
\begin{aligned}
(\partial_t + \Delta_x) u(x,t) &= 0, \ \forall (x,t) \in \Omega \subset \mathbb{R}^{n+1} \\
u(x,t) &> 0, \ \forall (x,t) \in \Omega \\
u(Q,\tau) &= 0, \ \forall (Q,\tau) \in \partial\Omega \\
\int_\Omega u(y,s)(\partial_s - \Delta_y)\varphi(y,s)\,dyds &= \int_{\partial\Omega} \varphi(Q,\tau)\,d\omega(Q,\tau), \ \forall \varphi \in C_c^\infty(\mathbb{R}^{n+1}).
\end{aligned}
\tag{5}
$$

The Radon measure, ω, above is the **caloric measure** (with pole at infinity) associated to Ω and we ask, as before, what the regularity of ω tells us about the regularity of $\partial\Omega$. While the literature on the one-phase problem for harmonic measure is quite extensive, many of the analogous results in the parabolic setting are yet unproven. In fact, even under the additional hypothesis

$$\frac{d\omega}{d\sigma} \geq 1 \ d\sigma\text{-almost everywhere and } |\nabla u| \leq 1 \text{ everywhere in } \Omega \tag{6}$$

(where σ is a parabolic analogue of surface measure on $\partial\Omega$) there was no description of u or $\partial\Omega$. The work of Alt and Caffarelli [AC81] (see also [KT04]) implies that any "flat" (i.e. without cone points) solution to (2) and (6) must be a half-plane solution (i.e. $\Omega = \{x_n > 0\}$ and $u = x_n^+$). They also showed that non-flat solutions exist (when $n \geq 3$). This result was crucial to the development of regularity theory for free boundary problems for harmonic measure and whether it remained true in the parabolic setting was an important open question for some time (see the discussion at the end of Section 5 in [HLN04] or [Nys06a], [Nys12]). In [Eng15], this was answered in the affirmative; all "flat" solutions to (5) and (6) are half-planes.

My investigation here is inspired by work of Andersson and Weiss [AW09] who considered solutions in the sense of "domain variations" to

$$(\partial_t - \Delta_x)u(x,t) = 0, (x,t) \in \{u > 0\} \text{ and } |\nabla u(x,t)| \equiv 1,\ (x,t) \in \partial\{u > 0\}. \tag{7}$$

The relationship between domain variation solutions to (7) and solutions to (5) and (6) remains unclear (see the end of Section 1 in [Nys12]). Thus we needed to adapt the methods of [AW09] to our setting.

Hofmann, Lewis and Nyström, [HLN04], first considered (5) and introduced the concept of a parabolic chord arc domain, which is an anisotropic version of the uniformly rectifiable domains in [DaS93]. More precisely, they defined a measure ν, which averages, in a scale invariant manner, how well $\partial\Omega$ is approximated by hyperplanes in an L^2 sense. A **parabolic chord-arc domain** is one in which the surface measure, σ, is Ahlfors-regular (i.e. is comparable to Hausdorff measure supported on a plane) and in which ν satisfies a Carleson measure condition. A parabolic chord arc domain is a **parabolic vanishing chord arc domain** if ν satisfies a vanishing Carleson measure condition. Sets of this type are ubiquitous in the study of GMT.

In analogy to the elliptic setting, it is proven in [HLN04] that in a parabolic vanishing chord arc domain, $\omega << \sigma$ and $k \equiv \frac{d\omega}{d\sigma}$ satisfies $\log(k) \in \text{VMO}(\partial\Omega)$. Kenig and Toro [KT03] proved the converse result for harmonic measure; that (under certain flatness conditions) $\log(k) \in \text{VMO}(\partial\Omega)$ implies that Ω is a vanishing chord arc domain. In [HLN04] they proved a partial parabolic analogue of this result, but were unable to prove the full result as they didn't know whether all "flat" solutions to (6) are half-plane solutions. With this classification of "flat" solutions, [Eng15] proved the full parabolic analogue of [KT03]:

Theorem 3. *[Theorem 1.9,* [Eng15]*] Let $\Omega \subset \mathbb{R}^{n+1}$ be a δ-Reifenberg flat parabolic chord arc domain with* $\log(k) \in \text{VMO}(\partial\Omega)$. *There $\exists\delta_n > 0$ such that if $\delta < \delta_n$, then Ω is a parabolic vanishing chord arc domain.*

To prove Theorem 3, we show, using a harmonic analysis and GMT argument, that any limit of rescaled and translated copies of Ω converges to an Ω_∞ (which will depend on the rescaling and translations) which satisfies (5) and (6). Our aforementioned classification of "flat" solutions to (5) and (6) implies Ω_∞ must be a plane. To finish the proof of Theorem 3 we must control the Carleson measure, ν; we do so through a GMT argument which may be of independent interest. Roughly, we approximate $\partial\Omega$ at each point and each scale by Lipschitz graphs (adapting arguments of [HLN03]), observe that the Carleson norms associated to these graphs locally bound the Carleson norm associated to $\partial\Omega$ and then use a compactness argument to show that the Carleson norm of these graphs, and thus $\partial\Omega$, vanishes.

Further Work on the Parabolic Problem: There are several interesting questions connected to the one-phase problem for caloric measure. The first is what role the *a priori* Reifenberg flat assumption plays in the proof. The elliptic theory implies that it is necessary in when $n \geq 3$ (as a non-flat solution to the harmonic measure problem is a stationary solution to the parabolic problem). In [Eng16b], we prove, under the assumption that Ω is a parabolic NTA domain, that the only solution to (5) and (6) when $n = 1$ is the half-plane solution. The key observation is that in one spatial dimension a parabolic NTA domain is necessarily a graph domain. In general dimensions, whether there are non-stationary solutions to (5) and (6) is unknown. We are currently investigating this problem when $n = 2$, where we believe "ancient" solutions to curve shortening flow should help us find non-stationary solutions which are parabolic NTA domains.

We may also study solutions to (5) and (6) under the assumption that, for each t_0, the slice $\Omega \cap \{t = t_0\}$ is convex (or empty). Together with David Jerison and Svitlana Mayboroda, I am working on a related question; under what conditions on the Poisson kernel, $\frac{d\omega}{d\sigma}(X,t)$, is there a unique domain, Ω, with convex time slices, such that the caloric Green function for Ω with pole at $(0,0)$ has the specified Poisson kernel. In [Jer89], [Jer92], Jerison fully solved the analogous question for harmonic measure by relating it to the Minkowski problem; we are interested in extending this relationship to the time dependent setting.

Regularity theory for Almost Minimizers: A function, u, is an **almost minimizer** to the Alt-Caffarelli functional, $J(u) = \int |\nabla u|^2 + q_+^2(x)\chi_{\{u>0\}}(x)\, dx$, inside a domain, Ω, if there exists a constant $C > 0$ and an exponent $\alpha \in (0,1)$ such that for all $B(x,r) \subset \Omega$ and v with $v|_{\partial B(x,r)} = u|_{\partial B(x,r)}$,

$$\int_{B(x,r)} |\nabla u|^2 + q_+^2(x)\chi_{\{u>0\}}(x)\, dx \leq \int_{B(x,r)} |\nabla v|^2 + q_+^2(x)\chi_{\{v>0\}}(x)\, dx + Cr^{\alpha+n}. \tag{8}$$

Note, if u is a minimizer, (8) holds without the "error" term, $Cr^{\alpha+n}$. These almost-minimizers were first studied by David and Toro in [DaT15], but have been considered before in others contexts (e.g. for the area functional, see [Alm68] and [DSS15a]). Almost-minimizers model situations in which measurement error or noise may be present. The key difficulty in proving regularity for any type of almost-minimizer is that the function need not satisfy an equation (unlike minimizers), so different methods are needed. Nonetheless, David and Toro prove $C^{1,\beta}$-regularity for almost-minimizers away from the free boundary, $\partial\{u > 0\}$, and Lipschitz regularity for almost-minimizers across the free boundary.

In work in progress, I, together with David and Toro, study the regularity of the free boundary, $\partial\{u > 0\}$.

Theorem 4. *[*[DET16]*] Let u be an almost minimizer to J such that, $q_+ \in C^{0,\alpha}$ and $q_+ \geq c > 0$. Then $\partial\{u > 0\}$ is locally given as the graph of a $C^{1,\beta}$ function near almost every point.*

The proof requires a combination of ideas and estimates from [AC81] but also from the study of rectifiable domains and harmonic measure. We first must show that $\{u > 0\}$ is an NTA domain using a blowup argument and the Weiss montonicity formula (see [Wei03]). To show Hölder regularity of the free boundary, we introduce a function, h_{x_0,r_0}, which has the same boundary values as u on $\partial(B(x_0,r_0) \cap \{u > 0\})$ but which is harmonic in the interior of $B(x_0,r_0) \cap \{u > 0\}$. The key estimate is to show that h is comparable to u in $B(x_0,r_0) \cap \{u > 0\}$ (with error depending on r_0). This allows us to conclude, at least at scales comparable to r_0 and near the point x_0, that the function h_{x_0,r_0} satisfies a perturbation of the Alt-Caffarelli free boundary problem. We then use the "improved flatness" lemma in [AC81] to say that, on an appropriately chosen smaller scale, $r_1 < r_0$, the set $\{h_{x_0,r_0} = 0\}$ (which agrees locally with $\partial\{u > 0\} \cap B(x_0,r_0)$) gets flatter. We repeat, creating h_{x_0,r_1} (which agrees with u on the boundary of $\{u > 0\} \cap B(x_0,r_1)$ but is harmonic on the interior), and running the argument again. Of course, this requires careful management of the errors involved. It is also interesting to note that proving $\{u > 0\}$ is NTA is not just an auxiliary result, but necessary in order to make the harmonic analysis estimates we need to bound h_{x_0,r_0}.

Career development, Choice of sponsor and Host Institution: David Jerison is a natural choice for my sponsoring scientist. He is a leading expert in applying harmonic analysis to the study of regularity theory for elliptic and parabolic PDE and working with him will deepen my investigations of FBPs.

Jerison is also a leading expert in the connections between geometric analysis and regularity theory for FBPs. This analogy, first observed by Alt and Caffarelli in [AC81], has been deepened by Jerison's recent work joint with De Silva, Kamburov and Savin (see [DeJ09], [DeJ11], [JS15], [JKa14]). Especially interesting is the possibility of extending this program to the parabolic setting. Along with Jerison, the presence of two experts in mean curvature flow, Colding and Minicozzi, makes MIT the perfect place to pursue this.

Broader Impacts: Throughout graduate school I have been involved in mathematical education and outreach at all levels. As a mentor in the University of Chicago REU (which I also co-organized in 2012 and 2016) and the directed reading program, I supervised math majors as they wrote expository papers about various advanced subjects. In the summer of 2014, I assisted professor Robert Fefferman in mentoring students from underrepresented groups who were about to begin graduate school in mathematics and statistics. During the summers of 2013 and 2015 I worked as an assistant instructor at MathILy; a summer program for mathematically inclined high school students. I've also tutored disadvantaged area high school students as part of the program "U Chicago Upward Bound". Now that I am in Cambridge, I hope to continue these activities as well as work closely with the undergraduate and graduate students at MIT. I've also strived to become an active member of the mathematical research community. I've disseminated my results at conferences in the UK and Argentina in addition to giving several contributed and invited talks around the United States. Furthermore, my time with Tatiana Toro at the University of Washington Seattle, and my time at MIT, has exposed me to many swaths of the research community (especially in GMT and geometric analysis). I plan to use this wide range of exposure to deepen the connections between these areas and advance collaboration (e.g. by organizing interdisciplinary meetings).

REFERENCES

[Alm68] F. J. Almgren, Jr., Existence and regularity almost everywhere of solutions to elliptic variational problems among surfaces of varying topological type and singularity structure. *Ann. Math. (2)* **87**, (1968) 321-391.

[Alm76] F. J. Almgren, Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints. *Mem. Amer. Math. Soc.* **165** (1976).

[Alm79] F.J. Almgren Jr., Dirichlet's problem for multiple valued functions and the regularity of mass minimizing integral currents. in "Minimal Submanifolds and Geodesics" North-Holland, Amstrerdam, 1979, pp 1-6.

[AC81] H. Alt and L. Caffarelli, Existence and Regularity for a minimum problem with free boundary. *J. Reine Angew. Math.* **325** (1981), pp 105-144.

[ACF84] H. Alt, L. Caffarelli and A. Friedman, Variational Problems with Two Phases and Their Free Boundaries. *Trans. Amer. Math. Soc.* **282** (1984), pp 431-461.

[AW09] J. Andersson and G. S. Weiss, A Parabolic Free Boundary Problem with Bernoulli Type Condition on the Free Boundary. *J. Reine Angew. Math.* **627** (2009), pp 213-235.

[AMT16] J. Azzam, M. Mourgoglou and X. Tolsa, Mutual Absolute Continuity of Interior and Exterior Harmonic Measure Implies Rectifiability. *C.P.A.M., to appear.* arXiv: 1602.01397v2. (2016).

[Bad11] M. Badger, Harmonic polynomials and tangent measures of harmonic measure, *Rev. Mat. Iberoamericana* **27** (2011), pp 841-870.

[Bad13] M. Badger, Flat points in zero sets of harmonic polynomials and harmonic measure from two sides. *J. London Math. Soc.* **87** (2013), pp 111-137.

[BET15] M. Badger, M. Engelstein and T. Toro, Structure of Sets which are Well Approximated by Zero Sets of Harmonic Polynomials. *Preprint*, arXiv:1509.03211. (2015).

[BL15] M. Badger and S. Lewis, Local set approximation: Mattila-Vuorinen type sets, Reifenberg type sets and tangent sets. *Forum Math. Sigma* **3** (2015) 63 pp.

[Caf87] L. Caffarelli, A Harnack inequality approach to the regularity of free boundaries. Part I, Lipschitz free boundaries are $C^{1,\alpha}$. *Rev. Math. Iberoamericana* **3** (1987), pp 139-162.

[Caf98] L. Caffarelli, The Obstacle Problem Revisited. *J. of Fourier Anal. and App.* **4** (1998), 383-402.

[CJK04] L. Caffarelli, D. Jerison, and C.E. Kenig, Global energy minimizers for free boundary problems and full regularity in three dimensions. In: Noncompact problems at the Intersection of Geometry, Analysis, and Topology. *Contemp. Math.*, **vol. 350** pp. 83?97. Am. Math. Soc., Providence (2004)

[CSaSi08] L. Caffarelli, S. Salsa and L. Silvestre, Regularity estimates for the solution and the free boundary of then obstacle problem for the fractional Laplacian. *Invent. Math.* **171** (2008), 425-461.

[CV95] L. Caffarelli and J.L. Vázquez, A free-boundary problem for the heat equation arising in flame propagation. *Trans. Am. Math. Soc.* **347** (1995) 411-441.

[CNV15] J. Cheeger, A. Naber and D. Valtorta, Critical sets of elliptic equations. *CPAM* **68** (2015), pp 173-209.

[CM15] T. Colding and W.P. Minicozzi, Uniqueness of blowups and Łojasiewicz inequalities. *Ann. Math.* **182** (2015) pp 443-471.

[Dah77] B. Dahlberg, Estimates of harmonic measure. *Arch. Rational Mech. Anal.* **65** (1977), 275-288.

[Dah79] B. Dahlberg, On the Poisson integral for Lipschitz and C^1-domains. *Studia Math.* **66** (1979), 13-24.

[Das14] P. Daskalopoulos, The regularity of solutions in degenerate geometric problems. *Surveys in Diff. Geom.* **19** (2014), 83-110.

[DH99] P. Daskalopoulos and R. Hamilton, The Free Boundary on the Gauss Curvature Flow with Flat Sides. *J. Reine Angenw. Math.* **510** (1999), 187-227.

[DET16] G. David, M. Engelstein and T. Toro, Free boundary regularity for almost minimizers. *Preprint* (2016).

[DaJ90] G. David and D. Jerison, Lipschitz approximation to hypersurfaces, harmonic measure, and singular integrals, *Ind. Univ. Math. J.* **39** (1990), 831-845.

[DaS93] G. David and S. Semmes, "Analysis of and on Uniformly Rectifiable Sets." Math. Surveys Monog. **38** Amer. Math. Soc., Providence, 1993.

[DaT15] G. David and T. Toro, Regularity of almost minimizers with free boundary. *Calc. Var. P.D.E.* **54** (2015), 455-524.

[DSS15a] C. De Lellis, E. Spadaro and L. Spolaor, Regularity theory for 2-dimensional almost minimal currents I: Lipschitz approximation. *Preprint* arXiv: 1508.05507, (2015).

[DFS14] D. De Silva, F. Ferrari and S. Salsa, Two-Phase Problems with Distributed Source: Regularity of the Free Boundary. *Anal. & PDE* **7** (2014), 267-310.

[DeJ09] D. De Silva and D. Jerison, A Singular Energy Minimizing Free Boundary. *J. Reine Angew. Math.* **635** (2009), 1-22.

[DeJ11] D. De Silva and D. Jerison, Gradient bound for energy minimizing free boundary graphs. *Comm. Pure Appl. Math.* **64** (2011) 538-555.

[Eng15] M. Engelstein, A Free Boundary Problem for the Parabolic Poisson Kernel. *Preprint* arXiv: 1510.05704, 2015.

[Eng16a] M. Engelstein, A Two-Phase Free Boundary Problem for Harmonic Measure. *Ann. Sci. École Norm. Sup.* **49** (2016), 859-905

[Eng16b] M. Engelstein, Parabolic NTA domains $\mathbb{R}^2$. *Preprint* arXiv:1607.08650 (2016).

[FKP91] R. Fefferman, C. Kenig and J. Pipher, The theory of weights and the Dirichlet problem for elliptic equations. *Ann. of Math.* **134** (1991), 65-124.

[FS16] M. Focardi and E. Spadaro, An epiperimetric inequality for the thin obstacle problem. *Adv. Differential Equations* **21** (2016), 153-200.

[GM05] J.B. Garnett and D.E. Marshall. "Harmonic Measure." Cambridge University Press, New York, 2005.

[GSP16] N. Garofalo, M. Smit Vega Garcia and A. Petrosyan, An epiperimetric inequality approach to the regularity of the free boundary in the Signorini problem with variable coefficients. *J. Math. Pures Appl.* **105** (2016) 745-787.

[GP09] N. Garofalo and A. Petrosyan, Some new monotonicity formulas and the singular set in the lower dimensional obstacle problem. *Inv. Math.* **177** (2009), 415-461.

[GT16] D. Girela-Sarrión and X. Tolsa, The Riesz transform and quantitative rectifiability for general Radon measures. *Preprint*, arXiv: 1601.08079 (2016).

[HKMP15] S. Hofmann, C. Kenig, S. Mayboroda and J. Pipher, Square function/non-tangential maximal function estimates and the Dirichlet problem for non-symmetric elliptic operators. *J.A.M.S.* **28** (2015), 483-529.

[HLN03] S. Hofmann, J.L. Lewis and K. Nyström, Existence of big pieces of graphs for parabolic problems. *Ann. Acad. Sci. Fenn. Math.* **28** (2003), 355-384.

[HLN04] S. Hofmann, J.L. Lewis and K. Nyström, Caloric measure in parabolic flat domains. *Duke Math. J.* **122** (2004), 281-346.

[HM14] S. Hofmann and J. M. Martell, Uniform rectifiability and harmonic measure I: Uniform rectifiability implies Poisson kernels in L^p. *Ann. Sci. École Norm. Sup.* **47** (2014) 577-654

[HMU14] S. Hofmann, J. M. Martell and I. Uriarte-Tuero, Uniform rectifiability and harmonic measure, II: Poisson kernels in L^p imply uniform rectifiability. *Duke Math. J.* **163** (2014) 1601-1654.

[Jer89] D. Jerison, Harmonic measure in convex domains. *Bull. Am. Math. Soc.* **21** (1989), 255-260.

[Jer90] D. Jerison, Regularity of the Poisson Kernel and Free Boundary Problems. *Coll. Math.* **60/61** (1990), 547-568.

[Jer92] D. Jerison, Prescribing harmonic measure on convex domains. *Invent. Math.* **105** (1991) 375-400.

[JKa14] D. Jerison and N. Kamburov, Structure of one-phase free boundaries in the plane. *I.M.R.N To Appear.* arXiv:1412.4106, 2014.

[JKe82] D. Jerison and C. Kenig, Boundary Behavior of Harmonic Functions in Non-Tangentially Accessible Domains. *Adv. Math.* **46** (1982), 80-147.

[JS15] D. Jerison and O. Savin, Some remarks on stability of cones for the one-phase free boundary problem. GAFA 25 (2015) 1240-1257

[KPT09] C. Kenig, D. Preiss, and T. Toro, Boundary structure and size in terms of interior and exterior harmonic measures in higher dimensions. *J.A.M.S.* **22** (2009), 771-796.

[KT97] C. Kenig and T. Toro, Harmonic Measure on Locally Flat Domains. *Duke Math. J.* **87** (1997), 509-550.

[KT99] C. Kenig and T. Toro, Free Boundary Regularity for Harmonic Measures and Poisson Kernels. *Ann. of Math.* **150** (1999), 369-454.

[KT03] C. Kenig and T. Toro, Poisson Kernel characterization of Reifenberg flat chord arc domains. *Ann. Sci. École Norm. Sup.* **36** (2003), 323-401.

[KT04] C. Kenig and T. Toro, On the Free Boundary Regularity Theorem of Alt and Caffarelli. *Disc. and Cont. Dyn. Systems* **10** (2004), 397-422.

[KT06] C. Kenig and T. Toro, Free boundary regularity below the continuous threshold: 2-phase problems. *J. Reine Angew. Math.* **596** (2006), 1-44.

[KNS78] D. Kinderlehrer, L. Nirenberg and J. Spruck, Regularity in Elliptic Free Boundary Problems I. *J. D' Analyse Math.* **34** (1978), 86-119.

[LM95] J.L. Lewis and M.A. Murray, The Method of Layer Potentials for the Heat Equation in Time-Varying Domains. *Mem. Am. Math. Soc.* **114** (1995), no 545.

[Nys06a] K. Nyström, On Blow-ups and the Classification of Global Solutions to a Parabolic Free Boundary Problems. *Ind. Univ. Math. J.* **55** (2006), 1233-1290.

[Nys06b] K. Nyström, Caloric Measure and Reifenberg Flatness. *Ann. Acad. Sci. Fenn. Math.* **31** (2006), 405-436.

[Nys12] K. Nyström, On an inverse type problem for the heat equation in parabolic regular graph domains. *Math. Zeitschrift* **270** (2012), 197-222.

[Pre87] D. Preiss, Geometry of measures in $\mathbb{R}^n$: distributions, rectifiability and densities. *Ann. of Math.* **125** (1987), 537-643.

[QPV14] F. Quiros, B. Perthame and J.-L. Vazquez, The Hele-Shaw Asymptotics for mechanical models of tumor growth. *ARMA* **212** (2014), 93-127.

[SP07] H. Shahgholian and A. Petrosyan (appendix by T. Arnarson), Parabolic obstacle problems applied to finance: free-boundary regularity approach in **D. Danielli ed.** Recent Developments in Nonlinear Partial Differential Equations. Contemporary Matematics **439** (2007), 117?133.

[Sim83] L. Simon, Asymptotics for a class of evolution equations, with applications to geometric problems, *Annals of Math.* **118** (1983), 525-571.

[Sim93] L. Simon, Cylindrical tangent cones and the singular set of minimal submanifolds. *J. Diff. Geom.* **38** (1993), 585-652.

[Sim95] L. Simon, Rectifiability of the singular sets of multiplicity 1 minimal surfaces and energy minimizing maps. In *Surveys in differential geometry*, **Vol. II** (Cambridge, MA, 1993), 246?305, Int. Press, Cambridge, MA, 1995.

[Tay76] J. Taylor, The structure of singularities in soap-bubble-like and soap-film-like minimal surfaces. *Ann. Math.* **103** (1976) 489-539.

[Tor10] T. Toro, Potential Analysis Meets Geometric Measure Theory. *Proc. of the ICM* Hyderabad, India, (2010), 1485-1497.

[We99a] G. Weiss, A homogeneity improvement approach to the obstacle problem. *Invent. Math.* **138** (1999) 23-50.

[We99b] G. Weiss, Partial Regularity for a Minimum Problem with Free Boundary. *Journal of Geom. Anal.* **9** (1999), 317-326.

[Wei03] G. Weiss, A singular limit arising in combustion theory: Fine properties of the free boundary. *Calc. Var. PDE* **17** (2003), 311-340.