THE UNIVERSITY OF CHICAGO

# FREE BOUNDARY PROBLEMS FOR HARMONIC AND CALORIC MEASURE

A DISSERTATION SUBMITTED TO
THE FACULTY OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

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CHICAGO, ILLINOIS
JUNE 2016

Dedication Text

To my family, Danny, Karen and Sophie. I owe them so much.

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# ACKNOWLEDGMENTS

My parents, Danny and Karen, have always encouraged me to follow my passions and I could not have come so far without their love and support. In the same vein, I thank my sister, Sophie, whose curiosity and compassion are inspirational to me. My soon-to-be wife, Caroline, found time during medical school and residency to be a sympathetic ear and a sounding board for ideas. I could not have finished graduate school without her.

Any success I have had is due to my great teachers and mentors. This list includes but is not limited to, sarah-marie belcastro, Jim Cocoros, Joseph Stern, Peter Jones, Frank Morgan, Marianna Csörnyei, Wilhelm Schlag and Tatiana Toro.

Finally, I thank my advisor, Carlos Kenig, who is brilliant, generous and kind. I was not always the easiest student, but Professor Kenig looked past my impatience and helped me develop not only as a mathematician but also as a person. He shared with me countless mathematical gems and also important lessons on professionalism, integrity and mathematical aesthetics. I could not have asked for a better advisor, and this thesis would not be half as good without his guidance.

## ABSTRACT

In this paper we consider two free boundary problems, which we solve using a combination of techniques and tools from harmonic analysis, geometric measure theory and partial differential equations. The first problem is a two-phase problem for harmonic measure, initially studied by Kenig and Toro (KT06). The central difficulty in that problem is the possibility of degeneracy; losing geometric information at a point where both phases vanish. We establish non-degeneracy by proving that the Almgren frequency formula, applied to an appropriately constructed function, is "almost monotone". In this way, we prove a sharp Hölder regularity result (this work was originally published in (Eng14)).

The second problem is a one-phase problem for caloric measure, initially posed by Hofmann, Lewis and Nyström (HLN04). Here the major difficulty is to classify the "flat blowups". We do this by adapting work of Andersson and Weiss (AW09), who analyzed a related problem arising in combustion. This classification allows us to generalize results of (KT03) to the parabolic setting and answer in the affirmative a question left open in the aforementioned paper of Hofmann et al. (this work was originally published in (Eng15)).

# CHAPTER 1

# INTRODUCTION

This thesis is concerned with free boundary problems; a class of partial differential equations in which an unknown function, $u$, satisfies a PDE on a domain, $\Omega$, which itself depends on $u$. The classical example of a free boundary problem is the Stefan problem, which describes an ice cube melting in water. Heat flows differently through water than it does through ice, thus the temperature solves two different partial differential equations (one in the water and one in the ice) on two disjoint domains which are evolving with time (as the ice melts). The interface between the water and ice (i.e. the layer of ice which is melting at any given second) is called the "free boundary" (as opposed to a "fixed boundary").

A prototypical free boundary problem for harmonic or caloric measure is the following; for an open, unbounded, $\Omega \subset \mathbb{R}^n$ and a Radon measure, $\omega$, consider the boundary value problem,

$$\begin{aligned} \Delta u(x) =& 0,\ x \in \Omega, \\ u(x) >& 0,\ x \in \Omega, \\ u(x) =& 0,\ x \in \partial\Omega, \\ \int_\Omega u \Delta \varphi dx =& \int_{\partial\Omega} \varphi d\omega,\ \forall \varphi \in C_0^\infty(\mathbb{R}^n). \end{aligned} \tag{1.0.1}$$

The function, $u$, (uniquely determined up to a constant multiple for sufficiently regular $\Omega$) is the *Green function of* $\Omega$ *with a pole at infinity* and the Radon measure, $\omega$, is the corresponding *harmonic measure.* (1.0.1) is overdetermined and does not have a solution for every combination of $\Omega$ and $\omega$ (as, in a generalized sense, both Neumann and Dirichlet boundary values for $u$ are prescribed). Therefore, *a priori* assumptions on $\omega$ could impose additional conditions on the "free boundary", $\partial\Omega$. Thus, it makes sense to ask the question:

$$\text{If we know } \omega \text{ is "regular" what can we say about } \partial\Omega? \tag{1.0.2}$$

## 1.1 Introduction to Harmonic and Caloric Measure

Intuitively, it is helpful to think of harmonic measure from a probabilistic perspective; given an open domain $\Omega \subset \mathbb{R}^n$, a point $X \in \Omega$ and a subset $E \subset \partial\Omega$, the *harmonic measure of E with a pole at X*, written $\omega^X(E)$, is the probability that a Brownian motion starting at $X$ will first exit $\Omega$ at a point inside of $E$.

While the above definition is useful, the following formulation is more suited to our purpose; let $\Omega \subset \mathbb{R}^n$ be a bounded open domain which admits a solution to the Dirichlet problem. That is, for every $f \in C(\partial\Omega)$ there exists a $u_f \in C^2(\Omega) \cap C(\overline{\Omega})$ such that

$$\begin{aligned} \Delta u_f(X) =& 0, X \in \Omega \\ \lim_{\Omega \ni X \to Q} u_f(X) =& f(Q), Q \in \partial\Omega. \end{aligned}$$

(the class of domains in which the Dirichlet problem can be solved with continuous data in quite broad see e.g. (Ken94), Chapter 1, Section 2 for more details).

The maximum principle then tells us that for every $X \in \partial\Omega$ the map $f \mapsto u_f(X)$ is a linear functional from $C(\partial\Omega) \to \mathbb{R}$ with norm 1. By the Riesz representation theorem, there is a probability measure, $\omega^X$, such that

$$\int_{\partial\Omega} f(Q) d\omega^X(Q) = u_f(X). \tag{1.1.1}$$

This probability measure, $\omega^X$, is the harmonic measure of $\Omega$ with a pole at $X$.

Caloric measure is defined similarly: we say that the parabolic Dirichlet problem is solvable in $\Omega \subset \mathbb{R}^{n+1}$ if, for every $f \in C(\partial_p\Omega)$, there exists a $u_f \in \mathbb{C}^{2,1}(\Omega) \cap C(\overline{\Omega})$ such that

$$\begin{aligned} \partial_t u_f(X,t) - \Delta u_f(X,t) =& 0, (X,t) \in \Omega \\ \lim_{\Omega \ni (X,t) \to (Q,\tau)} u_f(X,t) =& f(Q,\tau), (Q,\tau) \in \partial_p\Omega. \end{aligned}$$

Here (and throughout), $\partial_p\Omega$ represents the parabolic boundary of $\Omega$; the points in $\partial\Omega$ which can be approached by paths contained in $\Omega$ that are monotonically decreasing in time. In analogy to the elliptic situation, the caloric measure is the probability measure, $\omega^{(X,t)}$, given by the parabolic maximum principle and the Riesz representation theorem. That is to say,

$$u_f(X,t) = \int_{\partial_p \Omega} f(Q,\tau) d\omega^{(X,t)}(Q,\tau).$$

Harmonic measure is an object of interest in several different branches of mathematics. In complex analysis, the harmonic measure of a simply connected domain, $\Omega$, with a pole at $X$ is the pushfoward of the uniform distribution on $\mathbb{S}^1$ along a conformal map $\varphi : \mathbb{D} \to \Omega$ such that $\varphi(0) = X$. Thus, the complex geometry of a domain is intimately linked with the harmonic measures it supports (for a beautiful introduction to this area see (GM05)). In probability, hitting measure plays an important role in understanding the behavior of Brownian motion and random walks, see, e.g. (Law96). Additionally, from (1.1.1), it is clear that a better understanding of harmonic measures should give insight into the boundary behavior of solutions to elliptic boundary value problems. For example, Dahlberg, (Dah79), showed that the mutual absolute continuity of harmonic measure with surface measure implied the solvability of the Dirichlet problem in Lipschitz domains for data in $L^2$. For other elliptic operators, there has been important work by Fefferman, Kenig and Piper, (FKP91), and Hofmann, Kenig, Mayboroda and Pipher (HKMP15) along with many others, connecting the regularity of the L-harmonic measure to questions of solvability. For a more comprehensive survey of results in this area, we defer to (Tor10).

While the parabolic theory is much less well developed, there has been work linking caloric measure to the study of the boundary behavior of caloric functions (see, e.g. (FGS86)) and the solvability of the Dirichlet problem for parabolic operators (see, e.g., (LM95), (Nys97) and (HL01)). We hope that a deeper understanding of caloric measure will help transfer some of the above results for elliptic operators to the time dependent setting.

## 1.2 Content and Structure of the Thesis

This thesis is broken up into two additional chapters and an appendix (which contains supplementary material relevant to both chapters). Each chapter has its own introduction which addresses that chapter's contents and the relevant literature in some detail. To avoid redundancy, we will summarize the contents of each chapter as briefly as possible.

Chapter 2 studies a two-phase free boundary problem for harmonic measure: let $\Omega^\pm$ be two disjoint NTA domains, (roughly, NTA domains are quantitatively open and quantitatively path connected, see Definition 2.2.1 for more details) such that $\mathbb{R}^n = \overline{\Omega^+ \cap \Omega^-}$ and that $\partial\Omega^+ = \partial\Omega^-$. Further assume that the harmonic measure for $\Omega^+$ with pole $X^+ \in \Omega^+$, call it $\omega^+$, and the harmonic measure for $\Omega^-$ with pole $X^- \in \Omega^-$, call it $\omega^-$, are mutually absolutely continuous and that $h \equiv \frac{d\omega^-}{d\omega^+}$ is the Radon-Nikodym derivative. We ask, "what does the regularity of $h$ tell us about the regularity of $\partial\Omega^\pm$?"

Our main result says that, assuming some *a priori* flatness, if $\log(h) \in C^{k,\alpha}$, then $\partial\Omega$ is locally the graph of a $C^{k+1,\alpha}$ function. This result is sharp and examples show that the flatness assumption is necessary. This extends work of Kenig and Toro (KT03), who studied the same problem under the assumption that $\log(h) \in \mathrm{VMO}(d\omega^+)$.

The possibility of degeneracy is the largest obstacle to proving regularity in the two-phase setting. The condition, $\log(h) \in C^{0,\alpha}$ does not rule out the possibility that there is a portion of the boundary, $E \subset \partial\Omega$, such that $\omega^\pm(E) = 0$ but $\mathcal{H}^{n-1}(E) > 0$. On such an $E$, we would have no hope of recovering any geometric information. Thus, the bulk of our effort goes towards proving that such an $E$ cannot exist (in fact, we prove a quantitative statement, that $\omega^\pm(E)/\mathcal{H}^{n-1}(E)$ is bounded from below). We establish non-degeneracy by showing a certain formula is "almost-monotone" (i.e. has derivative bounded from below by a function which is integrable at zero). To prove "almost-monotonicity" we use estimates from harmonic analysis and geometric measure theory.

Later, we prove higher regularity of the free boundary using Schauder-type estimates for weak solutions of elliptic and coercive systems and the partial Hodograph transform. We hope that our exposition in that section will be of interest, as some of the results, while familiar to experts, do not seem to appear explicitly in the literature.

Chapter 3 studies a one-phase problem for caloric measure. Namely, we prove that the oscillation of the parabolic Poisson kernel controls the regularity of the free boundary. Let $\Omega$ be a parabolic chord arc domain (a generalization of the appropriate class of parabolic Lipschitz domains) and let $\omega$ be the caloric measure for $\Omega$ associated to a point $(X,t) \in \Omega$. Our final result is that with *a priori* assumed flatness, if $\log(\frac{d\omega}{d\sigma}) \in \mathbb{C}^{k+\alpha,(k+\alpha)/2}$, then $\partial\Omega \cap \{s < t\}$ is locally the graph of a $C^{k+1+\alpha,(k+1+\alpha)/2}$ function. This is sharp and examples show that the flatness assumption is necessary in three or more spatial dimensions.

This result is the parabolic analogue of theorems by Kenig and Toro, (KT03), and Alt and Caffarelli, (AC81). In many instances, we adapt their techniques to the time-dependent setting with only technical adjustments. However, one major obstacle was the lack of a classification of "global" solutions to the free boundary problem. Specifically, it was unknown whether an unbounded parabolic chord arc domain, $\Omega$, with additional assumed flatness and $\frac{d\omega}{d\sigma} \equiv 1$ must be a half plane. We show this is the case by adapting work of Andersson and Weiss, (AW09), who considered solutions, in the sense of "domain variations", to a related free boundary problem.

The definition of a parabolic chord arc domain is more complicated than the corresponding elliptic one (see Section 3.1), thus we also required novel arguments to prove free boundary regularity given that all flat "global" solutions are planes. To overcome this difficulty, we approximate $\Omega$ at every point and every scale by graph domains and use harmonic analysis and geometric measure theory to bound the relevant quantities for $\Omega$ by their counterparts in these graph domains (these techniques were adapted from (HLN03)).

Finally, the reader may also be interested in the appendix to Chapter 3 which proves parabolic counterparts of several potential-theoretic results for harmonic functions in rough domains. While many of these proofs mirror those for harmonic functions, some require new ideas, in particular, the construction of interior sawtooth domains (Lemma B.4.3).

# CHAPTER 2

# A TWO-PHASE FREE BOUNDARY PROBLEM FOR HARMONIC MEASURE

## 2.1 Introduction

In this paper we consider the following two-phase free boundary problem for harmonic measure: let $\Omega^+$ be an unbounded 2-sided non-tangentially accessible (NTA) domain (see Definition 2.2.1) such that $\log(h)$ is regular, e.g. $\log(h) \in C^{0,\alpha}(\partial\Omega)$. Here $h := \frac{d\omega^-}{d\omega^+}$ and $\omega^\pm$ is the harmonic measure associated to the domain $\Omega^\pm$ ($\Omega^- := \text{int}((\Omega^+)^c)$). We ask the question: what can be said about the regularity of $\partial\Omega$?

This question was first considered by Kenig and Toro (see (KT06)) when $\log(h) \in \text{VMO}(d\omega^+)$. They concluded, under the initial assumption of $\delta$-Reifenberg flatness, that $\Omega$ is a vanishing Reifenberg flat domain (see Definition 2.2.2). Later, the same problem, without the initial flatness assumption, was investigated by Kenig, Preiss and Toro (see (KPT09)) and Badger (see (Bad11) and (Bad13)). Our work is a natural extension of theirs, though the techniques involved are substantially different.

Our main theorem is:

**Theorem 2.1.1.** *Let $\Omega$ be a 2-sided NTA domain with $\log(h) \in C^{k,\alpha}(\partial\Omega)$ where $k \geq 0$ is an integer and $\alpha \in (0,1)$.*

- *When $n = 2$: $\partial\Omega$ is locally given by the graph of a $C^{k+1,\alpha}$ function.*
- *When $n \geq 3$: there is some $\delta_n > 0$ such that if $\delta < \delta_n$ and $\Omega$ is $\delta$-Reifenberg flat then $\partial\Omega$ is locally given by the graph of a $C^{k+1,\alpha}$ function.*

0. The contents of this chapter are taken from a paper of the same title, to appear in the *Annales scientifiques de l'École normale supérieure*. While writing that paper I was partially supported by the Department of Defense's National Defense Science and Engineering Graduate Fellowship as well as by the National Science Foundation's Graduate Research Fellowship, Grant No. (DGE-1144082). I'd also like to thank an anonymous referee for their comments.

*Similarly, if* $\log(h) \in C^\infty$ *or* $\log(h)$ *is analytic we can conclude (under the same flatness assumptions above) that* $\partial\Omega$ *is locally given by the graph of a* $C^\infty$ *(resp. analytic) function.*

When $n > 2$, the initial flatness assumption is needed; if $n \geq 4$, $\Omega = \{X \in \mathbb{R}^n \mid x_1^2 + x_2^2 > x_3^2 + x_4^2\}$ is a 2-sided NTA domain such that $\omega^+ = \omega^-$ on $\partial\Omega$ (where the poles are at infinity). As such, $h \equiv 1$ but, at zero, this domain is not a graph. In $\mathbb{R}^3$, H. Lewy (see (Lew77)) proved that, for $k$ odd, there are homogeneous harmonic polynomials of degree $k$ whose zero set divides $\mathbb{S}^2$ into two domains. The cones over these regions are NTA domains and one can calculate that $\log(h) = 0$. Again, at zero, $\partial\Omega$ cannot be written as a graph. However, these two examples suggest an alternative to the *a priori* flatness assumption.

**Theorem 2.1.2.** *Let* $\Omega$ *be a Lipschitz domain (that is,* $\partial\Omega$ *can be locally written as the graph of a Lipschitz function) and let* $h$ *satisfy the conditions of Theorem 2.1.1. Then the same conclusions hold.*

The corresponding one-phase problem, "Does regularity of the Poisson kernel imply regularity of the free boundary?", has been studied extensively. Alt and Caffarelli (see (AC81)) first showed, under suitable flatness assumptions, that $\log(\frac{d\omega}{d\sigma}) \in C^{0,\alpha}(\partial\Omega)$ implies $\partial\Omega$ is locally the graph of a $C^{1,s}$ function. Jerison (see (Jer90)) showed $s = \alpha$ above and, furthermore, if $\log(\frac{d\omega}{d\sigma}) \in C^{1,\alpha}(\partial\Omega)$ then $\partial\Omega$ is locally the graph of a $C^{2,\alpha}$ function (from here, higher regularity follows from classical work of Kinderlehrer and Nirenberg, (KN77)). Later, Kenig and Toro (see (KT03)) considered when $\log(\frac{d\omega}{d\sigma}) \in \text{VMO}(d\sigma)$ and concluded that $\partial\Omega$ is a vanishing chord-arc domain (see Definition 1.8 in (KT03)).

Two-phase elliptic problems are also an object of great interest. The paper of Alt, Caffarelli and Friedman (see (ACF84)) studied an "additive" version of our problem. Later, Caffarelli (see (Caf87) for part one of three) studied viscosity solutions to an elliptic free boundary problem similar to our own. This work was then extended to the non-homogenous setting by De Silva, Ferrari and Salsa (see (DFS14)). It is important to note that, while our problem is related to those studied above, we cannot immediately apply any of their results.

In each of the aforementioned works there is an *a priori* assumption of non-degeneracy built into the problem (either in the class of solutions considered or in the free boundary condition itself). Our problem has no such *a priori* assumption. Unsurprisingly, the bulk of our efforts goes into establishing non-degeneracy.

Even in the case of $n = 2$, where the powerful tools of complex analysis can be brought to bear, our non-degeneracy results seem to be new. We briefly summarize some previous work in this area: let $\Omega^+$ be a simply connected domain bounded by a Jordan curve and $\Omega^- = \overline{\Omega^+}^c$. Then $\partial\Omega = G^+ \cup S^+ \cup N^+$ where

- $\omega^+(N^+) = 0$
- $\omega^+ << \mathcal{H}^1 << \omega^+$ on $G^+$
- Every point of $G^+$ is the vertex of a cone in $\Omega^+$. Furthermore, if $C^+$ is the set of all cone points for $\Omega^+$ then $\mathcal{H}^1(C^+\backslash G^+) = 0 = \omega^+(C^+\backslash G^+)$.
- $\mathcal{H}^1(S^+) = 0$.
- For $\omega^+$ a.e $Q \in S^+$ we have $\limsup_{r\downarrow 0} \frac{\omega^+(B(Q,r))}{r} = +\infty$ and $\liminf_{r\downarrow 0} \frac{\omega^+(B(Q,r))}{r} = 0$

with a similar decomposition for $\omega^-$. These results are due to works by Makarov, McMillan, Pommerenke and Choi. See Garnett and Marshall (GM05), Chapter 6 for an introductory treatment and more precise references.

In our context, that is where $\omega^+ << \omega^- << \omega^+$, $\Omega$ is a 2-sided NTA domain and $\log(h) \in C^{0,\alpha}(\partial\Omega)$, one can use the Beurling monotonicity formula (see Lemma 1 in (BCGJ89)) to show $\limsup_{r\downarrow\infty} \frac{\omega^\pm(B(x,r))}{r} < \infty$. Therefore, $\omega^\pm(S^+ \cup S^-) = 0$ and we can write $\partial\Omega = \Gamma \cup N$ where $\omega^\pm(N) = 0$ and $\Gamma$ is 1-rectifiable (i.e. the image of countably many Lipschitz maps) and has $\sigma$-finite $\mathcal{H}^1$-measure. This decomposition is implied for $n > 2$ by the results of Section 2.5. In order to prove increased regularity one must bound from below $\liminf_{r\downarrow 0} \frac{\omega^+(B(Q,r))}{r}$, which we do in Corollary 2.6.4 and seems to be an original contribution to the literature.

The approach is as follows: after establishing some initial facts about blowups and the Lipschitz continuity of the Green's function (Sections 2.3 and 2.4) we tackle the issue of degeneracy. Our main tools here are the monotonicity formulae of Almgren, Weiss and Monneau which we introduce in Section 2.5. Unfortunately, in our circumstances these functionals are not actually monotonic. However, and this is the key point, we show that they are "almost monotonic" (see, e.g., Theorem 2.5.8). More precisely, we bound the first derivative from below by a summable function. From here we quickly conclude pointwise non-degeneracy. In Section 2.6, we use the quantitative estimates of the previous section to prove uniform non-degeneracy and establish the $C^1$ regularity of the free boundary.

At this point the regularity theory developed by De Silva et al. (see (DFS14)) and Kinderlehrer et al. (see (KN77) and (KNS78)) can be used to produce the desired conclusion. However, these results cannot be applied directly and some additional work is required to adapt them to our situation. These arguments, while standard, do not seem to appear explicitly in the literature. Therefore, we present them in detail here. Section 2.7 adapts the iterative argument of De Silva, Ferrari and Salsa (DFS14) to get $C^{1,s}$ regularity for the free boundary. In Section 2.8 we first describe how to establish optimal $C^{1,\alpha}$ regularity and then $C^{2,\alpha}$ regularity (in analogy to the aforementioned work of Jerison (Jer90)). This is done through an estimate in the spirit of Agmon et al. ((ADN59) and (ADN64)) which is proven in the appendix. Higher regularity then follows easily.

## 2.2 Notation and Definitions

Throughout this article $\Omega \subset \mathbb{R}^n$ is an open set and our object of study. For simplicity, $\Omega^+ := \Omega$ and $\Omega^- := \overline{\Omega}^c$. To avoid technicalities we will assume that $\Omega^\pm$ are both unbounded and let $u^\pm$ be the Green's function of $\Omega^\pm$ with a pole at $\infty$ (our methods and theorems apply to finite poles and bounded domains). Let $\omega^\pm$ be the harmonic measure of $\Omega^\pm$ associated to $u^\pm$; it will always be assumed that $\omega^- << \omega^+ << \omega^-$. Define $h = \frac{d\omega^-}{d\omega^+}$ to be the Radon-Nikodym derivative and unless otherwise noted, it will be assumed that $\log(h) \in C^{0,\alpha}(\partial\Omega)$.

Finally, for a measurable $f : \mathbb{R}^n \to \mathbb{R}$, we write $f^+(x) := |f(x)|\chi_{\{f>0\}}(x)$ and $f^-(x) := |f(x)|\chi_{\{f<0\}}(x)$. In particular, $f(x) = f^+(x) - f^-(x)$. Define $u^\pm$ outside of $\Omega^\pm$ to be identically zero and set $u(x) := u^+(x) - u^-(x)$ (so that these two notational conventions comport with each other).

Recall the definition of an non-tangentially accessible (NTA) domain.

**Definition 2.2.1.** *[See (JK82) Section 3] A domain $\Omega \subset \mathbb{R}^n$ is* **non-tangentially accessible**, *(NTA), if there are constants $M > 1, R_0 > 0$ for which the following is true:*

1. *$\Omega$ satisfies the corkscrew condition: for any $Q \in \partial\Omega$ and $0 < r < R_0$ there exists $A = A_r(Q) \in \Omega$ such that $M^{-1}r < \mathrm{dist}(A, \partial\Omega) \leq |A - Q| < r$.*
2. *$\overline{\Omega}^c$ satisfies the corkscrew condition.*
3. *$\Omega$ satisfies the Harnack chain condition: let $\varepsilon > 0, x_1, x_2 \in \Omega \cap B(R_0/4, Q)$ for a $Q \in \partial\Omega$ with $\mathrm{dist}(x_i, \partial\Omega) > \varepsilon$ and $|x_1 - x_2| \leq 2^k\varepsilon$. Then there exists a "Harnack chain" of overlapping balls contained in $\Omega$ connecting $x_1$ to $x_2$. Furthermore we can ensure that there are no more than $Mk$ balls and that the diameter of each ball is bounded from below by $M^{-1}\min_{i=1,2}\{\mathrm{dist}(x_i, \partial\Omega)\}$*

*When $\Omega$ is unbounded we also require that $\mathbb{R}^n \backslash \partial\Omega$ has two connected components and that $R_0 = \infty$.*

*We say that $\Omega$ is* **2-sided NTA** *if both $\Omega$ and $\overline{\Omega}^c$ are NTA domains. The constants $M, R_0$ are referred to as the "NTA constants" of $\Omega$.*

It should be noted that our analysis in this paper will be mostly local. As such we need only that our domains be "locally NTA" (i.e. that $M, R$ can be chosen uniformly on compacta). However, for the sake of simplicity we will work only with NTA domains. We now recall the definition of a Reifenberg flat domain.

**Definition 2.2.2.** *For $Q \in \partial\Omega$ and $r > 0$,*

$$\theta(Q,r) := \inf_{P \in G(n,n-1)} D[\partial\Omega \cap B(Q,r), \{P+Q\} \cap B(Q,r)],$$

*where $D[A,B]$ is the Hausdorff distance between $A, B$.*

*For $\delta > 0, R > 0$ we then say that $\Omega$ is $(\delta, R)$-**Reifenberg flat** if for all $Q \in \partial\Omega, r < R$ we have $\theta(Q,r) \leq \delta$. When $\Omega$ is unbounded we say it is $\delta$-**Reifenberg flat** if the above holds for all $0 < r < \infty$.*

*Additionally, if $K \subset\subset \mathbb{R}^n$ we can define*

$$\theta_K(r) = \sup_{Q \in K \cap \partial\Omega} \theta(Q,r).$$

*Then we say that $\Omega$ is **vanishing Reifenberg flat** if for all $K \subset\subset \mathbb{R}^n$, $\limsup_{r\downarrow 0} \theta_K(r) = 0$.*

**Remark 2.2.3.** *Recall that a $\delta$-Reifenberg flat NTA domain is not necessarily a Lipschitz domain, and a Lipschitz domain need not be $\delta$-Reifenberg flat. However, all Lipschitz domains are (locally) 2-sided NTA domains (see (JK82) for more details and discussion).*

Finally, let us make two quick technical points regarding $h$.

**Remark 2.2.4.** *For every $Q \in \partial\Omega$, we have $\lim_{r\downarrow 0} \frac{\omega^-(B(Q,r))}{\omega^+(B(Q,r))} = h(Q)$ (in particular the limit exists for every $Q \in \partial\Omega$).*

*Justification of Remark.* By assumption, $\frac{d\omega^-}{d\omega^+}$ agrees with a Hölder continuous function $h$ where defined (i.e. $\omega^+$-almost everywhere). For any $Q \in \partial\Omega$ we rewrite $\lim_{r\downarrow 0} \frac{\omega^-(B(Q,r))}{\omega^+(B(Q,r))} = \lim_{r\downarrow 0} \fint_{B(Q,r)} \frac{d\omega^-}{d\omega^+}(P) d\omega^+(P) = \lim_{r\downarrow 0} \fint_{B(Q,r)} h(P) d\omega^+(P)$. This final limit exists and is equal to $h(Q)$ everywhere because $h$ is continuous. $\square$

We also note that $h$ is only defined on $\partial\Omega$. However, by Whitney's extension theorem, we can extend $h$ to $\tilde{h} : \mathbb{R}^n \to \mathbb{R}$ such that $\tilde{h} = h$ on $\partial\Omega$ and $\log(\tilde{h}) \in C^\alpha(\mathbb{R}^n)$ (or, if $\log(h) \in C^{k,\alpha}(\partial\Omega)$ then $\log(\tilde{h}) \in C^{k,\alpha}(\mathbb{R}^n)$). For simplicity's sake, we will abuse notation and let $h$ refer to the function defined on all of $\mathbb{R}^n$.

## 2.3 Blowups on NTA and Lipschitz Domains

For any $Q \in \partial\Omega$ and any sequence of $r_j \downarrow 0$ and $Q_j \in \partial\Omega$ such that $Q_j \to Q$, define the **pseudo-blowup** as follows:

$$\begin{aligned} \Omega_j &:= \frac{1}{r_j}(\Omega - Q_j) \\ u_j^{\pm}(x) &:= \frac{u^{\pm}(r_j x + Q_j) r_j^{n-2}}{\omega^{\pm}(B(Q_j, r_j))} \\ \omega_j^{\pm}(E) &:= \frac{\omega^{\pm}(r_j E + Q_j)}{\omega^{\pm}(B(Q_j, r_j))}. \end{aligned} \tag{2.3.1}$$

A pseudo-blowup where $Q_j \equiv Q$, is a **blowup**. Kenig and Toro characterized pseudo-blowups of 2-sided NTA domains when $\log(h) \in \mathrm{VMO}(d\omega^+)$.

**Theorem 2.3.1.** *[(KT06), Theorem 4.4] Let $\Omega^{\pm} \subset \mathbb{R}^n$ be a 2-sided NTA domain, $u^{\pm}$ the associated Green's functions and $\omega^{\pm}$ the associated harmonic measures. Assume* $\log(h) \in \mathrm{VMO}(d\omega^+)$. *Then, along any pseudo-blowup, there exists a subsequence (which we shall relabel for convenience) such that (1) $\Omega_j \to \Omega_\infty$ in the Hausdorff distance uniformly on compacta, (2) $u_j^{\pm} \to u_\infty^{\pm}$ uniformly on compact sets (3) $\omega_j^{\pm} \rightharpoonup \omega_\infty^{\pm}$. Furthermore, $u_\infty := u_\infty^+ - u_\infty^-$ is a harmonic polynomial (whose degree is bounded by some number which depends on the dimension and the NTA constants of $\Omega$) and $\partial\Omega_\infty = \{u_\infty = 0\}$.*

*Additionally, if $n = 2$ or $\Omega$ is a $\delta$-Reifenberg flat domain with $\delta > 0$ small enough (depending on $n$) then $u_\infty(x) = x_n$ (possibly after a rotation). In particular, $\Omega$ is vanishing Reifenberg flat.*

This result plays a crucial role in our analysis. In particular, the key estimate in (2.5.5) follows from vanishing Reifenberg flatness. Therefore, in order to prove Theorem 2.1.2 we must establish an analogous result when $\Omega$ is a Lipschitz domain.

**Corollary 2.3.2.** *Let $\Omega \subset \mathbb{R}^n$ be as in Theorem 2.1.2. Then, along any pseudo-blowup we have (after a possible rotation) that $u_\infty(x) = x_n$. In particular, $\Omega^{\pm}$ is a vanishing Reifenberg flat domain.*

*Proof.* We first recall Remark 2.2.3, which states that any Lipschitz domain is a (locally) 2-sided NTA domain. Therefore, the conditions of Theorem 2.3.1 are satisfied. A result of Badger (Theorem 6.8 in (Bad13)) says that, under the assumptions of Theorem 2.3.1, the set of points where all *blowups* are 1-homogenous polynomials is in fact vanishing Reifenberg flat ("locally Reifenberg flat with vanishing constant" in the terminology of (Bad13)). Additionally, graph domains (i.e. domains whose boundaries are locally the graph of a function) are closed under blowups, so all blowups of $\partial\Omega$ can be written locally as the graph of a some function. Observe that the zero set of a $k$-homogenous polynomial is a graph domain if and only if $k = 1$. In light of all the above, it suffices to show that all blowups of $\partial\Omega$ are given by the zero set of a homogenous harmonic polynomial. We now recall another result of Badger.

**Theorem** ((Bad11), Theorem 1.1)**.** *If $\Omega$ is an NTA domain with harmonic measure $\omega$ and $Q \in \partial\Omega$, then $\mathrm{Tan}(\omega, Q) \subset P_d \Rightarrow \mathrm{Tan}(\omega, Q) \subset F_k$ for some $1 \leq k \leq d$. $P_d$ is the set of harmonic measures associated to a domain of the form $\{h > 0\}$, where $h$ is a harmonic polynomial of degree $\leq d$. $F_k$ is the set of harmonic measures associated to a domain of the form $\{h > 0\}$, where $h$ is a homogenous harmonic polynomial of degree $k$.*

In other words, if every blowup of an NTA domain is the zero set of a degree $\leq d$ harmonic polynomial, then every blowup of that domain is the zero set of a $k$-homogenous harmonic polynomial. This result, combined with Theorem 2.3.1, immediately implies that all blowups of $\partial\Omega$ are given by the zero set of a $k$-homogenous harmonic polynomial. By the arguments above, $k = 1$ and $\partial\Omega$ is vanishing Reifenberg flat.

That $u_\infty = x_n$ (as opposed to $kx_n$ for some $k \neq 1$) follows from $\omega_\infty(B(0,1)) = \lim_i \omega_i(B(0,1)) \equiv 1$, and that $u^\pm_\infty$ is the Green's function associated to $\omega_\infty$. □

Hereafter, we can assume, without loss of generality, that $\Omega$ is a vanishing Reifenberg flat domain and that all pseudo-blowups are 1-homogenous polynomials.

## 2.4 $u$ is Lipschitz

The main aim of this section is to prove that $u$ is locally Lipschitz.[1] We adapt the method of Alt, Caffarelli and Friedman ((ACF84), most pertinently Section 5) which uses the following monotonicity formula to establish Lipschitz regularity for an "additive" two phase free boundary problem.

**Theorem 2.4.1.** *[(ACF84), Lemma 5.1] Let $f \in C^0(B(x_0,R)) \cap W^{1,2}(B(x_0,R))$ where $f(x_0) = 0$ and $f$ is harmonic in $B(x_0,R)\backslash\{f=0\}$. Then*

$$J(x,r) := \frac{1}{r^2}\left(\int_{B(x,r)} \frac{|\nabla f^+|^2}{|x-y|^{n-2}}dy\right)^{1/2}\left(\int_{B(x,r)} \frac{|\nabla f^-|^2}{|x-y|^{n-2}}dy\right)^{1/2}$$

*is increasing in $r \in (0,R)$ and is finite for all $r$ in that range.*

In a 2-sided NTA domain, $u \in C^0(B(Q,R)) \cap W^{1,2}(B(Q,R))$ for any $Q \in \partial\Omega$ and any $R$ (as such domains are "admissible" see (KPT09), Lemma 3.6). This monotonicity immediately implies upper bounds on $\frac{\omega^\pm(B(Q,r))}{r^{n-1}}$.

**Corollary 2.4.2.** *Let $K \subset\subset \mathbb{R}^n$ be compact. There is a $0 < C \equiv C_{K,n} < \infty$ such that*

$$\sup_{0<r\leq 1}\sup_{Q\in K\cap\partial\Omega} \frac{\omega^\pm(B(Q,r))}{r^{n-1}} < C.$$

*Proof.* Using the Theorem 2.4.1 one can prove that

$$\frac{\omega^+(B(Q,r))}{r^{n-1}}\frac{\omega^-(B(Q,r))}{r^{n-1}} \leq C\|u\|_{L^2(B(Q,4))},\ \forall 0 < r \leq 1,$$

(see Remark 3.1 in (KPT09)). Note that

$$\sup_{1\geq r>0, Q\in\partial\Omega\cap K}\left(\frac{\omega^\pm(B(Q,r))}{r^{n-1}}\right)^2 = \sup_{1\geq r>0, Q\in\partial\Omega\cap K} \frac{\omega^+(B(Q,r))}{r^{n-1}}\frac{\omega^-(B(Q,r))}{r^{n-1}}\frac{\omega^\pm(B(Q,r))}{\omega^\mp(B(Q,r))}$$

1. NB: In this section we need only assume that $\log(h) \in C(\partial\Omega)$.

$$\leq \sup_{P\in\partial\Omega,\, \mathrm{dist}(P,K)\leq 1} h^{\mp 1}(P) \sup_{1\geq r>0, Q\in\partial\Omega\cap K} \frac{\omega^+(B(Q,r))}{r^{n-1}}\frac{\omega^-(B(Q,r))}{r^{n-1}}.$$

By continuity, $\log(h)$ is bounded on compacta and so we are done. □

Blowup analysis connects the Lipschitz continuity of $u$ to the boundedness of $\frac{\omega^{\pm}(B(Q,r))}{r^{n-1}}$.

**Lemma 2.4.3.** *Let $K \subset\subset \mathbb{R}^n$ be compact, $Q \in K \cap \partial\Omega$ and $1 \geq r > 0$. Then there is a constant $C > 0$ (which depends only on dimension and $K$) such that*

$$\frac{1}{r}⨍_{\partial B(Q,r)} |u| < C.$$

*Proof.* We rewrite $\frac{1}{r}⨍_{\partial B(Q,r)} |u| = \frac{1}{r}⨍_{\partial B(0,1)} |u(ry+Q)|d\sigma(y)$. Standard estimates on NTA domains imply $u^{\pm}(ry+Q) \leq C_K u^{\pm}(A_{\pm}(Q,r)) \leq C_K \frac{\omega^{\pm}(B(Q,r))}{r^{n-2}}$ (see (JK82), Lemmas 4.4 and 4.8). So

$$\frac{1}{r}⨍_{\partial B(Q,r)} |u| \leq C_K \left(\frac{\omega^+(B(Q,r))}{r^{n-1}} + \frac{\omega^-(B(Q,r))}{r^{n-1}}\right).$$

Corollary 2.4.2 implies the desired result. □

We then prove Lipschitz continuity around the free boundary.

**Proposition 2.4.4.** *If $K \subset\subset \mathbb{R}^n$ is compact then $|Du(x)| < C \equiv C(n,K) < \infty$ a.e. in $K$.*

*Proof.* As $u$ is analytic away from $\partial\Omega$ and $u \equiv 0$ on $\partial\Omega$ we can conclude that $Du$ exists a.e.

Pick $x \in K$ and, without loss of generality, let $x \in \Omega^+$. Define $\rho(x) := \mathrm{dist}(x, \partial\Omega)$ and let $Q \in \partial\Omega$ be such that $\rho(x) = |x - Q|$. If $\rho > 1/5$ then elliptic regularity implies $|Du(x)| \leq C(n,K)$.

So we may assume that $\rho < 1/5$. A standard estimate yields

$$|Du(x)| \leq \frac{C}{\rho}⨍_{\partial B(x,\rho)} |u(y)|d\sigma(y). \tag{2.4.1}$$

We may pick $3\rho < \sigma < 5\rho$ such that $y \in \partial B(x,\rho) \Rightarrow y \in B(Q,\sigma)$. As $|u|$ is subharmonic and $\text{dist}(y, \partial B(Q,\sigma)) > \sigma/3$ we may estimate

$$|u(y)| \leq c\int_{\partial B(Q,\sigma)} \frac{\sigma^2 - |y-Q|^2}{\sigma|y-z|^n}|u(z)|d\sigma(z) \leq c \fint_{\partial B(Q,\sigma)} |u(z)|d\sigma(z) \overset{\text{Lem 2.4.3}}{\leq} C\sigma \leq C'\rho.$$

This estimate, with (2.4.1), implies the Lipschitz bound. □

Consider any pseudo-blowup $Q_j \to Q, r_j \downarrow 0$. It is clear that $u_j$ is a Lipschitz function (though perhaps not uniformly in $j$). If $\phi \in C_c^\infty(B_1; \mathbb{R}^n)$ then Corollary 2.3.2 implies (after a possible rotation)

$$\int \phi \cdot \nabla u_j^\pm = -\int (\nabla \cdot \phi) u_j^\pm \overset{j\to\infty}{\to} -\int (\nabla\cdot\phi)(x_n)^\pm = \int \phi \cdot e_n \chi_{\mathbb{H}^\pm}.$$

Because $\nabla u_j^\pm$ converges in the weak-$*$ topology on $L^\infty(B_1; \mathbb{R}^n)$, $|\nabla u_j^\pm|$ is bounded in $L^\infty(B_1)$. Therefore, $|\nabla u_j^\pm|$ converges in the weak-$*$ topology on $L^\infty(B_1)$ to some function $f$. However, as $\nabla u_j^\pm \overset{*}{\rightharpoonup} e_n\chi_{\mathbb{H}^\pm}$ it must be true that $|\nabla u_j^\pm|$ converges pointwise to $\chi_{\mathbb{H}^\pm}$ and thus $f = \chi_{\mathbb{H}^\pm}$ (more generally, converges to the indicator function of some half space which may depend on the blowup sequence taken).

The existence of this weak-$*$ limit implies that $\Theta^{n-1}(\omega^\pm, Q) := \lim_{r\downarrow 0} \frac{\omega^\pm(B(Q,r))}{r^{n-1}}$ exists, and is finite, everywhere on $\partial\Omega$ (as opposed to $\mathcal{H}^{n-1}$-almost everywhere). Let $r_j \downarrow 0$; one can compute that $J(Q,r_j) = \frac{\omega^+(B(Q,r_j))}{r_j^{n-1}}\frac{\omega^-(B(Q,r_j))}{r_j^{n-1}} J_{Q,r_j}(0,1)$ where

$$J_{Q,r_j}(0,s) := \frac{1}{s^2}\left(\int_{B(0,s)} \frac{|\nabla u_j^+(y)|^2}{|y|^{n-2}}dy\right)^{1/2}\left(\int_{B(0,s)} \frac{|\nabla u_j^-(y)|^2}{|y|^{n-2}}dy\right)^{1/2}$$

and $u_j$ is a blowup along the sequence $Q_j \equiv Q$ and $r_j \downarrow 0$. By the arguments above, $|\nabla u_j^\pm|^2$ converges in the weak-$*$ topology to the indicator function of some halfspace. Therefore, $J_{Q,r_j}(0,1) \overset{j\to\infty}{\to} c(n)$, where $c(n)$ is some constant independent of $r_j \downarrow 0$ (the halfspace may depend on the sequence, but the integral does not). Furthermore, by Theorem 2.4.1

$J(Q,0) := \lim_{r\downarrow 0} J(Q,r)$ exists. It follows that

$$\lim_{r\downarrow 0} \frac{\omega^+(B(Q,r))}{r^{n-1}} \frac{\omega^-(B(Q,r))}{r^{n-1}} = \frac{J(Q,0)}{c(n)}.$$

In particular, the limit on the left exists for every $Q \in \partial\Omega$, which (given Remark 2.2.4) implies $\Theta^{n-1}(\omega^\pm, Q)$ exists for every $Q \in \partial\Omega$.

## 2.5 Non-degeneracy of $\Theta^{n-1}(\omega^\pm, Q)$

In this section we show $\Theta^{n-1}(\omega^\pm, Q) > 0$ for all $Q \in \partial\Omega$ (Proposition 2.5.10). Let

$$v^{(Q)}(x) := h(Q)u^+(x) - u^-(x),\ Q \in \partial\Omega. \tag{2.5.1}$$

For any $r_j \downarrow 0$, we define the blowup of $v^{(Q)}$ along $r_j$ to be $v_j^{(Q)}(x) := \frac{r_j^{n-2} v^{(Q)}(r_j x + Q)}{\omega^-(B(Q,r_j))}$. Let us make some remarks concerning $v^{(Q)}$ and its blowups.

**Remark 2.5.1.** *The following hold for any $Q \in \partial\Omega$.*

- *For any compact $K$, we have $\sup_{Q\in K\cap\partial\Omega} \|v^{(Q)}\|_{W^{1,\infty}_{loc}(\mathbb{R}^n)} < \infty$.*
- *$v_j^{(Q)}(x) \to x \cdot e_n$ uniformly on compacta (after passing to a subsequence and a possible rotation). Additionally (as above), we have $|\nabla v_j^{(Q)}| \overset{*}{\rightharpoonup} 1$ in $L^\infty$.*
- *If the non-tangential limit of $|\nabla v^{(Q)}|$ at $Q$ exists it is equal to $\Theta^{n-1}(\omega^-, Q)$.*

*Justification of Remarks.* The first two statements follow from the work in Section 2.4.

To prove the third statement we first notice

$$\nabla v_j^{(Q)}(x) = \frac{r_j^{n-1} \nabla v^{(Q)}(r_j x + Q)}{\omega^-(B(Q,r_j))}. \tag{2.5.2}$$

The second statement implies $\lim_{j\to\infty} |\nabla v_j^{(Q)}(x)| = 1$ almost everywhere. The result follows. □

### *2.5.1 Almgren's Frequency Formula*

Remark 2.5.1 hints at a connection between the degeneracy of $\Theta^{n-1}(\omega^-, Q)$ and that of the non-tangential limit of $\nabla v^{(Q)}$. This motivates the use of Almgren's frequency function (first introduced in (Alm79)).

**Definition 2.5.2.** *Let $f \in H^1_{\text{loc}}(\mathbb{R}^n)$ and pick $x_0 \in \{f = 0\}$. Define*

$$H(r, x_0, f) = \int_{\partial B_r(x_0)} f^2,$$

$$D(r, x_0, f) = \int_{B_r(x_0)} |\nabla f|^2,$$

*and finally*

$$N(r, x_0, f) = \frac{rD(r, x_0, f)}{H(r, x_0, f)}.$$

Almgren first noticed that when $f$ is harmonic, $r \mapsto N(r, x_0, f)$ is absolutely continuous and monotonically decreasing as $r \downarrow 0$. Furthermore, $N(0, x_0, f)$ is an integer and is the order to which $f$ vanishes at $x_0$ (these facts first appear in (Alm79). See (Mal09) for proofs and a gentle introduction).

Throughout the rest of this subsection we consider $v \equiv v^{(Q)}$ and, for ease of notation, set $Q = 0$. $v$ may not be harmonic and thus $N(r, 0, v)$ may not be monotonic. However, in the sense of distributions, the following holds:

$$\Delta v(x) = (h(0)d\omega^+ - d\omega^-)|_{\partial\Omega} = \left(\frac{h(0)}{h(x)} - 1\right) d\omega^-|_{\partial\Omega}. \tag{2.5.3}$$

Therefore, $\log(h) \in C^\alpha(\partial\Omega)$ implies that $|\Delta v(x)| \leq C|x|^\alpha d\omega^-|_{\partial\Omega}$. That $v$ is "almost harmonic" will imply that $N$ is "almost monotonic" (see Lemma 2.5.6).

When estimating $N'(r, 0, v)$ we reach a technical difficulty; *a priori* $v$ is merely Lipschitz, and so $\nabla v$ is not defined everywhere. To address this, we will work instead with $v_\varepsilon = v * \varphi_\varepsilon$, where $\varphi$ is a $C^\infty$ approximation to the identity (i.e. supp $\varphi \subset B_1$ and $\int \varphi = 1$). Let

$N_\varepsilon(r) := N(r, 0, v_\varepsilon)$ and similarly define $H_\varepsilon, D_\varepsilon$.

**Remark 2.5.3.** *The following are true:*

$$\begin{aligned}
&\lim_{r\downarrow 0} N(r,0,v) = 1\\
&D_\varepsilon(r) = \int_{\partial B_r} v_\varepsilon (v_\varepsilon)_\nu d\sigma - \int_{B_r} v_\varepsilon \Delta v_\varepsilon\\
&\frac{d}{dr} D_\varepsilon(r) = \frac{n-2}{r}\int_{B_r} |\nabla v_\varepsilon|^2 dx + 2\int_{\partial B_r} (v_\varepsilon)_\nu^2 - \frac{2}{r}\int_{B_r} \langle x, \nabla v_\varepsilon\rangle \Delta v_\varepsilon dx\\
&\frac{d}{dr} H_\varepsilon(r) = \frac{n-1}{r} H_\varepsilon(r) + 2\int_{\partial B_r} v_\varepsilon (v_\varepsilon)_\nu d\sigma.
\end{aligned}$$

*Proof.* The second equation follows from integration by parts and the third (originally observed by Rellich) can be obtained using the change of variables $y = x/r$. The final equation can be proven in the same way as the third.

To establish the first equality we take blowups. Pick any $r_j \downarrow 0$. One computes,

$$N(r_j, 0, v) = \frac{\int_{B_1} |\nabla v_j|^2}{\int_{\partial B_1} v_j^2}.$$

Recall Remark 2.5.1; $v_j \to x_n$ uniformly on compacta and $|\nabla v_j| \overset{*}{\rightharpoonup} 1$ in $L^\infty$ (perhaps passing to subsequences and rotating the coordinate system). Therefore, $\lim_{j\to\infty} N(r_j, 0, v) = \lim_{j\to\infty} N(1, 0, v_j) = N(1, 0, x_n)$. Almgren (in (Alm79)) proved that if $p$ is a 1-homogenous polynomial then $N(r, 0, p) \equiv 1$ for all $r$. It follows that $\lim_{j\to\infty} N(r_j, 0, v) = 1$. $\square$

With these facts in mind we calculate $N_\varepsilon'(r)$.

$$\begin{aligned}
H_\varepsilon^2(r) N_\varepsilon'(r) = 2r&\left(\int_{\partial B_r} (v_\varepsilon)_\nu^2 d\sigma \int_{\partial B_r} v_\varepsilon^2 d\sigma - \left[\int_{\partial B_r} v_\varepsilon (v_\varepsilon)_\nu d\sigma\right]^2\right)\\
&+ 2r\int_{B_r} v_\varepsilon \Delta v_\varepsilon dx \int_{\partial B_r} v_\varepsilon (v_\varepsilon)_\nu d\sigma - 2H_\varepsilon(r)\int_{B_r} \langle x, \nabla v_\varepsilon\rangle \Delta v_\varepsilon dx
\end{aligned} \tag{2.5.4}$$

*Derivation of* (2.5.4). By the quotient rule

$$H_\varepsilon^2(r)N_\varepsilon'(r) = D_\varepsilon(r)H_\varepsilon(r) + rD_\varepsilon'(r)H_\varepsilon(r) - rD_\varepsilon(r)H_\varepsilon'(r).$$

Using the formulae for $H_\varepsilon', D_\varepsilon'$ found in Remark 2.5.3 we rewrite the above as

$$H_\varepsilon^2(r)N_\varepsilon'(r) = D_\varepsilon(r)H_\varepsilon(r) - rD_\varepsilon(r)\left(\frac{n-1}{r}H_\varepsilon(r) + 2\int_{\partial B_r} v_\varepsilon(v_\varepsilon)_\nu d\sigma\right)$$

$$+rH_\varepsilon(r)\left(\frac{n-2}{r}\int_{B_r}|\nabla v_\varepsilon|^2dx + 2\int_{\partial B_r}(v_\varepsilon)_\nu^2 - \frac{2}{r}\int_{B_r}\langle x, \nabla v_\varepsilon\rangle \Delta v_\varepsilon dx\right).$$

Distribute and combine terms to get

$$H_\varepsilon^2(r)N_\varepsilon'(r) = \left(D_\varepsilon(r)H_\varepsilon(r) + (n-2)H_\varepsilon(r)\int_{B_r}|\nabla v_\varepsilon|^2dx - (n-1)D_\varepsilon(r)H_\varepsilon(r)\right)$$

$$+2r\left(H_\varepsilon(r)\int_{\partial B_r}(v_\varepsilon)_\nu^2 d\sigma - D_\varepsilon(r)\int_{\partial B_r} v_\varepsilon(v_\varepsilon)_\nu d\sigma\right) - 2H_\varepsilon(r)\int_{B_r}\langle x, \nabla v_\varepsilon\rangle \Delta v_\varepsilon dx.$$

The first set of parenthesis above is equal to zero (recalling the definition of $D_\varepsilon(r)$). In the second set of parenthesis use the formula for $D_\varepsilon(r)$ found in Remark 2.5.3. This gives us

$$H_\varepsilon^2(r)N_\varepsilon'(r) = 2r\left(H_\varepsilon(r)\int_{\partial B_r}(v_\varepsilon)_\nu^2 d\sigma - \left(\int_{\partial B_r}(v_\varepsilon)_\nu v_\varepsilon d\sigma\right)^2\right)$$

$$+2r\int_{B_r} v_\varepsilon \Delta v_\varepsilon dx \int_{\partial B_r} v_\varepsilon(v_\varepsilon)_\nu d\sigma - 2H_\varepsilon(r)\int_{B_r}\langle x, \nabla v_\varepsilon\rangle \Delta v_\varepsilon dx.$$

□

The difference in parenthesis on the right hand side of (2.5.4) is positive by the Cauchy-Schwartz inequality. Thus, to establish a lower bound on $N_\varepsilon'(r)$, it suffices to consider the other terms in the equation.

**Lemma 2.5.4.** *Let $\varepsilon < r$ and define $E_\varepsilon(r) = \int_{B_r}\langle x, \nabla v_\varepsilon\rangle \Delta v_\varepsilon dx$. Then there exists a constant $C$ (independent of $r, \varepsilon$) such that $|E_\varepsilon(r)| \leq Cr^{1+\alpha}\omega^-(B(0,r))$.*

*Proof.* Since $\Delta v_\varepsilon = (\Delta v) * \varphi_\varepsilon$ in terms of distributions, we can move the convolution from one term to the other:

$$\int_{B_r} \langle x, \nabla v_\varepsilon \rangle \, \Delta v_\varepsilon dx = \int [(\chi_{B_r}(x) \langle x, \nabla v_\varepsilon \rangle) * \varphi_\varepsilon] \Delta v dx.$$

Evaluate $\Delta v$, as in (2.5.3), to obtain

$$\left| \int_{B_r} \langle x, \nabla v_\varepsilon \rangle \, \Delta v_\varepsilon dx \right| = \left| \int (\chi_{B_r}(x) \langle x, \nabla v_\varepsilon \rangle)_\varepsilon \left( \frac{h(0)}{h(x)} - 1 \right) d\omega^- \right|$$

$$\leq Cr^{1+\alpha} \int_{B_{r+\varepsilon}} (|\nabla v|_\varepsilon)_\varepsilon d\omega^-,$$

where the last inequality follows from $\log(h) \in C^\alpha$, and $|x| < C(r+\varepsilon) < Cr$ on the domain of integration. The desired estimate then follows from the Lipschitz continuity of $v$ and that the harmonic measure of an NTA domain is doubling (see (JK82), Theorem 2.7). □

**Lemma 2.5.5.** *Let $\varepsilon << r$. Then $H_\varepsilon(r) > c\frac{\omega^-(B(0,r))^2}{r^{n-3}}$ for some constant $c > 0$ independent of $r, \varepsilon > 0$.*

*Proof.* By the corkscrew condition (see Definition 2.2.1 condition (1)) on $\Omega$, there is a point $x_0 \in \partial B_r \cap \Omega$ such that $\mathrm{dist}(x_0, \partial\Omega) > cr$ ($c$ depends only on the NTA properties of $\Omega$). The Harnack chain condition (see Definition 2.2.1 condition (3)) gives $v(x_0) \sim v(A_r(0))$. The Harnack inequality then implies that, for $\varepsilon << r$ there is a universal $k$ such that for $y \in B(x_0, kr)$ we have $v_\varepsilon(y) \sim v(x_0) \sim v(A_r(0))$.

Therefore, there is a subset of $\partial B_r$ (with surface measure $\approx k|\partial B_r|$) on which $v_\varepsilon$ is proportional to $v(A_r(0))$. We then recall that in an NTA domain we have $v(A_r(0)) \sim \frac{\omega^-(B(0,r))}{r^{n-2}}$ ((JK82), Lemma 4.8), which proves the desired result. □

It is useful now to establish bounds on the growth rate of $\omega^\pm(B(Q,r))$. As $\Omega$ is vanishing Reifenberg flat, $\omega^\pm$ is asymptotically optimally doubling ((KT97), Corollary 4.1). This implies a key estimate: for any $\delta > 0$ and $Q \in \partial\Omega$ we have

$$\lim_{r \downarrow 0} \frac{r^{n-1+\delta}}{\omega^-(B(Q,r))} = 0. \tag{2.5.5}$$

**Lemma 2.5.6.** *Let $\varepsilon << R$. There exists a function, $C(R,\varepsilon)$, such that*

$$\begin{aligned} \forall R/4 < r < R,\ N_\varepsilon(R) + C(R,\varepsilon)(R-r) &\geq N_\varepsilon(r) &(2.5.6)\\ C(R,\varepsilon)R &\leq kR^{\alpha/2} &(2.5.7)\end{aligned}$$

*where $k > 0$ is a constant independent of $\varepsilon, R$ (as long as $\varepsilon << R$).*

*Proof.* If $C(R,\varepsilon) := \sup_{R/4<r<R}(N_\varepsilon(r)')^-$, the first claim of our lemma is true by definition.

Recall (2.5.4):

$$H_\varepsilon^2(r)N_\varepsilon'(r) = 2r\left(\int_{\partial B_r}(v_\varepsilon)_\nu^2 d\sigma \int_{\partial B_r} v_\varepsilon^2 d\sigma - \left[\int_{\partial B_r} v_\varepsilon (v_\varepsilon)_\nu d\sigma\right]^2\right)$$

$$+2r\int_{B_r} v_\varepsilon \Delta v_\varepsilon dx \int_{\partial B_r} v_\varepsilon (v_\varepsilon)_\nu d\sigma - 2H_\varepsilon(r)E_\varepsilon(r).$$

As mentioned above, the difference in parenthesis is positive by the Cauchy-Schwartz inequality. Therefore

$$(N_\varepsilon'(r))^- \leq 2\left|\frac{E_\varepsilon(r)}{H_\varepsilon(r)}\right| + \left|\frac{2r\int_{B_r} v_\varepsilon \Delta v_\varepsilon dx \int_{\partial B_r} v_\varepsilon (v_\varepsilon)_\nu d\sigma}{H_\varepsilon(r)^2}\right|.$$

**(A) Estimating** $2r\int_{B_r} v_\varepsilon \Delta v_\varepsilon dx \int_{\partial B_r} v_\varepsilon (v_\varepsilon)_\nu d\sigma$: On $\partial B_r, |(v_\varepsilon)_\nu| < C, |v_\varepsilon| < Cr$ by Lipschitz continuity. Therefore, arguing as in Lemma 2.5.4, we can estimate

$$\left|2r\int_{B_r} v_\varepsilon \Delta v_\varepsilon dx \int_{\partial B_r} v_\varepsilon (v_\varepsilon)_\nu d\sigma\right| \leq Cr^{n+1}\int_{B_{r+\varepsilon}\cap\partial\Omega} |v_\varepsilon|_\varepsilon \left(\frac{h(0)}{h(x)} - 1\right) d\omega^-$$

$$\leq Cr^{n+\alpha+2}\omega^-(B(0,r)),$$

where the last inequality follows from $|v_\varepsilon|_\varepsilon \le C\varepsilon < Cr$ on $\partial\Omega$ (by Lipschitz continuity).

From Lemma 2.5.5 it follows that

$$\left|\frac{2r\int_{B_r} v_\varepsilon \Delta v_\varepsilon dx \int_{\partial B_r} v_\varepsilon (v_\varepsilon)_\nu d\sigma}{H_\varepsilon(r)^2}\right| \le \frac{Cr^{n+\alpha+2}r^{2n-6}}{\omega^-(B(0,r))^3} = C\left(\frac{r^{n-1+\alpha/6}}{\omega^-(B(0,r))}\right)^3 r^{\alpha/2-1}.$$

**(B) Estimating** $2\left|\frac{E_\varepsilon(r)}{H_\varepsilon(r)}\right|$: Lemma 2.5.4 and Lemma 2.5.5 imply

$$2\left|\frac{E_\varepsilon(r)}{H_\varepsilon(r)}\right| \le C\frac{r^{n-2+\alpha}}{\omega^-(B(0,r))} = C\left(\frac{r^{n-1+\alpha/2}}{\omega^-(B(0,r))}\right) r^{\alpha/2-1}.$$

From (2.5.5) we can conclude

$$\frac{R^{n-1+\alpha/2}}{\omega^-(B(0,R))}, \frac{R^{3n-3+\alpha/2}}{\omega^-(B(0,R))^3} \overset{R\downarrow 0}{\to} 0.$$

Combine the estimates in (A) and (B) to conclude that $C(\varepsilon,R)R \le o_R(1)R^{\alpha/2}$. □

We can now prove a lower bound on the size of $N_\varepsilon(r)$ for small $r$.

**Corollary 2.5.7.** $\limsup_{\varepsilon\downarrow 0}\frac{1}{r}(N_\varepsilon(r)-1) > -Cr^{\alpha/2-1}$.

*Proof.* As $\lim_{s\downarrow 0} N(s) = 1$ there is some $r' << r$ such that $|N(r')-1| < Cr^{\alpha/2}$. Now pick $\varepsilon << r'$ small enough that Lemma 2.5.6 applies for $\varepsilon$ and all $r' < R < r$ and such that $|N_\varepsilon(r') - N(r')| < Cr^{\alpha/2}$(recall $N_\varepsilon(\rho) \to N(\rho)$ for fixed $\rho$ as $\varepsilon \downarrow 0$).

Let $j$ be such that $2^{-j}r < r' < 2^{-j+1}r$. Then

$$N_\varepsilon(r) - N_\varepsilon(r') \ge \sum_{\ell=0}^{j-2}(N_\varepsilon(2^{-\ell}r) - N_\varepsilon(2^{-\ell-1}r)) + N_\varepsilon(2^{-j+1}r) - N_\varepsilon(r') \ge$$

$$-C(2^{-j+1}r,\varepsilon)(2^{-j+1}r-r') - \frac{1}{2}\sum_{\ell=0}^{j-2} C(2^{-\ell}r,\varepsilon)2^{-\ell}r \overset{\text{Lem 2.5.6}}{\ge} -kr^{\alpha/2}\sum_{\ell=0}^{j-1}2^{-\ell\alpha/2} \ge -C_\alpha r^{\alpha/2}.$$

Combining the inequalities above we have that $N_\varepsilon(r) - 1 > -Cr^{\alpha/2}$ for small $\varepsilon > 0$. □

### 2.5.2 Monneau Monotonicity and Non-degeneracy

Our main tool here will be the Monneau potential, defined for $f \in H^1_{\text{loc}}(\mathbb{R}^n)$ and $p \in C^\infty(\mathbb{R}^n)$,

$$M^{x_0}(r, f, p) := \frac{1}{r^{n+1}} \int_{\partial B_r} (f(x + x_0) - p)^2 d\sigma(x). \tag{2.5.8}$$

Monneau, (Mon09), observed that if $f$ is a harmonic function vanishing to first order at $x_0$ and $p$ is a 1-homogenous polynomial then $M^{x_0}$ is monotonically decreasing as $r \downarrow 0$.

We follow closely the methods of Garofalo and Petrosyan ((GP09), see specifically Sections 1.4-1.5) who studied issues of non-degeneracy in an obstacle problem. Their program, which we adapt to our circumstances, has two steps: first relate the growth of the Monneau potential to the growth of Almgren's frequency function. Second, use this relation to establish lower bounds on the growth of $M$ and the existence of a limit at zero for $M$. As before, $v \equiv v^{(Q)}$ and without loss of generality, $Q = 0 \in \partial\Omega$. Additionally, $p$ will always be a 1-homogenous polynomial. We drop the dependence of $M$ on $Q$ and $v$ when no confusion is possible. Again $v_\varepsilon = v * \varphi_\varepsilon$, where $\varphi$ is an approximation to the identity. Naturally, $M_\varepsilon(r, p) := M^0(r, v_\varepsilon, p)$.

First we derive equations (2.5.9) and (2.5.10).

$$M'_\varepsilon(r, p) = \frac{2}{r^{n+2}} \int_{\partial B_r} (v_\varepsilon - p)(x \cdot \nabla(v_\varepsilon - p) - (v_\varepsilon - p)) d\sigma. \tag{2.5.9}$$

*Derivation of* (2.5.9). Let $x = ry$ so that $M_\varepsilon(r, p) = \int_{\partial B_1} \left(\frac{v_\varepsilon(ry)}{r} - \frac{p(ry)}{r}\right)^2 d\sigma(y)$. Differentiating under the integral gives

$$M'_\varepsilon(r, p) = \int_{\partial B_1} 2 \left(\frac{v_\varepsilon(ry)}{r} - \frac{p(ry)}{r}\right) \left(\frac{y}{r} \cdot \nabla_x [v_\varepsilon(ry) - p(ry)] - \frac{1}{r^2}(v_\varepsilon(ry) - p(ry))\right) d\sigma(y).$$

Changing back to $x$ we have that

$$M'_\varepsilon(r, p) = \frac{2}{r^{n+2}} \int_{\partial B_r} (v_\varepsilon - p)(x \cdot \nabla(v_\varepsilon - p) - (v_\varepsilon - p)) d\sigma(x).$$

□

Next we establish a relation between the derivative of $M$ and the growth rate of $N$ (we emphasize that (2.5.10) is true only when $p$ is a 1-homogenous polynomial).

$$\frac{H_\varepsilon(r)}{r^{n+1}}\left(N_\varepsilon(r)-1\right) = -\frac{1}{r^n}\int_{B_r}(v_\varepsilon - p)\Delta v_\varepsilon dx + rM_\varepsilon'(r,p)/2 \tag{2.5.10}$$

*Derivation of* (2.5.10). Recall for all 1-homogenous polynomials $p$ we have $N(r, x_0, p) \equiv 1$. We "add zero" and distribute to rewrite

$$\frac{H_\varepsilon(r)}{r^{n+1}}\left(N_\varepsilon(r)-1\right) = \frac{1}{r^n}\int_{B_r}|\nabla(v_\varepsilon - p)|^2 + 2\nabla v_\varepsilon \cdot \nabla p dx - \frac{1}{r^{n+1}}\int_{\partial B_r}(v_\varepsilon - p)^2 + 2v_\varepsilon p d\sigma.$$

Transform the first integral on the right hand side using integration by parts,

$$\frac{H_\varepsilon(r)}{r^{n+1}}\left(N_\varepsilon(r)-1\right) = \frac{1}{r^n}\int_{\partial B_r}\frac{x}{r}\cdot\nabla(v_\varepsilon - p)(v_\varepsilon - p) + 2\left(\frac{x}{r}\cdot\nabla p\right)v_\varepsilon$$

$$-\frac{1}{r^n}\int_{B_r}(v_\varepsilon - p)\Delta(v_\varepsilon - p) + 2v_\varepsilon\Delta p dx - \frac{1}{r^{n+1}}\int_{\partial B_r}(v_\varepsilon - p)^2 + 2v_\varepsilon p d\sigma.$$

As $p$ is a 1-homogenous polynomial, $\Delta p = 0$ and $x \cdot \nabla p - p = 0$. The above simplifies to

$$\frac{H_\varepsilon(r)}{r^{n+1}}\left(N_\varepsilon(r)-1\right) = -\frac{1}{r^n}\int_{B_r}(v_\varepsilon - p)\Delta v_\varepsilon + \frac{1}{r^{n+1}}\int_{\partial B_r}(x\cdot\nabla(v_\varepsilon - p) - (v_\varepsilon - p))(v_\varepsilon - p)d\sigma.$$

In light of (2.5.9), we are finished. □

The above two equations, along with Corollary 2.5.7, allow us to control the growth of $M$ from below.

**Lemma 2.5.8.** *Let $p$ be any 1-homogenous polynomial. Then for any $R > 0$ there exists a constant $C$ (independent of $R$ and $p$) such that*

$$M(R,p) - M(r,p) \geq -(C + C\|p\|_{L^\infty(\partial B_1)})R^{\alpha/2}$$

*for any* $r \in [R/4, R]$.

*Proof.* Recall (2.5.10),

$$rM_\varepsilon'(r,p)/2 = \frac{H_\varepsilon(r)}{r^{n+1}}(N_\varepsilon(r) - 1) + \frac{1}{r^n}\int_{B_r}(v_\varepsilon - p)\Delta v_\varepsilon.$$

Consider first the integral on the right hand side and argue as before to estimate,

$$\left|\frac{1}{r^n}\int_{B_r}(v_\varepsilon - p)\Delta v_\varepsilon\right| \leq \frac{1}{r^n}\int_{\partial\Omega\cap B_{r+\varepsilon}} |v_\varepsilon - p|_\varepsilon \left(\frac{h(x)}{h(0)} - 1\right) d\omega^- \\ \leq C(1 + \|p\|_{L^\infty(\partial B_1)})\frac{\omega^-(B(0,r))r^{1+\alpha}}{r^n},$$

where $|v_\varepsilon| < Cr$ on $\partial\Omega$ because $v$ is Lipschitz and $|p(x)| \leq C\|p\|_{L^\infty(\partial B_1)}r$ because $p$ is 1-homogenous. By Corollary 2.4.2, $\frac{\omega^-(B(Q,r))}{r^{n-1}}$ is bounded uniformly in $r < 1$ and in $Q \in \partial\Omega$ on compacta. Therefore, $|\frac{1}{r^n}\int_{B_r}(v_\varepsilon - p)\Delta v_\varepsilon| \leq C(1 + \|p\|_{L^\infty(\partial B_1)})r^\alpha$.

Returning to (2.5.10),

$$\limsup_{\varepsilon\downarrow 0}\sup_{R/4<r<R}(M_\varepsilon(r,p)')^- \leq C(1+\|p\|_{L^\infty(\partial B_1)})R^{\alpha-1} + \limsup_{\varepsilon\downarrow 0}\sup_{R/4<r<R}\frac{1}{r}(N_\varepsilon(r) - 1).$$

The bounds on the growth of $N$ (Corollary 2.5.7) imply

$$\limsup_{\varepsilon\downarrow 0}\sup_{R/4<r<R}(M_\varepsilon(r,p)')^- \leq (C + C\|p\|_{L^\infty(\partial B_1)})R^{\alpha/2-1},$$

which is equivalent to the desired result. □

When it is not relevant to the analysis (e.g. in the proofs of Lemma 2.5.9 and Proposition 2.5.10 below), we omit the dependence of the constant in Lemma 2.5.8 on $\|p\|_{L^\infty(\partial B_1)}$.

**Lemma 2.5.9.** *Let $p$ be any 1-homogenous polynomial. Then $M(0,p) := \lim_{r\downarrow 0} M(r,p)$ exists.*

*Proof.* Let $a := \limsup_{r\downarrow 0} M(r,p)$. That $a < \infty$ follows from Lemma 2.5.8, applied iteratively (as $r^{\alpha/2-1}$ is integrable at zero). We claim that there exists a constant $C < \infty$ such that $M(r,p) - a > -Cr^{\alpha/2}$ for any $0 < r \leq 1$.

On the other hand, $a - M(r,p) > -o(1)$ as $r \downarrow 0$ by the definition of $\limsup$. This, with the claim above, implies that $\lim_{r\downarrow 0} M(r,p) = a$.

Let us now address the claim: take $r_0 < r$. Let $k$ be such that $2^{-k}r \geq r_0 \geq 2^{-k-1}r$. Then, by Lemma 2.5.8, we have

$$M(r,p) - M(r_0,p) = \sum_{\ell=0}^{k-1}(M(2^{-\ell}r,p) - M(2^{-\ell-1}r,p)) + M(2^{-k}r,p) - M(r_0,p)$$

$$\geq -Cr^{\alpha/2}\sum_{\ell=0}^{\infty}(2^{\alpha/2})^{-\ell} \geq -C_\alpha r^{\alpha/2}.$$

The claim follows if we pick $r_0$ small so that $M(r_0,p)$ is arbitrarily close to $a$. □

Finally, we can establish the pointwise non-degeneracy of $\Theta^{n-1}(\omega^\pm, Q)$.

**Proposition 2.5.10.** *For all $Q \in \partial\Omega$ we have $\Theta^{n-1}(\omega^\pm, Q) > 0$.*

*Proof.* It suffices to assume $Q = 0$ and to prove $\Theta^{n-1}(\omega^-, 0) > 0$.

We proceed by contradiction. Pick some $r_j \downarrow 0$ so that $v_j \to p$ uniformly on compacta (where $p$ is a 1-homogenous polynomial given by Corollary 2.3.2). Lemma 2.5.9 implies

$$M(0,p) = \lim_{j\to\infty} M(r_j,p) = \lim_{j\to\infty}\int_{\partial B_1}\left(v_j(x)\frac{\omega^-(B(0,r_j))}{r_j^{n-1}} - \frac{p(r_jx)}{r_j}\right)^2 d\sigma(x).$$

As $\frac{p(r_jx)}{r_j} = p(x)$ and $\Theta^{n-1}(\omega^-, 0) = 0$, by assumption, we conclude $M(0,p) = \int_{\partial B_1} p^2 d\sigma$.

For any $j$, the homogeneity of $p$ implies

$$M(r_j,p) - M(0,p) = \frac{1}{r_j^{n+1}}\int_{\partial B_{r_j}}(v-p)^2 - \int_{\partial B_1} p^2 = \int_{\partial B_1}\left(v_j(y)\frac{\omega^-(B(0,r_j))}{r_j^{n-1}} - p\right)^2 - p^2 d\sigma$$

$$= \int_{\partial B_1}\left(v_j(y)\frac{\omega^-(B(0,r_j))}{r_j^{n-1}}\right)^2 - 2v_j(y)\frac{\omega^-(B(0,r_j))}{r_j^{n-1}}p(y)d\sigma \geq -Cr_j^{\alpha/2},$$

where the last inequality follows from iterating Lemma 2.5.8 (as in the proof of Lemma 2.5.9).

Rewrite the above equation as

$$\frac{\omega^-(B(0,r_j))}{r_j^{n-1}}\int_{\partial B_1} v_j(y)^2\frac{\omega^-(B(0,r_j))}{r_j^{n-1}} - 2v_j(y)p(y)d\sigma \geq -Cr_j^{\alpha/2}.$$

Divide by $\omega^-(B(0,r_j))/r_j^{n-1}$ and let $j \to \infty$. By (2.5.5) the right hand side vanishes and, by assumption, $\omega^-(B(0,r_j))/r_j^{n-1} \to 0$. In the limit we obtain $-2\int_{\partial B_1} p^2 \geq 0$, a contradiction. $\square$

At this point we have proven that $\infty > \Theta^{n-1}(\omega^-, Q) > 0$ everywhere on $\partial\Omega$ and that $\Theta^{n-1}(\omega^-, Q)$ is bounded uniformly from above on compacta. Using standard tools from geometric measure theory this implies, for all dimensions, the decomposition mentioned in the introduction (for $n = 2$): $\partial\Omega = \Gamma \cup N$, where $\omega^\pm(N) = 0$ and $\Gamma$ is a $(n-1)$-rectifiable set with $\sigma$-finite $\mathcal{H}^{n-1}$ measure.

## 2.6 Uniform non-degeneracy and initial regularity

### *2.6.1 $\Theta^{n-1}(\omega^\pm, Q)$ is bounded uniformly away from 0.*

In order to establish greater regularity for $\partial\Omega$ we need a uniform lower bound. Again the method of Garofalo and Petrosyan ((GP09), specifically Theorems 1.5.4 and 1.5.5) guides us. Our first step is to show that there is a unique tangent plane at every point.

**Lemma 2.6.1.** *For each $Q \in \partial\Omega$ there exists a unique 1-homogenous polynomial, $p^Q$, such that for any $r_j \downarrow 0$ we have $v_j \to p^Q$ uniformly on compacta (i.e. the limit described in Corollary 2.3.2 is unique).*

*Proof.* We prove it for $Q = 0$. Pick $r_j \downarrow 0$ so that $v_{r_j} \to p$ uniformly on compacta for some 1-homogenous polynomial $p$. Let $\tilde{r}_j \downarrow 0$ be another sequence so that $v_{\tilde{r}_j} \to \tilde{p}$, where $\tilde{p}$ is also a 1-homogenous polynomial.

By Lemma 2.5.9, $M(0, \Theta^{n-1}(\omega^-, 0)p)$ exists. Therefore,

$$\begin{aligned} M(0, \Theta^{n-1}(\omega^-, 0)p) &= \lim_{j\to\infty} M(r_j, \Theta^{n-1}(\omega^-, 0)p) \\ &= \lim_{j\to\infty} \int_{\partial B_1} \left( \frac{\omega^-(B(0, r_j))}{r_j^{n-1}} v_{r_j}(x) - \Theta^{n-1}(\omega^-, 0)p \right)^2 d\sigma \qquad (2.6.1) \\ &= 0. \end{aligned}$$

The last equality above follows by the dominated convergence theorem and that $v_{r_j} \to p$.

Similarly,

$$\begin{aligned} M(0, \Theta^{n-1}(\omega^-, 0)p) &= \lim_{j\to\infty} M(\tilde{r}_j, \Theta^{n-1}(\omega^-, 0)p) \\ &= \lim_{j\to\infty} \int_{\partial B_1} \left( \frac{\omega^-(B(0, \tilde{r}_j))}{\tilde{r}_j^{n-1}} v_{\tilde{r}_j}(x) - \Theta^{n-1}(\omega^-, 0)p \right)^2 \\ &= (\Theta^{n-1}(\omega^-, 0))^2 \int_{\partial B_1} (\tilde{p} - p)^2 d\sigma. \end{aligned}$$

Again the last equality follows by dominated convergence theorem and that $v_{\tilde{r}_j} \to \tilde{p}$. As $\Theta^{n-1}(\omega^-, 0) > 0$ (Proposition 2.5.10), we have $p = \tilde{p}$. □

We should note that Lemma 2.5.9 (the existence of a limit at 0) and Lemma 2.5.8 (estimates on the derivatives of $M$) both hold for $M^Q(r, v^{(Q)}, p)$ where $p$ is any 1-homogenous polynomial and (as before) $v^{(Q)}(y) = h(Q)u^+(y) - u^-(y)$. Furthermore the constants in Lemma 2.5.8 are uniform for $Q$ in a compact set. We now prove the main result of this subsection.

**Proposition 2.6.2.** *The function* $Q \mapsto \tilde{p}^Q := \Theta^{n-1}(\omega^-, Q)p^Q$ *is a continuous function from* $\partial\Omega \to C(\mathbb{R}^n)$.

*Proof.* As $\tilde{p}^Q$ is a 1-homogenous polynomial, it suffices to show that $Q \mapsto \tilde{p}^Q$ is a continuous function from $\partial\Omega \to L^2(\partial B_1)$.

Pick $\varepsilon > 0$ and $Q \in \partial\Omega$. Equation 2.6.1 implies that $M^Q(0, v^{(Q)}, \tilde{p}^Q) = 0$. In particular, there is a $r_\varepsilon > 0$ such that if $r \leq r_\varepsilon$ then $M^Q(r, v^{(Q)}, \tilde{p}^Q) < \varepsilon$. Shrink $r_\varepsilon$ so that $r_\varepsilon^{\alpha/2} < \varepsilon$.

$v^{(Q)} \in W^{1,\infty}_{loc}(\mathbb{R}^n)$ (uniformly for $Q$ in a compact set) and $h \in C^\alpha(\partial\Omega)$, so there exists a $\delta = \delta(r_\varepsilon, \varepsilon) > 0$ such that for all $P \in B_\delta(Q)$ and $x \in B_1(0)$ we have

$$|v^{(Q)}(x + Q) - v^{(P)}(x + P)| < \varepsilon r_\varepsilon. \tag{2.6.2}$$

Since $\sup_{P \in B_\delta(Q)} \|v^{(P)}(- + P)\|_{L^\infty(\partial B_{r_\varepsilon})} < r_\varepsilon$, (2.6.2) immediately implies that

$$\left| M^Q(r_\varepsilon, v^{(Q)}, \tilde{p}^Q) - \frac{1}{r_\varepsilon^{n+1}} \int_{\partial B_{r_\varepsilon}} (v^{(P)}(x + P) - \tilde{p}^Q)^2 \right| < C\varepsilon, \ \forall P \in B_\delta(Q).$$

By definition, $M^Q(r_\varepsilon, v^{(Q)}, \tilde{p}^Q) < \varepsilon$, so it follows that

$$M^P(r_\varepsilon, v^{(P)}, \tilde{p}^Q) \equiv \frac{1}{r_\varepsilon^{n+1}} \int_{\partial B_{r_\varepsilon}} (v^{(P)}(x + P) - \tilde{p}^Q)^2 < C\varepsilon, \ \forall P \in B_\delta(Q).$$

Repeated application of Lemma 2.5.8 yields,

$$M^P(r_\varepsilon, v^{(P)}, \tilde{p}^Q) - M^P(0, v^{(P)}, \tilde{p}^Q) > -(C + C\|\tilde{p}^Q\|_{L^\infty(\partial B_1)}) r_\varepsilon^{\alpha/2}, \ \forall P \in B_\delta(Q) \Rightarrow$$

$$C\varepsilon > M^P(0, v^{(P)}, \tilde{p}^Q) = \int_{\partial B_1} (\tilde{p}^P - \tilde{p}^Q)^2, \ \forall P \in B_\delta(Q).$$

That the first line implies the second follows from $\|\tilde{p}^Q\|_{L^\infty(\partial B_1)} = \Theta^{n-1}(\omega^-, Q) < C$ uniformly on compacta, $r_\varepsilon^{\alpha/2} < \varepsilon$ and $M^P(r_\varepsilon, v^{(P)}, \tilde{p}^Q) < C\varepsilon$. The equality in the second line follows from the standard blowup argument (see the proof of Lemma 2.6.1) and allows us to conclude that $Q \mapsto \tilde{p}^Q$ is continuous from $\partial\Omega \to L^2(\partial B_1)$. □

**Corollary 2.6.3.** *The function $Q \mapsto \Theta^{n-1}(\omega^-, Q)$ is continuous. Additionally, the function $Q \mapsto \{p^Q = 0\}$ is continuous (from $\partial\Omega$ to $\mathrm{G}(n, n-1)$).*

*Proof.* Clearly the first claim, combined with Proposition 2.6.2, implies the second.

For $Q_1, Q_2 \in \partial\Omega$, if $P_1 = \{p^{Q_1} = 0\}, P_2 = \{p^{Q_2} = 0\}$ are distinct hyperplanes with normals $\hat{n}_1, \hat{n}_2$, then both $(\hat{n}_1 + \hat{n}_2)^\perp$ and $(\hat{n}_1 - \hat{n}_2)^\perp$ consist of points equidistant from $P_1$ and $P_2$. Elementary geometry then shows that there is some constant $c > 0$ such that

$$\max\{D[(\hat{n}_1 + \hat{n}_2)^\perp \cap B_1(0), P_1 \cap B_1(0)], D[(\hat{n}_1 - \hat{n}_2)^\perp \cap B_1(0), P_1 \cap B_1(0)]\} \geq c.$$

Let $P_3(Q_1, Q_2)$ be the plane which achieves this maximum. If $P_1 = P_2$ then pick $P_3(Q_1, Q_2)$ to be any hyperplane such that $D[P_1 \cap B_1(0), P_3 \cap B_1(0)] \geq c$.

Recall Corollary 2.3.2, which implies that $p^Q$ is a monic 1-homogenous polynomial for all $Q \in \partial\Omega$. So if $y \in P_3(Q_1, Q_2) \cap \partial B_1(0)$, there is an universal $\tilde{c} > 0$ such that $\tilde{c} < |p^{Q_1}(y)| = |p^{Q_2}(y)|$.

Therefore,

$$\begin{aligned} \|\tilde{p}^{Q_1} - \tilde{p}^{Q_2}\|_{L^\infty(\partial B_1)} \geq & |\Theta^{n-1}(\omega^-, Q_1)p^{Q_1}(y) - \Theta^{n-1}(\omega^-, Q_2)p^{Q_2}(y)| \\ \geq & \tilde{c}|\Theta^{n-1}(\omega^-, Q_1) - (\text{sgn}\, p^{Q_1}(y)p^{Q_2}(y))\Theta^{n-1}(\omega^-, Q_2)|. \end{aligned}$$

If sgn $p^{Q_1}(y)p^{Q_2}(y) = -1$ ($p^{Q_1}(y)$ and $p^{Q_2}(y)$ have opposite signs), then

$$\|\tilde{p}^{Q_1} - \tilde{p}^{Q_2}\|_{L^\infty(\partial B_1)} \geq \tilde{c}(\Theta^{n-1}(\omega^-, Q_1) + \Theta^{n-1}(\omega^-, Q_2)) \geq \tilde{c}\Theta^{n-1}(\omega^-, Q_1).$$

Letting $Q_2 \to Q_1$, the continuity of $Q \mapsto \tilde{p}^Q$ (Proposition 2.6.2) implies $0 \geq \Theta^{n-1}(\omega^-, Q_1)$. This contradicts the non-degeneracy of $\Theta^{n-1}(\omega^-, Q_1)$ (Proposition 2.5.10).

On the other hand, if sgn $p^{Q_1}(y)p^{Q_2}(y) = 1$ ($p^{Q_1}(y)$ and $p^{Q_2}(y)$ share the same sign), then

$$\|\tilde{p}^{Q_1} - \tilde{p}^{Q_2}\|_{L^\infty(\partial B_1)} \geq \tilde{c}|\Theta^{n-1}(\omega^-, Q_1) - \Theta^{n-1}(\omega^-, Q_2)|,$$

and the continuity of $Q \mapsto \tilde{p}^Q$ implies that $Q \mapsto \Theta^{n-1}(\omega^-, Q)$ is continuous. □

Uniform non-degeneracy immediately follows.

**Corollary 2.6.4.** *For any $K \subset\subset \mathbb{R}^n$ there is a $c = c(K) > 0$ such that, for all $Q \in K \cap \partial\Omega$,*

$$\Theta^{n-1}(\omega^{\pm}, Q) > c.$$

### *2.6.2 $\partial\Omega$ is a $C^1$ domain*

We define for $Q_0 \in \partial\Omega$ and $r > 0$

$$\beta(Q_0, r) = \inf_P \frac{1}{r} \sup_{Q \in \partial\Omega \cap B_r(Q_0)} \mathrm{dist}(Q, P) \tag{2.6.3}$$

where the infimum is taken over all $(n-1)$-dimensional hyperplanes through $Q_0$ (these are a variant of Jones' $\beta$-numbers, see (Jon90)). David, Kenig and Toro (see (DKT01), Proposition 9.1) show that, under suitable assumptions, $\beta(Q_0, r) \lesssim r^\gamma$ implies that $\partial\Omega$ is locally the graph of a $C^{1,\gamma}$ function for any $1 > \gamma > 0$. We will adapt this proof to show that $\partial\Omega$ is locally the graph of a $C^1$ function.

For any $Q_0 \in \partial\Omega$,

$$P(Q_0) := \{p^{Q_0} = 0\}$$

(where $p^{Q_0}$ is the 1-homogenous polynomial guaranteed to exist by Corollary 2.3.2 and which is unique by Lemma 2.6.1). By the definition of blowups, we know that $P(Q_0) + Q_0$ approximates $\partial\Omega$ near $Q_0$. The following lemma shows that this approximation is uniformly tight in $Q_0$.

**Lemma 2.6.5.** *[Compare to (DKT01), equation 9.14] Let $K \subset\subset \mathbb{R}^n$ and $\varepsilon > 0$. Then there is an $R = R(K, \varepsilon) > 0$ such that $r < R$ and $Q_0 \in K \cap \partial\Omega$ implies*

$$\sup_{Q \in \partial\Omega \cap B_r(Q_0)} \frac{1}{r} \mathrm{dist}\left(Q - Q_0, P(Q_0)\right) < \varepsilon. \tag{2.6.4}$$

*Proof.* The proof hinges on the following estimate (see (GP09) Theorem 1.5.5); for any $K$

compact there exists a modulus of continuity $\sigma_K$ with $\lim_{t\downarrow 0} \sigma_K(t) = 0$ such that

$$|v^{(Q_0)}(x + Q_0) - \tilde{p}^{Q_0}(x)| \leq \sigma_K(|x|)|x| \tag{2.6.5}$$

for any $Q_0 \in K \cap \partial\Omega$.

Assume this estimate is true; let $Q \in B_r(Q_0) \cap \partial\Omega$ and write $Q = Q_0 + x$. As $\Theta^{n-1}(\omega^-, Q_0) > c$ for all $Q_0 \in K \cap \partial\Omega$ (Corollary 2.6.4) it follows that $\mathrm{dist}(Q - Q_0, P(Q_0)) \lesssim |\tilde{p}^{Q_0}(x)|$. Then (2.6.5) yields that $\mathrm{dist}(Q - Q_0, P(Q_0)) \lesssim |\tilde{p}^{Q_0}(x)| \leq |x|\sigma_K(|x|) = r\sigma_K(r)$. Set $R$ to be small enough so that $r < R$ implies $\sigma_K(r) < \varepsilon$ to prove (2.6.4).

Thus it suffices to establish (2.6.5). Let $|x| = r$ and write $x = ry$ with $|y| = 1$. If we divide by $r$, (2.6.5) is equivalent to

$$|v^{(Q_0)}(ry + Q_0)/r - \tilde{p}^{Q_0}(y)| \leq \sigma_K(r). \tag{2.6.6}$$

As $v^{(Q)}(ry+Q)/r$ is locally Lipschitz (uniformly in $Q$ on compacta), the uniform estimate (2.6.6) follows from an $L^2$ estimate: for all $\varepsilon > 0$, there exists a $R = R_{K,\varepsilon} > 0$ such that if $r < R$ and $Q_0 \in K \cap \partial\Omega$ then

$$M^{Q_0}(r, v^{(Q_0)}, \tilde{p}^{Q_0}) \equiv \|v^{(Q_0)}(ry + Q_0)/r - \tilde{p}^{Q_0}(y)\|^2_{L^2(\partial B_1)} < \varepsilon.$$

For each point $Q \in K \cap \partial\Omega$ we can find an $R = R_\varepsilon(Q)$ such that $R << \varepsilon$ and for all $r < R$, $|M^Q(r, v^{(Q)}, \tilde{p}^Q)| < \varepsilon/4$. Furthermore, for every $r > 0$ there is a $\delta(r) > 0$ such that for $Q, Q' \in K \cap \partial\Omega$ we have

$$|Q - Q'| < \delta(r) \Rightarrow |M^{Q'}(r, v^{(Q')}, \tilde{p}^{Q'}) - M^Q(r, v^{(Q)}, \tilde{p}^Q)| < \varepsilon/4.$$

The existence of $\delta(r)$ follows from the uniform Lipschitz continuity of $v^{(Q)}$, the Hölder continuity of $h$ and the continuity of $Q \mapsto \tilde{p}^Q$.

As $K$ is compact we can find $Q_1, ..., Q_n \in K \cap \partial\Omega$ such that if $\delta_1 := \delta(R_\varepsilon(Q_1)), ..., \delta_n := \delta(R_\varepsilon(Q_n))$ then $K \cap \partial\Omega \subset \bigcup B_{\delta_i}(Q_i)$. By the definition of $\delta_i$, if $Q' \in B_{\delta_i}(Q_i)$, then $M^{Q'}(R_\varepsilon(x_i), v^{(Q')}, \tilde{p}^{Q'}) < \varepsilon/2$. Recall, $R_\varepsilon(Q_i) << \varepsilon$ and Lemma 2.5.8 to conclude that for all $Q' \in B_{\delta_i}(Q_i)$ and $r < R_\varepsilon(Q_i), M^{Q'}(r, v^{(Q')}, \tilde{p}^{Q'}) < \varepsilon$. Therefore, setting $R_{K,\varepsilon} \equiv \min_i\{R_\varepsilon(Q_i)\}$ gives the $L^2$ estimate $r < R_{K,\varepsilon}, Q' \in K \cap \partial\Omega \Rightarrow M^{Q'}(r, v^{(Q')}, \tilde{p}^{Q'}) < \varepsilon$. □

We should note, (2.6.5) (along with the Whitney extension theorem) allows for an alternative proof that $\partial\Omega$ is a $C^1$ domain (see (GP09) Theorem 1.3.8). We will, however, continue our proof in the vein of (DKT01).

**Proposition 2.6.6.** *Let $\Omega \subset \mathbb{R}^n$ satisfy the conditions of Theorem 2.1.1 or Theorem 2.1.2. If $\log(h) \in C^{0,\alpha}(\partial\Omega)$ then $\Omega$ is a $C^1$ domain.*

*Proof.* For $Q_0 \in \partial\Omega$, equation 2.6.4 shows that $P(Q_0) + Q_0$ is a tangent plane to $\partial\Omega$ at $Q_0$. Furthermore, $Q_0 \mapsto P(Q_0)$ is continuous (Corollary 2.6.3). Under the assumptions of Theorem 2.1.2, $\Omega$ is a Lipschitz domain with a tangent plane at every $Q \in \partial\Omega$ that varies continuously in $Q$; thus we are done.

If we simply assume that $\Omega$ is Reifenberg flat (Theorem 2.1.1), we still need to show that $\Omega$ is a graph domain (in fact we will show it is a Lipschitz domain). Let $R = R_{K,\varepsilon} > 0$ be chosen later and let $r < R$. If $R$ is small enough, vanishing Reifenberg flatness (Corollary 2.3.2), along with Lemma 2.6.5, implies

$$\pi(\{\partial\Omega \cap B(Q_0, r) - Q_0\}) \supset P(Q_0) \cap B(0, \frac{r}{2}),\ \forall Q_0 \in K \cap \partial\Omega, r < R.$$

Here $\pi : \mathbb{R}^n \to P(Q_0)$ is a projection (for more details see the proof of (DKT01) Lemma 8.3 or (KT97) Remark 2.2).

We need only to show that $\pi^{-1}$ is a well defined function with bounded Lipschitz norm on $P(Q_0) \cap B(0, r/2)$. Let $\Sigma := (\partial\Omega - Q_0) \cap B(0, r) \cap \pi^{-1}(B(0, r/2))$ and pick distinct $Q_1, Q_2 \in \Sigma$. Perhaps shrinking $R$ again, the continuity of $Q \mapsto P(Q)$, combined with

Lemma 2.6.5, implies

$$\frac{1}{|Q_1 - Q_2|}\text{dist}(Q_1 - Q_2, P(Q_0)) < \varepsilon. \tag{2.6.7}$$

Therefore, $\pi^{-1}$ is well defined and $\|\pi^{-1}\|_{\text{Lip}(P(Q_0)\cap B(0,r/2))} < (1-\varepsilon)^{-1}$. □

It should be noted that if $\Omega$ is a $C^1$ domain it is not necessarily true that $u \in C^1(\overline{\Omega})$ (see (Pom92), pg 45). However, as $\Theta^{n-1}(\omega^{\pm}, Q)$ is continuous, we can establish the following.

**Corollary 2.6.7.** *Let $\Omega, \log(h)$ be as in Proposition 2.6.6. Then $u^{\pm} \in C^1(\overline{\Omega^{\pm}})$.*

*Proof.* For $Q \in \partial\Omega$, let $\nu(Q)$ be the inward pointing normal to $\Omega$ at $Q$. We will prove that

$$\lim_{\substack{X \to Q \\ X \in \Omega^+}} D_i u^+(X) = (\nu(Q) \cdot e_i)\Theta^{n-1}(\omega^+, Q), \forall i = 1, ..., n.$$

The desired result follows from $\Theta^{n-1}(\omega^+, -), \nu(-) \in C(\partial\Omega)$ (Corollary 2.6.3 and Proposition 2.6.6). The proof for $u^-$ is identical.

Pick $r$ small so that $B(Q,r) \cap \partial\Omega$ can be written as the graph of a $C^1$ function. Then construct a bounded NTA domain $\Omega_B \subset \Omega$ such that $\partial\Omega_B \cap \partial\Omega = B(Q,r) \cap \partial\Omega$ (see (JK82) Lemma 6.3 and (KT03) Lemma A.3.3). For $X_0 \in \Omega_B$, let $\omega_B^{X_0}$ be the harmonic measure of $\Omega_B$ with a pole at $X_0$. By local Lipschitz continuity, $|D_i u^+| < C$ on $\Omega_B$ and, therefore, $D_i u^+$ has a non-tagential limit $g(P)$ for $\omega_B^{X_0}$-a.e. $P$ in $\partial\Omega_B$ (see Section 5 in (JK82)). Furthermore, if $K(X,P) := \frac{d\omega_B^X}{d\omega_B^{X_0}}(P)$ we have the following representation (see (JK82) Corollary 5.12),

$$D_i u^+(X) = \int_{\partial\Omega_B} g(P)K(X,P)d\omega_B^{X_0}(P).$$

Using blowup analysis, one computes $g(P) = (\nu(P) \cdot e_i)\Theta^{n-1}(\omega^+, P)$ for $P \in \partial\Omega_B \cap B(Q, r/2)$. As $g(P)$ is continuous on $B(Q,r/2) \cap \partial\Omega_B$, there is some $s < r/2$ such that $P \in B(Q,s) \cap \partial\Omega_B \Rightarrow |g(P) - g(Q)| < \varepsilon$. On the other hand, Jerison and Kenig (Lemma

4.15) proved that $\lim_{X\in\Omega_B, X\to Q}\sup_{P\in\partial\Omega_B\setminus B(Q,s)} K(X,P) = 0$. This allows us to estimate,

$$\lim_{\substack{X\to Q\\X\in\Omega^+}} |D_i u^+(X) - (\nu(Q)\cdot e_i)\Theta^{n-1}(\omega^+,Q)| = \lim_{\substack{X\to Q\\X\in\Omega_B}} |D_i u^+(X) - g(Q)|$$

$$\leq \lim_{\substack{X\to Q\\X\in\Omega_B}} \int K(X,P)|g(P)-g(Q)|(\chi_{\partial\Omega_B\setminus B(Q,s)}(P) + \chi_{\partial\Omega_B\cap B(Q,s)}(P))d\omega_B^{X_0}(P)$$

$$\leq \lim_{\substack{X\to Q\\X\in\Omega_B}} C\omega_B^{X_0}(\partial\Omega_B\setminus B(Q,s)) \sup_{P\notin B(Q,s)} K(X,P) + \varepsilon\omega_B^X(B(Q,s)) \leq \varepsilon.$$

The first equality follows from the fact that any sequence in $\Omega^+$ approaching $Q$ must, apart from finitely many terms, be contained in $\Omega_B$. The last line follows first from $|g(P)| < C$ and then from the fact that $\omega_B^X$ is a probability measure for any $X\in\Omega_B$. □

## 2.7 Initial Hölder regularity: $\partial\Omega$ is $C^{1,s}$

In this section we will prove that $\partial\Omega$ is locally the graph of a $C^{1,s}$ function for some $0 < s \leq \alpha$. Note that, in general, the best one can hope for is $s = \alpha$ (if $\partial\Omega$ is the graph of a $C^{1,\alpha}$ function then $\log(h)\in C^{0,\alpha}$).

Here we will borrow heavily from the arguments of De Silva et al. (DFS14), who prove $C^{1,\gamma}$ regularity for a wide class of non-homogenous free boundary problems. We cannot immediately apply their results, as they assume a non-degeneracy in the free boundary condition that our problem does not have (see condition (H2) in Section 7 of (DFS14)). It should also be noted that our main result in this section is not immediately implied by the remark at the end of Caffarelli's paper, (Caf87). Indeed, Caffarelli's free boundary condition also contains an *a priori* non-degeneracy condition (see condition (a) at the top of page 158 in (Caf87)) which our problem lacks.

### 2.7.1 The Iterative Argument

In this section we shall state the main lemma and show how that lemma, through an iterative argument, implies our desired result. First we need two definitions.

**Definition 2.7.1.** *Let* $g : \mathbb{R}^n \to \mathbb{R}$*. Then* $w \in C(B_1(0))$ *is* **a solution to the free boundary problem associated to** $g$ *if:*

- $w \in C^2(\{w > 0\}) \cap C^2(\{w < 0\})$
- $w \in C^1(\overline{\{w > 0\}}) \cap C^1(\overline{\{w < 0\}})$
- *w satisfies, in* $B_1(0)$*, the following:*

$$\begin{aligned} \Delta w(x) &= 0,\ x \in \{w \neq 0\} \\ (w^+)_{\nu_x}(x) g(x) &= -(w^-)_{\nu_x}(x),\ x \in \{w = 0\} \end{aligned} \tag{2.7.1}$$

  *where* $\nu_x$ *is the normal to* $\{w = 0\}$ *at* $x$*.*

One observes that Corollary 2.6.7 implies that $u$ is a solution to the free boundary problem associated to $h$. We now need the notion of a "two-plane solution".

**Definition 2.7.2.** *Let* $\gamma > 0$ *and* $g : \mathbb{R}^n \to \mathbb{R}$*. Then for any* $x_0 \in B_1(0)$ *we can define the* **two-plane solution associated to** $g$ **at** $x_0$*:*

$$U_\gamma^{(x_0)}(t) := \gamma t^+ - g(x_0)\gamma t^-,\ t \in \mathbb{R}.$$

*When no confusion is possible we drop the dependence on* $x_0$*. It should also be clear from context to which function* $g$ *our* $U$ *is associated.*

The following remark, which follows immediately from Corollary 2.3.2 and (2.6.5), elucidates the relationship between a two-plane solution and our function $u$.

**Remark 2.7.3.** *Let $x_0 \in \partial\Omega$. As $r \to 0$ it is true that*

$$u_{r,x_0}(x) := \frac{u(rx + x_0)}{r} \to U^{(x_0)}_{\Theta^{n-1}(\omega^+,x_0)}(x \cdot \nu_{x_0})$$

*uniformly on compacta. Here $U$ is the two-plane solution associated to $h$. Furthermore, the rate of this convergence is independent of $x_0 \in K \cap \partial\Omega$ for $K$ compact.*

Intuitively, the faster the rate of this convergence, the greater the regularity of $\partial\Omega$. This relationship motivates the following lemma (compare with (DFS14), Lemma 8.3), which says roughly that if $u$ is close to a two-plane solution in a large ball, then $u$ is in fact even closer to a, possibly different, two-plane solution in a smaller ball.

**Lemma 2.7.4.** *Let $\infty > C_1, c_1 > 0$ and $\tilde{k} > 0$. Let $v$ be a solution to a free boundary problem associated to $g$ such that $\inf_{x \in B_2(x_0)} g(x) \geq \tilde{k} > 0$ and such that $v(x_0) = 0$. Let $\varepsilon > 0, C_1 > \gamma > c_1, \nu \in \mathbb{S}^{n-1}$ and assume*

$$U^{(x_0)}_{\gamma}(x \cdot \nu - \varepsilon) \leq v(x + x_0) \leq U^{(x_0)}_{\gamma}(x \cdot \nu + \varepsilon), \; x \in B_1(0). \tag{2.7.2}$$

*Also, assume that $\sup_{x,y \in B_1(x_0)} \frac{|g(x)-g(y)|}{|x-y|^\alpha} < \varepsilon^2$.*

*Then there exists some $R_0 = R_0(C_1, c_1, n) > 0$ such that for all $r < R_0$ there is a $\tilde{\varepsilon} = \tilde{\varepsilon}(r, C_1, c_1, n) > 0$ so that if the $\varepsilon$ above satisfies $\varepsilon \leq \tilde{\varepsilon}$ then*

$$U^{(x_0)}_{\gamma'}(x \cdot \nu' - r\frac{\varepsilon}{2}) \leq v(x + x_0) \leq U^{(x_0)}_{\gamma'}(x \cdot \nu' + r\frac{\varepsilon}{2}), \; x \in B_r(0), \tag{2.7.3}$$

*where $|\nu'| = 1, |\nu' - \nu| \leq \tilde{C}\varepsilon$ and $|\gamma - \gamma'| \leq \tilde{C}\gamma\varepsilon$. Here $\tilde{C} = \tilde{C}(C_1, c_1, n) > 0$.*

With this lemma we can prove Hölder regularity by way of an iterative argument.

**Proposition 2.7.5.** *Let $\Omega \subset \mathbb{R}^n$ be a 2-sided NTA domain with $\log(h) \in C^{0,\alpha}(\partial\Omega)$.*

- *If $n = 2$, then $\Omega$ is a $C^{1,s}$ domain for some $s > 0$.*

- *If $n \geq 3$, assume that either $\Omega$ is a $\delta$-Reifenberg flat domain for some $0 < \delta$ small enough or that $\Omega$ is a Lipschitz domain. Then $\Omega$ is a $C^{1,s}$ domain for some $s > 0$.*

*Proof of Proposition 2.7.5 assuming Lemma 2.7.4.* Without loss of generality let $0 \in \partial\Omega$ and $e_n$ be the inward pointing normal to $\Omega$ at $x_0 \in B_1(0) \cap \partial\Omega$. We will show that $\beta(x_0, t) \leq C'' t^s$ for some $s > 0$ and some $C'' > 0$ independent of $t > 0, x_0 \in \partial\Omega \cap B_1(0)$. A theorem of David, Kenig and Toro ((DKT01), Proposition 9.1) then implies that $\partial\Omega$ is locally the graph of a $C^{1,s}$ function.

Set $\gamma = \Theta^{n-1}(\omega^+, x_0)$ and let

$$C_1 := 2 \sup_{z \in \partial\Omega \cap B_4(0)} \Theta^{n-1}(\omega^+, z),\ c_1 := \frac{1}{2} \inf_{z \in \partial\Omega \cap B_4(0)} \Theta^{n-1}(\omega^+, z).$$

By Corollary 2.6.4 and the work of Section 2.4 we have $\infty > C_1 \geq c_1 > 0$.

Lemma 2.7.4 gives us an $R_0$. Pick $0 < \overline{r} \leq R_0$ small enough so that $\overline{r}^\alpha < \frac{1}{4}$. We then get a $\tilde{\varepsilon} > 0$ depending on $\overline{r}$. Pick $\varepsilon < \tilde{\varepsilon}$ such that

$$1/2 \leq \left( \prod_{k=0}^{\infty} (1 - \tilde{C}\varepsilon/2^k) \right) < \left( \prod_{k=0}^{\infty} (1 + \tilde{C}\varepsilon/2^k) \right) \leq 2$$

where $\tilde{C}$ is the constant from Lemma 2.7.4.

Recall Remark 2.7.3, that $u_{\rho,x_0}(x) \to U_\gamma(x_n)$ for $x \in B_1$ as $\rho \downarrow 0$. Thus, for small enough $\rho$, we have

$$\|u_{\rho,x_0}(x) - U_\gamma(x_n)\|_{L^\infty(B_1)} < K\varepsilon,$$

where $K \leq \min\{c_1, \inf_{x \in B_1} |h(x)| c_1\}$. This implies

$$U_\gamma(x_n - \varepsilon) \leq u_{\rho,x_0}(x) \leq U_\gamma(x_n + \varepsilon), x \in B_1(0).$$

$u_{\rho,x_0}$ is a solution to the free boundary problem associated to $g(x) = h(\rho x + x_0)$. In particular, if $\rho$ is small enough such that $\rho^\alpha \|h\|_{C^{0,\alpha}} < \varepsilon^2$ then $g$ satisfies the growth and lower bound assumptions of Lemma 2.7.4.

If $u^0(x) := u_{\rho,x_0}(x)$, then we can apply Lemma 2.7.4 to $u^0$ in the direction $e_n$ with $\gamma, C_1, c_1, \overline{r}, \varepsilon$ as above. This gives us a $\nu_1 \in \mathbb{S}^{n-1}$ and a $\gamma_1 > 0$ such that

$$U_{\gamma_1}(x \cdot \nu_1 - \overline{r}\frac{\varepsilon}{2}) \leq u^0(x) \leq U_{\gamma_1}(x \cdot \nu_1 + \overline{r}\frac{\varepsilon}{2}),\ x \in B_{\overline{r}}(0).$$

Write $x = \overline{r}y$ and divide the above equation by $\overline{r}$ to obtain,

$$U_{\gamma_1}(y \cdot \nu_1 - \frac{\varepsilon}{2}) \leq u^0(\overline{r}y)/\overline{r} \leq U_{\gamma_1}(y \cdot \nu_1 + \frac{\varepsilon}{2}),\ y \in B_1(0).$$

Let $u^1(z) := u^0(\overline{r}z)/\overline{r}$ so that

$$U_{\gamma_1}(y \cdot \nu_1 - \varepsilon/2) \leq u^1(y) \leq U_{\gamma_1}(y \cdot \nu_1 + \varepsilon/2),\ y \in B_1(0)$$

Apply Lemma 2.7.4 to $u^1$ in direction $\nu_1$ with $C_1, c_1, \gamma_1, \varepsilon/2, \overline{r}$ and iterate.

In this way, we create a sequence of $u^k(y), \theta_k, \gamma_k, \nu_k$ such that

$$U_{\gamma_k}(y \cdot \nu_k - \varepsilon/2^k) \leq u^k(y) \leq U_{\gamma_k}(y \cdot \nu_k + \varepsilon/2^k),\ y \in B_1(0)$$

and $|\nu_k - \nu_{k+1}| < \tilde{C}\varepsilon/2^k$. We must prove that it is valid to apply Lemma 2.7.4 at each step.

By Lemma 2.7.4 and construction,

$$c_1 \leq \frac{1}{2}\gamma \leq \prod_{i=0}^{k-1}(1 - \tilde{C}\varepsilon/2^k)\gamma \leq \gamma_k \leq \prod_{i=0}^{k-1}(1 + \tilde{C}\varepsilon/2^k)\gamma \leq 2\gamma \leq C_1,$$

so $\gamma_k$ is always in the acceptable range for another application of Lemma 2.7.4. Also in the $k$th step we apply the lemma with $\varepsilon/2^k < \varepsilon < \tilde{\varepsilon}$ and the same $\overline{r}$.

Finally, in the $k$th step we have $u^k(y) = u_{\rho\overline{r}^k,x_0}(y)$. Thus we need to make sure that $(\rho\overline{r}^k)^\alpha \|h\|_{C^{0,\alpha}} < (\varepsilon/2^k)^2$. By construction, $\rho^\alpha \|h\|_{C^{0,\alpha}} < \varepsilon^2$ and $\overline{r}^{k\alpha} \leq \frac{1}{4}^k$ and so the conditions of Lemma 2.7.4 are satisfied for each $k$.

After $k$ steps,

$$U_{\gamma_k}(y\cdot\nu_k-\varepsilon/2^k)\le u^k(y)\le U_{\gamma_k}(y\cdot\nu_k+\varepsilon/2^k),\ y\in B_1(0)\Rightarrow$$

$$U_{\gamma_k}(x\cdot\nu_k-\rho\overline{r}^k\varepsilon/2^k)\le u(x+x_0)\le U_{\gamma_k}(x\cdot\nu_k+\rho\overline{r}^k\varepsilon/2^k),\ x\in B_{\rho\overline{r}^k}(0).$$

If $x\in B_{\rho\overline{r}^k}(0)$ is taken such that $x+x_0\in\partial\Omega$ then the above equation implies

$$x\cdot\nu_k-\rho\overline{r}^k\varepsilon/2^k<0<x\cdot\nu_k+\rho\overline{r}^k\varepsilon/2^k\Rightarrow$$

$$|x\cdot\nu_k|\le\rho\overline{r}^k\varepsilon/2^k\Rightarrow\beta(x_0,\rho\overline{r}^k)\le\varepsilon/2^k.$$

If $s:=-\log_{\overline{r}}(2)>0$, we have shown $\beta(x_0,\rho\overline{r}^k)\le\frac{\varepsilon}{\rho^s}(\rho\overline{r}^k)^s\le C'(\rho\overline{r}^k)^s$ (Remark 2.7.3 implies that we can we can take $\rho$ uniformly in $x_0\in B_1(0)$). If $t$ is such that $\rho\overline{r}^{k+1}<t\le\rho\overline{r}^k$ we can estimate

$$\beta(x_0,t)<\frac{\rho\overline{r}^k}{t}\beta(x_0,\rho\overline{r}^k)<C'\frac{\rho\overline{r}^k}{t}(\rho\overline{r}^k)^s=C'\frac{\rho\overline{r}^k}{t}t^s\left(\frac{\rho\overline{r}^k}{t}\right)^s\le\frac{C'}{\overline{r}^{1+s}}t^s\equiv C''t^s,$$

where we used that $\frac{\rho\overline{r}^k}{t}<\frac{1}{\overline{r}}$. □

It is worthwhile to note that the condition $\overline{r}^\alpha<1/4$ implies $s=-\log_{\overline{r}}(2)<\alpha/2$. So this argument does not give optimal Hölder regularity.

### *2.7.2 Harnack Inequalities*

It remains to prove Lemma 2.7.4. We first define a subsolution to the free boundary problem (see Definition 2.7.1).

**Definition 2.7.6.** *Let $\mathcal{O}$ be an open set in $\mathbb{R}^n$ and $g:\mathbb{R}^n\to\mathbb{R}$. We say that $z\in C(\overline{\mathcal{O}})$ is a* **strict-subsolution to the free boundary problem associated with** *$g$ in $\mathcal{O}$ if:*

- *$\{z=0\}$ is locally the graph of a $C^2$ function.*

- $z \in C^1(\overline{\{z > 0\} \cap \mathcal{O}}) \cap C^1(\overline{\{z < 0\} \cap \mathcal{O}})$.
- *On the set* $\{z \neq 0\}$ *we have* $\Delta z > 0$.
- *For* $x_0 \in \{z = 0\}$ *we have*

$$g(x_0)(z^+)_{\nu_{x_0}}(x_0) + (z^-)_{\nu_{x_0}}(x_0) > 0,$$

*where* $\nu_{x_0}$ *is the inward pointing normal at* $x_0$ *to* $\{z > 0\}$.

*We define a strict supersolution analogously.*

With this definition we need a comparison principle (note that this comparison principle can also be taken to be the definition of a sub/super solution, see e.g. (DFS14)).

**Lemma 2.7.7.** *[Compare to (CS05) Lemma 2.1, (DFS14) Definition 7.2] Let* $\mathcal{O}$ *be an open set in* $\mathbb{R}^n$. *Let* $w, z$ *be a solution and strict subsolution respectively to the free boundary problem associated to a positive* $g$ *in* $\mathcal{O}$. *If* $w \geq z$ *in* $\overline{\mathcal{O}}$ *then* $w > z$ *in* $\mathcal{O}$.

*The analogous statement holds for supersolutions.*

*Proof.* We proceed by contradiction and let $\tilde{x}$ be a touching point. There are three cases:

**Case 1:** $\tilde{x} \in \{z = 0\}$. $\{z = 0\}$ is locally the graph of a $C^2$ function so there is a tangent ball $B \subset \{z > 0\}$ with $B \cap \{z = 0\} = \tilde{x}$. Since $\{z > 0\} \subseteq \{w > 0\}$ we have $B \cap \{w = 0\} = \tilde{x}$ and $B \subset \{w > 0\}$. As such $\{z = 0\}, \{w = 0\}$ share a normal vector $\nu$ at $\tilde{x}$.

Since $w \geq z, z \neq w$ we have that $z - w$ attains a local maximum at $\tilde{x}$. Thus $(z^+ - w^+)_\nu \leq 0$ and $(-z^- + w^-)_{-\nu} = (z^- - w^-)_\nu \leq 0$. We then have $0 \geq g(\tilde{x})(z^+ - w^+)_\nu + (z^- - w^-)_\nu = g(\tilde{x})(z^+)_\nu + (z^-)_\nu > 0$ a contradiction.

**Case 2:** $\tilde{x} \in \{z > 0\}$. As $\{z > 0\} \subseteq \{w > 0\}$, both $-w, z$ are subharmonic on $\{z > 0\}$. So $z - w$ cannot attain a local maximum on $\{z > 0\}$ which implies $w > z$ on $\{z > 0\} \cap \mathcal{O}$.

**Case 3:** $\tilde{x} \in \{z < 0\}$. In this case $\tilde{x} \in \{w < 0\}$. As $\{w < 0\} \subseteq \{z < 0\}$, we have $-w, z$ are both subharmonic on $\{w < 0\}$. We can then argue as in **Case 2**. □

With this comparison lemma we can prove a “one-sided” Harnack type inequality.

**Lemma 2.7.8.** *[Compare with (DFS14), Lemmas 4.3 and 8.1] Let $w$ be a solution to the free boundary problem associated to a positive continuous function $g$ on $B_1(0)$ (see Definition 2.7.1). Let $\tilde{k} > 0$ and assume $\inf_{x\in B_1(0)} g(x) \geq \tilde{k}$. Also assume $w$ satisfies*

$$w(x) \geq U_\gamma^{(0)}(x \cdot \nu),\ x \in B_1(0)$$

*(where $\nu \in \mathbb{S}^{n-1}$ and $\gamma > 0$) and that at $\overline{x} = \frac{1}{5}\nu$*

$$w(\overline{x}) \geq U_\gamma^{(0)}(1/5 + \varepsilon). \tag{2.7.4}$$

*Finally, assume that $\sup_{x\in B_1} |g(0) - g(x)| \leq 10\varepsilon^2$.*

*Then there exists $\overline{\varepsilon} > 0$ and $0 < c < 1$ (which depend only on the dimension and $k$), such that if the above $\varepsilon < \overline{\varepsilon}$ we can conclude*

$$w(x) \geq U_\gamma^{(0)}(x \cdot \nu + c\varepsilon),\ x \in \overline{B}_{1/2}(0).$$

*Analogously, if $w(x) \leq U_\gamma(x\cdot\nu),\ x \in B_1$ and $w(\overline{x}) \leq U_\gamma(1/5-\varepsilon)$ then $w(x) \leq U_\gamma(x\cdot\nu - c\varepsilon)$ in $\overline{B}_{1/2}(0)$.*

*Proof.* For ease of notation we will drop the dependence of $U$ on $\gamma, 0$ and let $\nu = e_n$. We prove the inequality from below; the inequality from above, and the result for general $\nu$, is proven similarly. Our first step is to widen the gap between $w$ and $U$:

**Claim:** There exists a universal $c_1 > 0$ such that $w(x) \geq (1 + c_1\varepsilon)\gamma x_n^+ - g(0)\gamma x_n^-$ for all $x \in \overline{B}_{19/20}(0)$ and for universal $c_1 > 0$.

*Proof of Claim:* In $\overline{B}_{1/20}(\overline{x})$ there is a universal constant $c_0 > 0$ such that $w(x) - U_\gamma(x) \geq c_0\gamma\varepsilon \geq c_0\gamma\varepsilon x_n$ by the Harnack inequality and (2.7.4).

Define $\mathcal{O} = (B_1 \cap \{x_n > 0\})\backslash\overline{B}_{1/20}(\overline{x})$ and let $\phi$ be the harmonic function in $\mathcal{O}$ such that $\phi = 0$ on $\partial(B_1 \cap \{x_n > 0\})$ and $\phi = 1$ on $\partial B_{1/20}(\overline{x})$.

We have

$$w(x) - \gamma x_n \geq 0 = \gamma c_0 \phi(x)\varepsilon/2, x \in \partial(B_1 \cap \{x_n > 0\}).$$

Also, note

$$w(x) - \gamma x_n \geq c_0 \gamma \varepsilon \geq \gamma c_0 \varepsilon \phi(x)/2, x \in \partial B_{1/20}(\overline{x}).$$

As $w - \gamma x_n$ and $\gamma c_0 \varepsilon \phi(x)/2$ are both harmonic on $\mathcal{O}$ we have that $w - \gamma x_n \geq \gamma c_0 \varepsilon \phi(x)/2$ on all of $\mathcal{O}$. Finally, by the boundary Harnack principle there is a $\tilde{c} > 0$ such that $\phi \geq \tilde{c} x_n$ on $\overline{\mathcal{O}} \cap B_{19/20}$. Therefore, $c_1 = \min\{c_0, c_0\tilde{c}/2, 5/2\}$ is such that $w - \gamma x_n^+ \geq \gamma \varepsilon c_1 x_n^+$ on $\overline{B}_{19/20}$, proving the claim.

Recall $w(\overline{x}) - U(\overline{x}_n) \geq \gamma\varepsilon > 0$. Thus $w(\overline{x}) - (1 + c_1\varepsilon)\gamma(\overline{x}_n)^+ \geq \gamma\varepsilon - c_1\gamma\varepsilon/5 \geq \gamma\varepsilon/2$. The Harnack inequality tells us that

$$w(x) - (1 + c_1\varepsilon)\gamma(x_n)^+ \geq c'\varepsilon\gamma,\ x \in \overline{B}_{1/20}(\overline{x}),$$

for $c'$ universal depending on dimension. If $c_2$ is small enough that $(1 + c_1\varepsilon)c_2 \leq c'$, then

$$w(x) - (1 + c_1\varepsilon)\gamma(x_n + c_2\varepsilon)^+ \geq 0,\ x \in \overline{B}_{1/20}(\overline{x}). \tag{2.7.5}$$

Now we create a strict subsolution in the annulus

$$A := B_{3/4}(\overline{x}) \backslash \overline{B}_{1/20}(\overline{x})$$

and then use this subsolution to transfer the gap in (2.7.5) to a neighborhood of 0.

Let

$$\psi(x) := 1 - c(|x - \overline{x}|^{-n} - (3/4)^{-n}),\ x \in A,$$

where $c$ is such that $\psi = 0$ on $\partial B_{1/20}(\overline{x})$. Then $0 \leq \psi \leq 1$ and $-\Delta\psi \geq k(n) > 0$ in $A$. We can extend $\psi \equiv 0$ on $B_{1/20}(\overline{x})$.

For $t \geq 0$ we write

$$v_t(x) := (1 + c_1\varepsilon)\gamma(x_n - \varepsilon c_2\psi(x) + t\varepsilon)^+ - g(0)\gamma(x_n - \varepsilon c_2\psi(x) + t\varepsilon)^-, \ x \in \overline{B}_{3/4}(\overline{x}). \quad (2.7.6)$$

We will prove later that this is a family of strict subsolutions.

By the claim, $v_0(x) \leq (1+c_1\varepsilon)\gamma x_n^+ - g(0)\gamma x_n^- \leq w(x)$ for $x \in \overline{B}_{3/4}(\overline{x})$. So we can define $t^* = \sup\{t \mid v_t(x) \leq w(x), \ \forall x \in \overline{B}_{3/4}(\overline{x})\}$. If $t^* \geq c_2$ we get

$$w(x) \geq v_{c_2}(x) \geq U_\gamma(x_n - \varepsilon c_2\psi + c_2\varepsilon) \geq U_\gamma(x_n + c\varepsilon), \ x \in B_{1/2}(0)$$

where $c := c_2(1 - \sup_{x \in B_{1/2}} \psi)$. This is the desired result.

Assume, to obtain a contradiction, $t^* < c_2$. There must be some point $\tilde{x} \in \overline{B}_{3/4}(\overline{x})$ such that $v_{t^*}(\tilde{x}) = w(\tilde{x})$ (and everywhere else in $\overline{B}_{3/4}(\overline{x})$ we have $v_{t^*}(x) \leq w(x)$) .

**Case 1:** $\tilde{x} \in \partial B_{3/4}(\overline{x})$. As $\psi(\tilde{x}) = 1$,

$$v_{t^*}(\tilde{x}) = (1 + c_1\varepsilon)\gamma(\tilde{x}_n + (t^* - c_2)\varepsilon)^+ - g(0)\gamma(\tilde{x}_n + (t^* - c_2)\varepsilon)^-$$

$$< (1 + c_1\varepsilon)\gamma(\tilde{x}_n)^+ - g(0)\gamma(\tilde{x}_n)^-.$$

Note, $\overline{B}_{3/4}(\overline{x}) \subset \overline{B}_{19/20}$, so the claim implies $w(\tilde{x}) \geq (1+c_1\varepsilon)\gamma(\tilde{x}_n)^+ - g(0)\gamma(\tilde{x}_n)^- > v_{t^*}(\tilde{x})$, a contradiction.

**Case 2:** $\tilde{x} \in \overline{B}_{1/20}(\overline{x})$. Here $\psi \equiv 0$ so $v_{t^*}(\tilde{x}) = (1 + c_1\varepsilon)\gamma(\tilde{x}_n + t^*\varepsilon)^+ < (1 + c_1\varepsilon)\gamma(\tilde{x}_n + c_2\varepsilon)^+$, as $t^* < c_2$. But (2.7.5) implies $w(\tilde{x}) \geq (1+c_1\varepsilon)\gamma(\tilde{x}_n + c_2\varepsilon)^+$, which is a contradiction.

**Case 3:** $\tilde{x} \in A$. If $v_t$ is a strict subsolution to the free boundary problem associated with $g$ in $A$, then Lemma 2.7.7 (the comparison lemma) gives the desired contradiction.

**Proof that $v_t$ is a strict subsolution**: Note that in $(\{v_{t^*} > 0\} \cap A) \cup (\{v_{t^*} < 0\} \cap A)$ we have $\Delta v_{t^*} \geq -m\varepsilon c_2 \Delta\psi \geq m\varepsilon c_2 k(n) > 0$ where $m = \gamma \min\{1, \tilde{k}\}$.

We then need to show that $\{v_{t^*} = 0\}$ is locally the graph of a $C^2$ function. Observe $\{v_{t^*} =$

$0\} = \{x_n - \varepsilon c_2\psi(x) + t^*\varepsilon = 0\}$. As $\psi \in C^\infty(\overline{A})$ it suffices to show that $|e_n - \varepsilon c_2 \nabla\psi(x)| \neq 0$ on $A$. But this is accomplished simply by picking $\overline{\varepsilon} < \frac{1}{c_2 M}$ where $M = \sup_{x\in A} |\nabla\psi(x)|$. $M$ depends only on dimension so $\overline{\varepsilon}$ can still be chosen universally.

To verify the boundary condition, let $x_0 \in \{v_t = 0\}$ and $\nu$ the unit normal pointing into $\{v_t > 0\}$ at $x_0$. Then $g(x_0)(v_t^+)_\nu + (v_t^-)_\nu = ((1 + c_1\varepsilon)g(x_0)\gamma - g(0)\gamma)(e_n - \varepsilon c_2\nabla\psi) \cdot \nu$. As $\nu$ points into $\{v_t > 0\}$ it must be the case that $(e_n - \varepsilon c_2\nabla\psi) \cdot \nu > 0$. So it is enough to prove that $(1 + c_1\varepsilon)g(x_0) - g(0) > 0$. By assumption $|g(x_0) - g(0)| \leq 10\varepsilon^2$ which means it suffices to show $c_1\varepsilon g(x_0) > 10\varepsilon^2$. By picking $\overline{\varepsilon} > 0$ small enough (now depending on $\tilde{k}$) this is true on $B_1(0)$ and we are done. □

Using the one-sided Harnack inequality we can prove a two-sided Harnack type inequality.

**Lemma 2.7.9.** *[Compare with (DFS14), Theorem 4.1] Let $\tilde{k} > 0$ and let $g \in C(B_2(0))$ such that $\inf_{x\in B_2(0)} g(x) \geq \tilde{k}$. Let $w$ be a solution to the free boundary problem associated to $g$ in $B_2(0)$. Assume $w$ satisfies at some point $x_0 \in B_2$,*

$$U_\gamma^{(0)}(x \cdot \nu + a_0) \leq w(x) \leq U_\gamma^{(0)}(x \cdot \nu + b_0),\ \forall x \in B_r(x_0) \subset B_2(0)$$

*where $\nu \in \mathbb{S}^{n-1}, \gamma > 0$ and $b_0 - a_0 \leq \varepsilon r, \sup_{x\in B_2} |g(x) - g(0)| \leq \varepsilon^2$ for some $\varepsilon > 0$.*

*Then there exists some $\overline{\varepsilon} = \overline{\varepsilon}(n, \tilde{k}) > 0$ such that if $\varepsilon \leq \overline{\varepsilon}$ we can conclude*

$$U_\gamma^{(0)}(x \cdot \nu + a_1) \leq w(x) \leq U_\gamma^{(0)}(x \cdot \nu + b_1),\ \forall x \in B_{r/20}(x_0),$$

*where $a_0 \leq a_1 \leq b_1 \leq b_0$ and $b_1 - a_1 \leq (1 - c)\varepsilon r$. Here $c = c(n, \tilde{k}) > 0$.*

*Proof.* Without loss of generality $x_0 = 0, r = 1, \nu = e_n$. There are three cases, each of which produces a universal $0 < \tilde{c} < 1$. Take $c$ to be the minimum of these three.

**Case 1:** $a_0 < -1/5$. For small $\varepsilon > 0$ we have $x_n + b_0 < 0$ on $B_{1/10}$. Therefore, by the assumed inequality on $w$,

$$0 \leq v(x) := \frac{w(x) - g(0)\gamma(x_n + a_0)}{g(0)\gamma\varepsilon} \leq 1, \ \forall x \in B_{1/10}.$$

Additionally, $\Delta v = 0$ on $B_{1/10}$.

So by the Harnack inequality there are constants $1 \geq k_1 \geq k_2 \geq 0$ such that $k_1 - k_2 = 1 - \tilde{c} < 1$ where $\tilde{c}$ is universal (though $k_1, k_2$ may depend on $w$) and $k_1 \geq v(x) \geq k_2$ on $B_{1/20}$.

This implies

$$U_\gamma(x_n + a_0 + k_2\varepsilon) \leq w(x) \leq U_\gamma(x_n + a_0 + k_1\varepsilon), \ \forall x \in B_{1/20}.$$

Set $a_1 = a_0 + k_2\varepsilon$ and $b_1 = a_0 + k_1\varepsilon$, so that $a_0 \leq a_1 \leq b_1 \leq b_0$ and $b_1 - a_1 \leq (k_1 - k_2)\varepsilon = (1 - \tilde{c})\varepsilon$.

**Case 2:** $a_0 > 1/5$. In this case $a_0 + x_n > 0$ on $B_{1/10}$ and so

$$0 \leq v(x) := \frac{w(x) - \gamma(x_n + a_0)}{\gamma\varepsilon} \leq 1$$

on $B_{1/10}$. The rest of the argument follows exactly as in **Case 1**.

**Case 3:** $|a_0| < 1/5$. We can rewrite the main assumption as

$$U_\gamma(x_n + a_0) \leq w(x) \leq U_\gamma(x_n + a_0 + \varepsilon), \ x \in B_1(0).$$

Without loss of generality, assume that

$$w(\overline{x}) \geq U_\gamma(\overline{x}_n + a_0 + \varepsilon/2) \tag{2.7.7}$$

where $\overline{x} = 4e_n/25 - a_0 e_n$ (the case with the reverse inequality is similar).

If $v(x) := w(x - a_0 e_n)$ for $x \in B_{4/5}(0)$, then the above can be rewritten as

$$\begin{gathered} U_\gamma(x_n) \leq v(x) \leq U_\gamma(x_n + \varepsilon),\ \forall x \in B_{4/5}(0). \\ v(4e_n/25) \geq U_\gamma(4/25 + \varepsilon/2). \end{gathered} \tag{2.7.8}$$

Note that $v$ satisfies the free boundary problem associated to $\tilde{g}$ which is a translate of $g$. Thus we can apply Lemma 2.7.8 with $\varepsilon/2$ and inside $B_{4/5}$ to get that

$$v(x) \geq U_\gamma(x_n + \tilde{c}\varepsilon), x \in \overline{B}_{2/5}(0) \Rightarrow$$

$$w(x) \geq U_\gamma(x_n + a_0 + \tilde{c}\varepsilon), x \in \overline{B}_{1/5}(0),$$

for some universal $0 < \tilde{c} < 1$. Letting $a_1 = a_0 + \tilde{c}\varepsilon$ and $b_1 = b_0$ we have $b_1 - a_1 = b_0 - a_0 - \tilde{c}\varepsilon \leq (1 - \tilde{c})\varepsilon$. □

With these lemmata in hand we can prove the following regularity result. This will be crucial in the proof of Lemma 2.7.4 (the iterative step).

**Corollary 2.7.10.** *[Compare with (DFS14), Corollary 8.2] Let $w, \gamma, g, \nu, \varepsilon, x_0$ satisfy the assumptions of Lemma 2.7.9 with $r = 1$. Define*

$$\tilde{w}_\varepsilon := \begin{cases} \dfrac{w(x) - \gamma x \cdot \nu}{\gamma\varepsilon},\ x \in B_2(0) \cap \{w \geq 0\} \\ \dfrac{w(x) - g(0)\gamma x \cdot \nu}{g(0)\gamma\varepsilon},\ x \in B_2(0) \cap \{w < 0\} \end{cases} \tag{2.7.9}$$

*Then $\tilde{w}_\varepsilon$ has a Hölder modulus of continuity at $x_0$ outside the ball of radius $\varepsilon/\overline{\varepsilon}$, i.e. for all $x \in B_1(x_0)$ with $|x - x_0| \geq \varepsilon/\overline{\varepsilon}$*

$$|\tilde{w}_\varepsilon(x) - \tilde{w}_\varepsilon(x_0)| \leq C|x - x_0|^\chi$$

*where $C, \chi$ depend only $n, \tilde{k}$.*

*Proof.* Let $\nu = e_n$. Repeated application of Lemma 2.7.9 gives

$$U_\gamma(x_n + a_m) \le w(x) \le U_\gamma(x_n + b_m),\ x \in B_{20^{-m}}(x_0),$$

with $b_m - a_m \le (1-c)^m \varepsilon$. However, we may only apply Lemma 2.7.9 when $m$ is such that $(1-c)^m 20^m \varepsilon \le \bar{\varepsilon}$ (as we are taking $r = 20^{-m}$ at the $m$th step).

If $20^{-\chi} = (1-c)$ then we have, for each acceptable $m$, that $x \in B_{20^{-m}}(x_0) \backslash B_{20^{-m-1}}(x_0)$ implies $|\tilde{w}_\varepsilon(x) - \tilde{w}_\varepsilon(x_0)| \le C|x - x_0|^\chi$. As above, $m$ must satisfy $20^{-m} \ge (1-c)^m \frac{\varepsilon}{\bar{\varepsilon}}$, which is true if $20^{-m} \ge \frac{\varepsilon}{\bar{\varepsilon}}$. So we have the desired continuity outside $B_{\frac{\varepsilon}{\bar{\varepsilon}}}(x_0)$. □

### *2.7.3 The Transmission Problem and Proof of Lemma 2.7.4*

In order to prove Lemma 2.7.4, we will argue by contradiction and analyze the limit of the $\tilde{w}_\varepsilon$ (see (2.7.9)) as $\varepsilon \downarrow 0$. This limit will be the solution to a transmission problem which we introduce now.

**Definition 2.7.11.** *We say that $W \in C(B_\rho)$ is a* **classical solution to the transmission problem at 0** *in $B_\rho$ if:*

- $W \in C^\infty(B_\rho \cap \{x_n \ge 0\}) \cap C^\infty(B_\rho \cap \{x_n \le 0\})$
- *$W$ satisfies*

$$\begin{aligned} \Delta W = 0,\ x \in B_\rho(0) \cap \{x_n \neq 0\} \\ \lim_{t \downarrow 0} W_n(x', t) - \lim_{t \uparrow 0} W_n(x', t) = 0,\ x \in B_\rho(0) \cap \{x_n = 0\} \end{aligned} \tag{2.7.10}$$

*When no confusion is possible, we will simply say that $W$ is a classical solution to the transmission problem or a classical solution to* (2.7.10).

We can deduce the following immediately from the definition:

**Lemma 2.7.12.** *Let $W$ be a classical solution to the transmission problem in $B_1$. Then there is a universal constant $C$ and a constant $p$ (which depend on $W$) such that*

$$|W(x) - W(0) - (\nabla_{x'} W(0) \cdot x' + p x_n^+ - p x_n^-)| \leq C \|W\|_{L^\infty(B_1)} r^2, \ \forall x \equiv (x', x_n) \in B_r(0). \tag{2.7.11}$$

Unfortunately, the conditions of Definition 2.7.11 are too difficult to verify directly. It will be more convenient to work with viscosity solutions.

**Definition 2.7.13.** *Let $\widetilde{W} \in C(B_\rho)$. We say that $\widetilde{W}$ is a viscosity solution to the transmission problem,* (2.7.10)*, if:*

- $\Delta \widetilde{W}(x) = 0$*, in the viscosity sense, when* $x \in \{x_n \neq 0\} \cap B_\rho$.
- *Let $\phi$ be any function of the form*

$$\phi(x) = A + p x_n^+ - q x_n^- + BQ(x - y)$$

*where*

$$Q(x) = \frac{1}{2}[(n-1)x_n^2 - |x'|^2], \ y = (y', 0), \ A \in \mathbb{R}, B > 0$$

*and $p - q > 0$. Then $\phi$ cannot touch $\widetilde{W}$ strictly from below at a point $x_0 = (x_0', 0) \in B_\rho$.*

- *If $p - q < 0$ then $\phi$ cannot touch $\widetilde{W}$ strictly from above on $\{x_n = 0\}$.*

The following result allows us to estimate the growth rate of viscosity solutions. We will omit the proof as it is identical to the one provided by De Silva, Ferrari and Salsa in (DFS14).

**Theorem 2.7.14.** *[Theorem 3.3 and Theorem 3.4 in (DFS14)] Let $\widetilde{W}$ be a viscosity solution to* (2.7.10) *in $B_1$ such that $\|\widetilde{W}\|_{L^\infty} \leq 1$. Then, in $B_{1/2}$, $\widetilde{W}$ is actually a classical solution to* (2.7.10)*. In particular, $\widetilde{W}$ satisfies the estimate* (2.7.11)*.*

With this machinery in hand we are ready to prove Lemma 2.7.4.

*Proof of Lemma 2.7.4.* It suffices to assume that $x_0 = 0$ and $\nu = e_n$ (by the rotation invariance of the conditions). Fix any $r > 0$ small and let $\{\gamma_k\}, \{\varepsilon_k\}, \{w_k\}, \{g_k\}$ be such that $C_1 > \gamma_k > c_1, \varepsilon_k \downarrow 0$ and $w_k$ is a classical solution to the free boundary problem associated to $g_k$. Furthermore, $\inf_{x\in B_1(0)} g_k(x) \geq \tilde{k}$, $\sup_{x,y\in B_1(0)} \frac{|g_k(x)-g_k(y)|}{|x-y|^\alpha} < \varepsilon_k^2$ and $w_k(x)$ satisfies

$$U_{\gamma_k}^{(0)}(x_n - \varepsilon_k) \leq w_k(x) \leq U_{\gamma_k}^{(0)}(x_n + \varepsilon_k),\ x \in B_1(0). \tag{2.7.12}$$

However, to obtain a contradiction, assume the desired $\nu_k, \gamma_k'$ do not exist.

Define $\tilde{w}_k$ as in (2.7.9). Then (2.7.12) implies that $\{\tilde{w}_k = 0\} \to \{x_n = 0\}$ in the Hausdorff distance norm and $\|\tilde{w}_k\|_{L^\infty} \leq 1$. These observations, combined with Corollary 2.7.10 and the Arzelà-Ascoli theorem, show that $\tilde{w}_k \to \tilde{w}$ uniformly in $C(B_1(0))$ (after passing to subsequences). Furthermore, Corollary 2.7.10 implies that $\tilde{w}$ is a $C^{0,\chi}$ function defined on $B_{1/2}(0)$.

**Claim:** *$\tilde{w}$ is a viscosity solution in $B_{1/2}$ to the transmission problem.*

If this is the case, $\tilde{w}$ satisfies the estimate (2.7.11). So there is a $p$ such that

$$|\tilde{w}(x) - \tilde{w}(0) - (\nabla_{x'}\tilde{w}(0) \cdot x' + px_n^+ - px_n^-)| \leq Cr^2,\ \forall x = (x', x_n) \in B_r(0).$$

Because $\|\tilde{w}\|_{L^\infty} \leq 1$ we have $|p| < 10$. We will also pick $r$ small enough so that $8Cr < 1$.

As $\tilde{w}_k$ converges uniformly to $\tilde{w}$, for large enough $k$ (depending on $r$ possibly) we have

$$|\tilde{w}_k(x) - (\nabla_{x'}\tilde{w}(0) \cdot x' + px_n^+ - px_n^-)| \leq 2Cr^2,\ \forall x = (x', x_n) \in B_r(0). \tag{2.7.13}$$

Let $\nu_k := \frac{1}{\sqrt{1+\varepsilon_k^2|\nabla_{x'}\tilde{w}(0)|^2}}(\varepsilon_k\nabla_{x'}\tilde{w}(0), 1)$ and $\gamma_k' := \gamma_k(1 + \varepsilon_k p)$. We will now prove

$$U_{\gamma_k'}(x \cdot \nu_k - r\frac{\varepsilon_k}{2}) \leq w_k(x) \leq U_{\gamma_k'}(x \cdot \nu_k + r\frac{\varepsilon_k}{2}),\ x \in B_r(0) \tag{A}$$

and also

$$|\gamma_k' - \gamma_k| \leq \tilde{C}\varepsilon_k\gamma_k,\ |e_n - \nu_k| \leq \tilde{C}\varepsilon_k, \tag{B}$$

for some universal $\tilde{C}$. This is the desired contradiction.

**Proof of** (A)**:** Assume $w_k(x) \geq 0$ (the other case follows similarly). (2.7.13) implies

$$(\nabla_{x'}\tilde{w}(0)\cdot x' + px_n^+ - px_n^-) - 2Cr^2 \leq \frac{w_k(x) - \gamma_k x_n}{\gamma_k\varepsilon_k} \leq 2Cr^2 + (\nabla_{x'}\tilde{w}(0)\cdot x' + px_n^+ - px_n^-)$$

for $x \in B_r(0)$. Consider the inequality on the left. Some algebraic manipulation yields

$$\gamma_k x_n + \gamma_k\varepsilon_k((\nabla_{x'}\tilde{w}(0)\cdot x' + px_n^+ - px_n^-) - 2Cr^2) \leq w_k(x),\ \forall x \in B_r(0)\cap\{w_k \geq 0\}.$$

We can rewrite this again to obtain, for all $x \in B_r(0)\cap\{w_k \geq 0\}$,

$$\sqrt{1+\varepsilon_k^2|\nabla_{x'}\tilde{w}(0)|^2}U_{\gamma_k'}(x\cdot\nu_k) - \gamma_k p\varepsilon_k^2|\nabla_{x'}\tilde{w}(0)\cdot x| - 2Cr^2\gamma_k\varepsilon_k \leq w_k(x).$$

The Cauchy-Schwartz inequality, followed by some more algebraic manipulation, gives

$$U_{\gamma_k'}(x\cdot\nu_k) - \gamma_k' r\frac{\varepsilon_k}{2}\left(\frac{2p\varepsilon_k|\nabla_{x'}\tilde{w}(0)| + 4Cr}{1+\varepsilon_k p}\right) \leq w_k(x),\ \forall x \in B_r(0)\cap\{w_k \geq 0\}.$$

Recall that $r$ was chosen so that $8Cr < 1$. Now pick $k$ large enough so that $20\varepsilon_k|\nabla_{x'}\tilde{w}(0)| < 1/2$. Together this implies $\left(\frac{2p\varepsilon_k|\nabla_{x'}\tilde{w}(0)|+4Cr}{1+\varepsilon_k p}\right) < 1$. In conclusion,

$$U_{\gamma_k'}(x\cdot\nu_k - r\frac{\varepsilon_k}{2}) \leq w_k(x),\ \forall x \in B_r(0)\cap\{w_k \geq 0\}.$$

The upper bound on $w_k$ and the inequalities for when $w_k < 0$ follow in the same fashion.

**Proof of** (B)**:** We compute $|\gamma_k' - \gamma_k| = \varepsilon_k p\gamma_k \leq 10\varepsilon_k\gamma_k$. Also $|\nu_k - e_n|^2 = (\nu_k - e_n, \nu_k - e_n) = 2 - 2(e_n, \nu_k) = 2 - \frac{2}{\sqrt{1+\varepsilon_k^2|\nabla_{x'}\tilde{w}(0)|^2}}$. For large $k$ (so that $\varepsilon_k|\nabla_{x'}\tilde{w}(0)| < 1/2$) the taylor series expansion of $\sqrt{1+x^2}$ yields the estimate $|\nu_k - e_n|^2 \leq \varepsilon_k^2|\nabla_{x'}\tilde{w}(0)|^2$. Let

$\tilde{C} = \max\{|\nabla_{x'}\tilde{w}(0)|, 10\}$ and we are done.

**Proof of Claim:** We want to establish that $\tilde{w}$ is a viscosity solution to the transmission problem. As $\Delta\tilde{w}_k = 0$, wherever $\{\tilde{w}_k \neq 0\}$, it is clear that $\Delta\tilde{w} = 0$, in the viscosity sense, when $\{x_n \neq 0\}$. It remains to verify the boundary condition.

So assume, in order to reach a contradiction, that there is a function

$$\tilde{\phi}(x) := A + px_n^+ - qx_n^- + BQ(x-y),$$

with $p-q > 0$, which touches $\tilde{w}$ strictly from below at $x_0 = (x_0', 0)$ (the case where $p-q < 0$ and $\tilde{\phi}$ touches from above follows similarly). Recall $Q(x) := \frac{1}{2}[(n-1)x_n^2 - |x'|^2], y = (y', 0), B > 0$ and $A \in \mathbb{R}$. We now construct a family of functions which converge uniformly to $\tilde{\phi}$. Define

$$\Gamma(x) := \frac{1}{n-2}[(|x|'^2 + |x_n - 1|^2)^{\frac{2-n}{2}} - 1] \text{ and } \Gamma_k(x) := \frac{1}{B\varepsilon_k}\Gamma(B\varepsilon_k(x-y) + AB\varepsilon_k^2 e_n).$$

Additionally, let

$$\phi_k(x) := \gamma_k(1+\varepsilon_k p)\Gamma_k^+(x) - g(0)\gamma_k(1+\varepsilon_k q)\Gamma_k^-(x) + \gamma_k(d_k^+(x))^2\varepsilon_k^{3/2} + g(0)\gamma_k(d_k^-(x))^2\varepsilon_k^{3/2},$$

where $d_k$ is the signed distance from $x$ to $\partial B_{\frac{1}{B\varepsilon_k}}(y + e_n(A\varepsilon_k - \frac{1}{B\varepsilon_k}))$. Finally, we can define $\tilde{\phi}_k$ as in (2.7.9).

A taylor series expansion gives $\Gamma(x) = x_n + Q(x) + O(|x|^3)$ and thus

$$\Gamma_k(x) = A\varepsilon_k + x_n + B\varepsilon_k Q(x-y) + O(\varepsilon_k^2), \ x \in B_1.$$

Therefore, $\tilde{\phi}_k$ converges uniformly to $\tilde{\phi}$. The existence of a touching point $x_0$ implies a sequence of constants, $c_k$, and points, $x_k \in B_{1/2}$, such that $\psi_k(x) := \phi_k(x + \varepsilon_k c_k e_n)$ touches $w_k$ from below at $x_k$. We will get the desired contradiction if $\psi_k$ is a strict subsolution to the free boundary problem associated to $g_k$.

When $\psi_k \neq 0$ we have $\Delta\psi_k \gtrsim \Delta d_k^2(x+\varepsilon_k c_k e_n) > 0$. If $\psi_k = 0$ a straightfoward computation shows $\Gamma_k(x+\varepsilon_k c_k e_n) = d_k(x+\varepsilon_k c_k e_n) = 0$. Thus, $|\nabla d_k^2| = 0$ whenever $\psi_k = 0$. We can also compute $(\nabla\Gamma_k^{\pm})_\nu = \pm 1$ on $\psi_k = 0$. Putting this together, $g_k(x)(\psi_k(x)^+)_\nu + (\psi_k(x)^-)_\nu = g_k(x)\gamma_k(1+\varepsilon_k p) - g(0)\gamma_k(1+\varepsilon_k q)$. Recall, $|g_k(x)-g(0)| = |g_k(x)-g_k(0)| \leq \varepsilon_k^2$ which implies, $g_k(x) \geq g(0) - \varepsilon_k^2$. Therefore, $g_k(x)(\psi_k(x)^+)_\nu + (\psi_k(x)^-)_\nu \geq g(0)\gamma_k\varepsilon_k(p - q) - \varepsilon_k^2\gamma_k(1+\varepsilon_k p)$. We are done if this last term is $> 0$. It is easy to see

$$g(0)\gamma_k\varepsilon_k(p-q) - \varepsilon_k^2\gamma_k(1+\varepsilon_k p) > 0 \Leftrightarrow g(0)(p-q) > \varepsilon_k(1+\varepsilon_k p)$$

which is clearly true for $k$ large enough.

□

## 2.8 Optimal Hölder regularity and higher regularity

Proposition 2.7.5 tells us that if $\log(h) \in C^{0,\alpha}(\partial\Omega)$ then $\partial\Omega$ is locally the graph of a $C^{1,s}$ function for some $s > 0$. In this section we will introduce tools from elliptic regularity theory in order to establish the sharp estimate $s = \alpha$. These tools will also allow us to analyze the case when $\log(h) \in C^{k,\alpha}(\partial\Omega)$ for $k \geq 1$.

### *2.8.1 Partial Hodograph Transform and Elliptic Systems*

We begin by recalling the partial hodograph transform (see (KS80), Chapter 7 for a short introduction). Here, and throughout the rest of the paper, we assume that $0 \in \partial\Omega$ and that, at 0, $e_n$ is the inward pointing normal to $\partial\Omega$.

Define $F^+ : \Omega^+ \to \mathbb{H}^+$ by $(x', x_n) = x \mapsto y = (x', u^+(x))$. Because $u_n^+(0) = \frac{d\omega^+}{d\sigma}(0) \neq 0$ (Proposition 2.5.10), $DF^+(0)$ is invertible. So, by the inverse function theorem, there is some neighborhood, $\mathcal{O}^+$, of 0 in $\Omega^+$ that is mapped diffeomorphically to $U$, a neighborhood of 0 in the upper half plane. Furthermore, this map extends in a $C^1$ fashion from $\overline{\mathcal{O}^+}$ to $\overline{U}$

(by Corollary 2.6.7).

Similarly, define $F^- : \Omega^- \to \mathbb{H}^+$ by $(x', x_n) = x \mapsto y = (x', u^-(x))$. Again $u_n^-(0) \neq 0$ so $DF^-(0)$ is invertible. We can conclude, as above, that there is a neighborhood, $\mathcal{O}^-$, of 0 in $\Omega^-$ that is mapped diffeomorphically to $U$ (perhaps after shrinking $U$) and that this map extends in a $C^1$ fashion from $\overline{\mathcal{O}^-}$ to $\overline{U}$.

Let $\psi : \overline{U} \to \mathbb{R}$ be given by $\psi(y) = x_n$, where $F^+(x) = y$. Because $F^+$ is locally one-to-one, $\psi$ is well defined. Similarly, define $\phi : \overline{U} \to \mathbb{R}$ by $\phi(y) = -x_n$ where $F^-(x) = y$. Again, $F^-$ is locally one-to-one, so $\phi$ is well defined.

If $\nu_y$ denotes the normal vector to $\partial\Omega$ pointing into $\Omega$ at $y$, then $u$ satisfies

$$\begin{aligned} \Delta u^+(x) &= 0,\ x \in \Omega^+ \\ \Delta u^-(x) &= 0,\ x \in \Omega^- \\ (u^+)_{\nu_x}(x)h(x) &= -(u^-)_{\nu_x}(x),\ x \in \partial\Omega. \end{aligned}$$

After our change of variables these equations become

$$\begin{aligned} 0 &= \frac{1}{2}\left(\frac{1}{\psi_n^2}\right)_n + \sum_{i=1}^{n-1}\left(-\left(\frac{\psi_i}{\psi_n}\right)_i + \frac{1}{2}\left(\frac{\psi_i^2}{\psi_n^2}\right)_n\right) \\ 0 &= \frac{1}{2}\left(\frac{1}{\phi_n^2}\right)_n + \sum_{i=1}^{n-1}\left(-\left(\frac{\phi_i}{\phi_n}\right)_i + \frac{1}{2}\left(\frac{\phi_i^2}{\phi_n^2}\right)_n\right), \end{aligned} \tag{2.8.1}$$

with both equations taking place for $y \in U$. On the boundary we have

$$\begin{aligned} \phi(y) + \psi(y) = 0,\ y \in \{y_n = 0\} \cap \overline{U} \\ \left(\frac{\tilde{h}(y)}{\psi_n(y)}\right) - \frac{1}{\phi_n(y)} = 0,\ y \in \{y_n = 0\} \cap \overline{U}, \end{aligned} \tag{2.8.2}$$

where $\tilde{h}((y', 0)) = h((y', \psi(y)))$.

**Remark 2.8.1.** *The following are true of* $\phi, \psi$*:*

- *Assume* $\psi, \phi \in C^{k,s}(\overline{U} \cap \{y_n = 0\})$ *with* $k \geq 1, s \in (0,1)$ *Then* $u^{\pm} \in C^{k,s}(\overline{\mathcal{O}^{\pm}}) \Leftrightarrow \psi, \phi \in C^{k,s}(\overline{U})$.

- *If* $h \in C^{k,\alpha}(\partial\Omega)$ *and* $\psi, \phi \in C^{k+1,s}(\overline{U})$ *for any* $s, \alpha \in (0,1)$, *then* $\tilde{h} \in C^{k,\alpha}(\overline{U} \cap \{y_n = 0\})$ *with norm depending only on the Hölder norms of* $h$ *and* $\psi, \phi$.

- $\phi_n, \psi_n > 0$ *in* $\overline{U}$.

*Proof.* Let us address the first statement; when $k \geq 2$ this follows from standard elliptic regularity applied to the function $\tilde{u}^+(x) = u^+(x + \phi(x', 0))$ (and a similarly defined $\tilde{u}^-$). When $k = 1$, a theorem of Kellogg (Kel29) says that $\nabla u^{\pm}$ has non-tangential limit everywhere on $\partial\Omega \cap \overline{\mathcal{O}^{\pm}}$ and that this non-tangential limit is in $C^{0,s}$. We can then argue as in the proof of Corollary 2.6.7 to see that $\nabla u^{\pm} \in C^{0,s}(\overline{\mathcal{O}^{\pm}})$; the desired result.

To prove the second statement when $k = 0$, one computes

$$|\tilde{h}(y_1) - \tilde{h}(y_2)| = |h((y_1', \psi(y_1'))) - h((y_2', \psi(y_2')))| \leq C|(y_1', \psi(y_1')) - (y_2', \psi(y_2'))|^{\alpha} \leq C|y_1 - y_2|^{\alpha}$$

where that last inequality follows because $\psi \in C^{1,s}(\overline{U})$. So $\tilde{h} \in C^{0,\alpha}(\{y_n = 0\} \cap \overline{U})$. When $k \geq 1$ we note that $\partial_i \tilde{h}(y, 0) = \partial_i h(y, \psi(y)) + \partial_n h(y, \psi(y))\partial_i \psi(y)$. By assumption $\partial_i \psi(y)$ is at least as regular as $\partial_n h(y, \psi(y))$ so the result follows by induction.

The third claim follows immediately from construction. □

We now recall the concepts of an elliptic system of equations and coercive boundary conditions. For the sake of brevity, our Definition 2.8.2 is not fully general—it considers only a specific type of system in "divergence form". A comprehensive introduction to elliptic systems can be found in Morrey ((Mor66)), Chapter 6 (weak solutions in particular are covered in Section 6.4).

**Definition 2.8.2.** *Let* $u^k$, $k = 1, 2$, *satisfy*

$$\begin{aligned} \int_U \sum_{\substack{|\chi|\le m_1 \\ |\gamma|\le t_1+s_1-m_1}} a^1_{\chi\gamma}(x) D^\gamma u^1 D^\chi \zeta &= \int_U \sum_{|\chi|\le m_1} f^1_\chi D^\chi \zeta \\ \int_U \sum_{\substack{|\chi|\le m_2 \\ |\gamma|\le t_2+s_2-m_2}} a^2_{\chi\gamma}(x) D^\gamma u^2 D^\chi \zeta &= \int_U \sum_{|\chi|\le m_2} f^2_\chi D^\chi \zeta \end{aligned} \tag{2.8.3}$$

*for all* $\zeta \in C_0^\infty(U)$. *Additionally assume,*

$$\begin{aligned} \int_{\partial U\cap\{y_n=0\}} \sum_{|\chi|\le p_1} \left( \sum_{k=1}^{2} B^1_{k\chi}(D_x, D_y, x) u^k \right) D^\chi_x \xi dx &= \int_{\partial U\cap\{y_n=0\}} \sum_{|\chi|\le p_1} g^1_\chi D^\chi_x \xi dx \\ \int_{\partial U\cap\{y_n=0\}} \sum_{|\chi|\le p_2} \left( \sum_{k=1}^{2} B^2_{k\chi}(D_x, D_y, x) u^k \right) D^\chi_x \xi dx &= \int_{\partial U\cap\{y_n=0\}} \sum_{|\chi|\le p_2} g^2_\chi D^\chi_x \xi dx \end{aligned} \tag{2.8.4}$$

*for all* $\xi \in C_0^\infty(\partial U \cap \{y_n = 0\})$. *Throughout,* $\gamma, \chi$ *are multi-indices. Let* $h_1, h_2$, *be such that* $B^1_{k\chi}$ *is of order* $\le t_k - h_1 - p_1$ *and* $B^2_{k\chi}$ *is of order* $\le t_k - h_2 - p_2$. *This system has* **a proper assignment of weights** *if there exists an* $h_0$ *such that* $h_0$ *and the* $t_k, m_j, s_j, h_r, p_r$, $k, j, r = 1, 2$ *satisfy the following conditions:*

- $\min_{j,k} s_j + t_k \ge 1$ *and* $\min_{j,k} t_k + s_j - m_j \ge 0$
- $\min m_j \ge 0$ *and* $\max s_j = 0$.
- $\min p_r \ge 0$ *and* $\min h_0 + h_r + p_r \ge 1$
- $\min t_k + h_0 \ge 0$ *and* $\min h_0 - s_j + m_j \ge 0$.

*We say the above system is* **elliptic** *if the block diagonal matrix*

$$M = \begin{pmatrix} (a^1_{\gamma\chi})_{|\chi|=m_1,|\gamma|=t_1+s_1-m_1} & 0 \\ 0 & (a^2_{\gamma\chi})_{|\chi|=m_2,|\gamma|=t_2+s_2-m_2} \end{pmatrix}$$

*is an elliptic matrix for any* $x_0 \in U$. *Additionally, when* $n = 2$, *we require that, for any* *linearly independent* $\xi, \eta \in \mathbb{R}^2$, *half the roots of the equation*

$$\det\begin{pmatrix} a^1_{\gamma\chi}(\xi + z\eta)_{\chi+\gamma} & 0 \\ 0 & a^2_{\gamma\chi}(\xi + z\eta)_{\chi+\gamma} \end{pmatrix} = 0$$

*have positive imaginary part and the other half have negative imaginary part (above we are using summation notation, where the upper left corner is taken over* $|\chi| = m_1, |\gamma| = t_1 + s_1 - m_1$ *and the lower right corner is taken over* $|\chi| = m_2$ *and* $|\gamma| = t_2 + s_2 - m_2$*).*

*Finally, we say that the boundary equations are* **coercive** *if for all* $y_0 \in \overline{U} \cap \{y_n = 0\}$ *the system*

$$\begin{aligned} \sum_{|\chi|=m_1, |\gamma|=t_1+s_1-m_1} a^1_{\chi\gamma}(y_0) D^{\gamma+\chi} v^1(y) &= 0 \\ \sum_{|\chi|=m_2, |\gamma|=t_2+s_2-m_2} a^2_{\chi\gamma}(y_0) D^{\gamma+\chi} v^2(y) &= 0 \\ \sum_{|\chi|=p_1} \sum_{k=1}^{2} \tilde{B}^1_{k\chi}(D_x, D_y, y_0) v^k((y', 0)) &= 0 \\ \sum_{|\chi|=p_2} \sum_{k=1}^{2} \tilde{B}^2_{k\chi}(D_x, D_y, y_0) v^k((y', 0)) &= 0 \end{aligned} \tag{2.8.5}$$

*has no solutions of the form* $v^k((y', y_n)) = e^{iy' \cdot \xi'} \tilde{v}^k(y_n), k = 1, 2$ *where* $\tilde{v}^k(y_n) \to 0$ *as* $y_n \to +\infty$ *and* $\xi' \in \mathbb{R}^{n-1}$*. Above,* $\tilde{B}^r_{k\chi}$ *denotes the part of the operator* $B^r_{k\chi}$ *which has order* $t_k - h_r - p_r$ *(the* **principle part***).*

**Definition 2.8.3.** *We define the* $h - \mu$*-conditions on the coefficients above:*

**(1) The** $a^j_{\chi\gamma}$ **satisfy the** $h - \mu$**-conditions**, $0 < \mu < 1$, *in some open* $\Gamma$*:*

1. *if* $|\gamma| = t_j + s_j - m_j$ *and* $|\chi| = m_j$ *then* $a^j_{\chi\gamma} \in C^{0,\mu}(\overline{\Gamma})$
2. *if* $h - s_j + |\chi| > 0$ *then* $a^j_{\chi\gamma} \in C^{h-s_j+|\chi|,\mu}(\overline{\Gamma})$
3. *else, the as are in* $C^{0,\mu}(\overline{\Gamma})$.

**(2) The operators $B^r_{k\chi}$ satisfy the $h-\mu$-conditions**, $0<\mu<1$, *in some open* $\Gamma$, *if* $B^r_{k\gamma}(D_x, D_y, -) \in C^{h+h_r+p_r,\mu}(\overline{\Gamma \cap \{y_n = 0\}})$.

With these definitions in mind, we can state Theorem 6.4.8 of (Mor66) (note the theorem in Morrey refers to a slightly more general class of elliptic systems). Our wording differs in order to comport with the notation used above.

**Theorem 2.8.4.** *[Theorem 6.4.8, (Mor66)] Let $u^k, k = 1, 2$ satisfy an elliptic and coercive system of equations on $U$ (a neighborhood of $0$ in the upper half plane with $C^\infty$ boundary) with a proper assignment of weights $h_0, h_r, p_r, t_k, s_j, m_j$. Let $\Gamma \supset \overline{U}$ be an open domain. Suppose the a's and the coefficients in the $B_{rk\gamma}$ satisfy the $h-\mu$-conditions on $\Gamma$, $0 < \mu < 1$, and suppose the* a priori *estimates: $f^j_\alpha \in C^{\rho,\mu}(U)$, $\rho = \max\{0, h - s_j + |\alpha|\}, g^r_\gamma \in C^{\tau,\mu}(U)$ with $\tau = \max\{0, h + h_r + |\gamma|\}$ and $u^k \in C^{t_k+h,\mu}(U)$. Then*

$$\sum_k \|u^k\|_{C^{t_k+h,\mu}(U)} \leq C\left(\sum_{j,\alpha}\|f^j_\alpha\|_{C^{\rho,\mu}(U)} + \sum_{r,\gamma}\|g^r_\gamma\|_{C^{\tau,\mu}(U)} + \sum_k \|u^k\|_{C^0(U)}\right). \quad (2.8.6)$$

*Here $C$ is, again, independent of $u^k$ the $f$'s and the $g$'s.*

### *2.8.2 Sharp $C^{1,\alpha}$ regularity and $C^{2,\alpha}$ regularity*

It should be noted that in (Mor66) it is not explicitly made clear if Theorem 2.8.4 applies when $h < h_0$ (nor if there should be additional restrictions on $h$). For the sake of completeness we include a proof of Theorem 2.8.4 with $h_0 = 0, h = -1$ in Appendix A.1. This is exactly the result we need to establish optimal $C^{1,\alpha}$ regularity.

**Proposition 2.8.5.** *Let $\Omega \subset \mathbb{R}^n$ be a 2-sided NTA domain with $\log(h) \in C^{0,\alpha}(\partial\Omega), \alpha \in (0, 1)$. In addition, if $n \geq 3$ also assume that $\Omega$ is $\delta$-Reifenberg flat, for $\delta > 0$ small, or that $\Omega$ is a Lipschitz domain. Then $\partial\Omega$ is locally the graph of a $C^{1,\alpha}$ function.*

*Proof.* Recall the functions $\phi, \psi$ which satisfy the system (2.8.1) with boundary conditions (2.8.2). For $t = (t', 0) \in \mathbb{R}^n$ we consider $u^{1,t}(x) := \psi(x + t) - \psi(x)$ and $u^{2,t}(x) := \phi(x +$

$t) - \phi(x)$; our plan is to show that $u^{1,t}, u^{2,t}$ satisfy a system like the one in Definition 2.8.2. Repeated applications of Theorem 2.8.4 will then give the desired result. Our proof has three steps.

**Step 1: constructing the elliptic and coercive system** Both $\phi$ and $\psi$ satisfy

$$\text{div}\vec{A}(Du) = 0$$

where $\vec{A}(Du) := \left(-\frac{u_1}{u_n}, -\frac{u_2}{u_n}, ..., \frac{1}{2}\left(\sum_{i=1}^{n-1}\left(\frac{u_i}{u_n}\right)^2 + \frac{1}{u_n^2}\right)\right)$. As such

$$\text{div}\int_0^1 \frac{d}{ds}\vec{A}(D(\psi(x) + s(\psi(x+t) - \psi(x))))ds = 0 \Rightarrow$$

$$\text{div}\int_0^1 a_{ij}(D(\psi(x) + s(\psi(x+t) - \psi(x))))D_i u^{1,t}(x)ds = 0$$

where $a_{ij}(\vec{p}) = \frac{d}{dp_j}A_i(\vec{p})$. $\phi$ and $u^{2,t}$ satisfy an analogous equation. Therefore, $u^{1,t}, u^{2,t}$ satisfy (2.8.3) with $a^1_{ij}(x) := a_{ij}(D\psi(x))$ and

$$f_j^1 := \sum_i \left(a_{ij}(D\psi(x)) - \int_0^1 a_{ij}(D(\psi(x) + s(\psi(x+t) - \psi(x))))ds\right) D_i u^{1,t}$$

(and with corresponding definitions for $f^2, a^2$ in terms of $\phi$). Note $m_1 = m_2 = 1, t_1 = t_2 = 2$ and $s_1 = s_2 = 0$. On the boundary $u^{1,t} + u^{2,t} = 0$ and $\frac{\tilde{h}(x)}{\phi_n(x+t)}D_n u^{2,t} - \frac{1}{\phi_n(x+t)}D_n u^{1,t} = \tilde{h}(x) - \tilde{h}(x+t)$. Therefore, $h_1 = 2, h_2 = 1$ and $p_1 = p_2 = 0$. Set $h_0 = 0$. It is then easy to see that this is a system with a proper assignment of weights. We will check in Step 3 that our system satisfies the ellipticity, coercivity and regularity conditions of Definition 2.8.3.

**Step 2: the iterative process** By Proposition 2.7.5, $u^{i,t} \in C^{1,s}(\overline{U})$. In particular, the $a^k_{ij}$'s and the $B$s satisfy the $h - \mu$-conditions with $h = -1$ and $\mu = s$. It is also easy to see that the $f$'s and $g$'s satisfy the conditions of Theorem 2.8.4 (we assume, of course, that

$\alpha \geq s$; otherwise the result is immediate). We conclude

$$\|u^{i,t}\|_{C^{1,s}(\overline{U})} \lesssim \sum_{i=1}^{2}\sum_{j=1}^{n}\|f_j^i\|_{C^s(\overline{U})} + \|\tilde{h}(-) - \tilde{h}(- + t)\|_{C^s(\overline{U}\cap\{y_n=0\})} + \sum_{k=1}^{2}\|u^{k,t}\|_{C(\overline{U})}, \quad (2.8.7)$$

where the constant implicit in $\lesssim$ is independent of $t$. Some additional justification is needed here: in Theorem 2.8.4 the constant may depend on the $C^{0,s}$ norm of the $a$'s and $B$'s. However, these coefficients have norms which can be bounded independently of $t$ and so the constant intself may be taken to be independent of $t$.

For any $x, y \in U$,

$$\begin{aligned} 2\|\tilde{h}\|_{C^\alpha}|x-y|^s|t|^{\alpha-s} &\geq \min\{2|x-y|^\alpha\|\tilde{h}\|_{C^\alpha}, 2|t|^\alpha\|\tilde{h}\|_{C^\alpha}\} \\ &\geq |\tilde{h}(x) - \tilde{h}(x+t) - \tilde{h}(y) + \tilde{h}(y+t)|. \end{aligned} \quad (2.8.8)$$

Thus $\|\tilde{h}(-) - \tilde{h}(- + t)\|_{C^{0,s}(\overline{U}\cap\{y_n=0\})} \leq C|t|^{\alpha-s}$. We also claim that if $w, v \in C^{0,s}$ then

$$\|(w(-) - w(- + t))(v(-) - v(- + t))\|_{C^{0,s}} \leq 4|t|^s\|w\|_{C^{0,s}}\|v\|_{C^{0,s}}$$

(this follows immediately from the triangle inequality and the fact that $\sup|w(-) - w(- + t)| < |t|^s\|w\|_{C^{0,s}}$). From here we conclude $\|f_j^i\|_{C^{0,s}(\overline{U})} \leq C|t|^s$. Plugging these estimates into (2.8.7) we obtain $\|u^{i,t}\|_{C^{1,s}(\overline{U})} \leq K(|t|^s + |t|^{\alpha-s} + |t|)$ (as $\|u^{k,t}\|_{C^0(\overline{U})} \leq C|t|$).

Therefore, for $j = 1, ..., n$, we have that

$$\begin{aligned} |D_j\psi(x+t) + D_j\psi(x-t) - 2D_j\psi(x)| &= |D_ju^{1,t}(x) - D_ju^{1,t}(x-t)| \\ &\leq \|u^{1,t}\|_{C^{1,s}}|t|^s \leq K(|t|^{2s} + |t|^\alpha). \end{aligned} \quad (2.8.9)$$

This implies $\psi|_{\overline{U}\cap\{y_n=0\}} \in C^{1,\beta}$ where $\beta = \min\{\alpha, 2s\}$ (see (Ste70), Chapter 5, Proposition 8). Remark 2.8.1 gives $\psi, \phi \in C^{1,\beta}(\overline{U})$. Iterate until $\beta = \alpha$.

**Step 3: verifying the conditions of Definition 2.8.2** It is easy to calculate the symmetric $(n \times n)$-matrix

$$DA(\vec{p}) = \begin{pmatrix} \frac{-1}{p_n} & 0 & 0 & \dots & \frac{p_1}{p_n^2} \\ 0 & \frac{-1}{p_n} & 0 & \dots & \frac{p_2}{p_n^2} \\ \vdots & 0 & \ddots & \dots & \vdots \\ \frac{p_1}{p_n^2} & \dots & \frac{p_i}{p_n^2} & \dots & -\left(\frac{1}{p_n}\right)^3 \left(1 + \sum_{i=1}^{n-1} p_i^2\right) \end{pmatrix}.$$

If $\vec{p} = D\phi, D\psi$, then $p_n > 0$ in $\overline{U}$. Thus the matrices $DA(D\phi)$ and $DA(D\psi)$ are both elliptic (justifying our above use of the Schauder estimates) and the system is also elliptic (with the obvious weights $t_1 = t_2 = 2, s_1 = s_2 = 0$). Addtionaly, when $n = 2$ we have the equation

$$-\frac{1}{p_2}(\xi_1 + z\eta_1)^2 + 2\frac{p_1}{p_2^2}(\xi_1 + z\eta_1)(\xi_2 + z\eta_2) - \frac{1}{p_2^3}(1 + p_1^2)(\xi_2 + z\eta_2)^2 = 0.$$

All the coefficients of this polynomial are real, so if $\alpha, \beta$ are its roots it must be the case that $\alpha = \overline{\beta}$ which is exactly the desired result.

We must check coercivity at an arbitrary $y_0 \in \overline{U} \cap \{y_n = 0\}$. If $u^1 = e^{iy' \cdot \xi'} \tilde{u}^1(y_n)$ solves $a_{ij}(D\psi(y_0))D_{ij}u^1 = 0$ then $\tilde{u}^1(y_n)$ is a linear combination of functions of the form $e^{ry_n}$ where $r$ is a root of

$$\frac{\sum |\xi'|^2}{p_n} + 2\frac{p_j}{p_n^2} \sum i\xi_j' x - \frac{1}{p_n^3}(1 + \sum p_i^2)x^2 = 0.$$

This equation has at most one root, call it $r_1$, with strictly negative real part (as the sum of the roots is purely imaginary). That $\tilde{u}^1(y_n) \to 0$ as $y_n \to \infty$ implies $\tilde{u}^1(y_n) = \alpha_1 e^{y_n r_1}$. Similarly, we define $\tilde{u}^2(y_n)$ and conclude $\tilde{u}^2(y_n) = \alpha_2 e^{y_n r_2}$, where $r_2$ has strictly negative real part (if such an $r_1$ or $r_2$ does not exist then we are done).

As $u^1 + u^2 = 0$ on the boundary it must be true that $\alpha_1 + \alpha_2 = 0$. Furthermore

$$\tilde{h}(0)D_n u^2 - D_n u^1 = 0 \Rightarrow$$

$$\tilde{h}(0)\alpha_2 r_2 - \alpha_1 r_1 = 0 \Rightarrow \tilde{h}(0) r_2 + r_1 = 0.$$

But $\tilde{h}(0)r_2$ has strictly negative real part and $r_1$ has strictly negative real part, so their sum must have strictly negative real part and the system is coercive. □

If $\log(h) \in C^{k,\alpha}$ for $k \geq 1$, the above argument can be modified slightly to give that $\partial\Omega$ is locally the graph of a $C^{2,\frac{\alpha}{2+\alpha}}$ function.

**Proposition 2.8.6.** *Let $\partial\Omega$ be a 2-sided NTA domain with $\log(h) \in C^{1,\alpha}(\partial\Omega)$ for $0 < \alpha < 1$. If $n \geq 3$ also assume either that $\Omega$ is $\delta$-Reifenberg flat for $\delta > 0$ small or that $\Omega$ is a Lipschitz domain. Then $\partial\Omega$ is locally the graph of a $C^{2,\frac{\alpha}{2+\alpha}}$ function.*

*Proof.* We follow the proof of Proposition 2.8.5; consider again $u^{1,t}, u^{2,t}$. We have already shown these functions satisfy an elliptic system with coercive boundary conditions. Note, by Proposition 2.8.5, $u^{i,t} \in C^{1,s}(\overline{U})$ for all $s \in (0,1)$. In particular, the $a_{ij}^k$'s and the $B$s satisfy the $h-\mu$-conditions with $h = -1$ and $\mu = s \in (0,1)$ to be choosen later. Furthermore, the $f$'s and $g$'s satisfy the conditions of Theorem 2.8.4.

Follow **Step 2** in the proof of Proposition 2.8.5 until we reach (2.8.8). Here we need an estimate which incorporates the higher regularity of $\tilde{h}$. By Remark 2.8.1, $\tilde{h} \in C^{1,\alpha}(\overline{U} \cap \{y_n = 0\})$. For any $x, y \in \mathbb{R}^n$ write, for the sake of brevity,

$$\delta_y^2 f(x) \equiv f(x+y) + f(x-y) - 2f(x).$$

We can then estimate, for $x, y \in \overline{U} \cap \{y_n = 0\}$,

$$\begin{aligned} |\delta_y^2 \tilde{h}(x+t) - \delta_y^2 \tilde{h}(x)| &\leq \|\tilde{h}\|_{C^{1+\alpha}} \min\{3|t|, 2|y|^{1+\alpha}\} \\ &\leq C\|\tilde{h}\|_{C^{1+\alpha}} |y|^s |t|^{1-\frac{s}{1+\alpha}}. \end{aligned} \tag{2.8.10}$$

Consequently, $\|\tilde{h}(-) - \tilde{h}(-+t)\|_{C^{0,s}} \leq C|t|^{1-\frac{s}{1+\alpha}}$.

Proceed as in **Step 2** of the proof of Proposition 2.8.5 until we reach (2.8.9), which now reads

$$\begin{aligned} |D_j\psi(x+t)+D_j\psi(x-t)-2D_j\psi(x)| &= |D_j u^{1,t}(x)-D_j u^{1,t}(x-t)| \\ &\leq \|u^{1,t}\|_{C^{1,s}}|t|^s \leq K(|t|^{2s}+|t|^{1+s-\frac{s}{1+\alpha}}). \end{aligned} \tag{2.8.11}$$

Pick $s \in (0,1)$ such that

$$1+s-\frac{s}{1+\alpha}=2s \Rightarrow s=\frac{1+\alpha}{2+\alpha}.$$

Then $\psi|_{\overline{U}\cap\{y_n=0\}} \in C^{2,\frac{\alpha}{2+\alpha}}$. By Remark 2.8.1 we can conclude that $u \in C^{2,\frac{\alpha}{2+\alpha}}(\overline{\Omega})$ and, ergo, $\psi, \phi \in C^{2,\frac{\alpha}{2+\alpha}}(\overline{U})$. □

### *2.8.3 Higher regularity*

Once we have shown $\phi, \psi \in C^{2,s}(\overline{U})$ for some $s \in (0,1)$, we can apply classical non-linear "Schauder" type estimates (which require the $C^{2,s}$, *a priori*, assumption). First we need to define a non-linear, elliptic and coercive system.

**Definition 2.8.7.** *Let* $u^k, k=1,2$ *satisfy*

$$\begin{aligned} F_1(y,u^1,u^2,Du^1,Du^2...,D^{t_1+s_1}u^1,D^{t_2+s_1}u^2)=0, y\in U \\ F_2(y,u^1,u^2,Du^1,Du^2...,D^{t_1+s_2}u^1,D^{t_2+s_2}u^2)=0, y\in U \end{aligned} \tag{2.8.12}$$

*and on the boundary satisfy*

$$\begin{aligned} B_1(y,u^1,u^2,Du^1,Du^2...,D^{t_1-h_1}u^1,D^{t_2-h_1}u^2)=0, y\in \overline{U}\cap\{y_n=0\} \\ B_2(y,u^1,u^2,Du^1,Du^2...,D^{t_1-h_2}u^1,D^{t_2-h_2}u^2)=0, y\in \overline{U}\cap\{y_n=0\}. \end{aligned} \tag{2.8.13}$$

*Where,* $\max s_i = 0$ *and* $\min\{t_k+s_i\}, \min\{t_k-h_i\} \geq 0$.

*For a solution,* $v$*, to* (2.8.12)*, we say that the system is* **elliptic** *along* $v$ *at a point* $y_0 \in U$

*if the linear system*

$$\begin{aligned} L_1^1(y_0,D)\phi^1 + L_1^2(y_0,D)\phi^2 &= \frac{d}{dt}F_1(y_0, v^1+t\phi^1, ..., D^{t_2+s_1}(v^2+t\phi^2))|_{t=0} \\ L_2^1(y_0,D)\phi^1 + L_2^2(y_0,D)\phi^2 &= \frac{d}{dt}F_2(y_0, v^1+t\phi^1, ..., D^{t_2+s_2}(v^2+t\phi^2))|_{t=0} \end{aligned} \tag{2.8.14}$$

*is elliptic. That is to say, if the block matrix A, where* $A_{ij} = \tilde{L}_j^i$, *is elliptic. Here* $\tilde{L}_j^i$ *is the principle part of the operator L (for more details see Definition 3.1, Chapter 6 of (KS80)).*

*For a solution, v, to both equations* (2.8.12) *and* (2.8.13) *we say that the boundary conditions are* **coercive** *along v at a point* $y_0 \in \overline{U} \cap \{y_n = 0\}$ *if the linear boundary conditions*

$$\begin{aligned} \Phi_1^1(y_0,D)\phi^1 + \Phi_1^2(y_0,D)\phi^2 &= \frac{d}{dt}B_1(y_0, v^1+t\phi^1, ..., D^{t_2-h_1}(v^2+t\phi^2))|_{t=0} \\ \Phi_2^1(y_0,D)\phi^1 + \Phi_2^2(y_0,D)\phi^2 &= \frac{d}{dt}B_2(y_0, v^1+t\phi^1, ..., D^{t_2-h_2}(v^2+t\phi^2))|_{t=0} \end{aligned} \tag{2.8.15}$$

*are coercive for the* (2.8.14) *(see Definition 2.8.2, above, for the definition of coercive linear boundary values. See Definition 3.2, Chapter 6 in (KS80) for more details). Note in all of the above* $D^n v$ *is short hand for all nth-order derivatives of v.*

We now recall the Schauder estimates for non-linear elliptic systems.

**Theorem 2.8.8.** *[see Theorem 12.2, (ADN64), Theorem 3.3 in Chapter 6, (KS80) and Chapter 6.8, (Mor66)] Assume* $u^k, k = 1, 2$ *satisfy an elliptic and coercive non linear system with proper weights like in Definition 2.8.7. Let* $0 < \alpha < 1$ *and* $\ell_0 = \max(0, -h_r)$ *and assume* $u^k \in C^{\ell_0+t_k,\alpha}(\overline{U})$ *for* $k = 1, 2$. *Then for any* $\ell \geq \ell_0$ *if* $F_i \in C^{\ell-s_i,\alpha}$ *and* $B_r \in C^{\ell+h_r,\alpha}$, *in all arguments, then* $u^k \in C^{\ell+t_k,\alpha}(\overline{U})$.

*Additionally if* $F, G$ *are* $C^\infty$ *(analytic) functions, then* $u^k$ *is* $C^\infty$ *(analytic).*

Our main theorem follows:

**Theorem** (Main Theorem)**.** *Let* $\Omega$ *be a 2-sided NTA domain with* $\log(h) \in C^{k,\alpha}(\partial\Omega)$ *where* $k \geq 0$ *is an integer and* $\alpha \in (0, 1)$. *Then:*

- *when* $n = 2$*:* $\partial\Omega$ *is locally given by the graph of a* $C^{k+1,\alpha}$ *function.*

- *when $n \geq 3$: there is some $\delta_n > 0$ such that if $\delta < \delta_n$ and $\Omega$ is $\delta$-Reifenberg flat or if $\Omega$ is a Lipschitz domain then $\partial\Omega$ is locally given by the graph of a $C^{k+1,\alpha}$ function.*

*Similarly, if $\log(h) \in C^\infty$ or $\log(h)$ is analytic we can conclude (under the same flatness assumptions above) that $\partial\Omega$ is locally given by the graph of a $C^\infty$ (resp. analytic) function.*

*Proof.* For $k = 0$ this result is contained in Proposition 2.8.5. For $k = 1$ Proposition 2.8.6 tells us that $\partial\Omega$ is $C^{2,s}$, $u^\pm \in C^{2,s}(\overline{\Omega}^\pm)$ for some $0 < s < \alpha$. Theorem 2.8.8, applied as below, combined with a standard difference quotient argument, like the ones above, gives the optimal regularity; $\partial\Omega$ given by the graph of a $C^{2,\alpha}$ function and $u^\pm \in C^{2,\alpha}(\overline{\Omega}^\pm)$

Let $k \geq 2$, and set $\ell_0 = 0, \ell = k-1, t_1 = t_2 = 2, s_1 = s_2 = 0$ and $h_1 = 2, h_2 = 1$. First, we will show that $\psi, \phi$ satisfy an elliptic and coercive non-linear system with the above weights (as defined in Definition 2.8.7). Argue similarly to prove $C^\infty$ or analytic regularity.

Recall, $\mathrm{div}\vec{A}(D\phi) = 0 = \mathrm{div}\vec{A}(D\psi)$, where,

$$\vec{A}(Dw) := \left(-\frac{w_1}{w_n}, -\frac{w_2}{w_n}, ..., -\frac{w_{n-1}}{w_n}, \frac{1}{2}\left(\sum_{i=1}^{n-1}\left(\frac{w_i}{w_n}\right)^2 + \frac{1}{w_n^2}\right)\right).$$

Therefore, the associated linear system at $y_0$ is $L_1^1 v^1 = \frac{d}{dp_i}A_j(\psi(y_0))v^1_{ij}, L_1^2 \equiv 0, L_2^1 \equiv 0$ and $L_2^2 v^2 = \frac{d}{dp_i}A_j(\phi(y_0))v^2_{ij}$. We have already established, in the proof of Proposition 2.8.5, that this is an elliptic system.

We have $B_1(y, \psi, \phi, ....) = \phi + \psi$ and $B_2(y, \psi, \phi, ...) = h((y', \psi(y)))\phi_n - \psi_n$, which are unchanged by linearization. Again, in the proof of Proposition 2.8.5, we have shown that these boundary conditions are coercive for the above linear equations. Furthermore, the above values give a proper assignment of weights.

Finally, $F_1, F_2, B_1$ are analytic in all arguments (recall that $\psi_n, \phi_n \neq 0$ in $U$) and $B_2$ is analytic in $D\psi, D\phi$ but has the same regularity in $y$ and $\psi$ that $h$ has in $x$. By assumption, $h \in C^{k,\alpha} = C^{\ell+h_2,\alpha}$ so $B_2$ has the desired regularity. Additionally, by Proposition 2.8.6, $u$ has the required initial smoothness. Thus, applying Theorem 2.8.8 yields the desired result. □

# CHAPTER 3

# A FREE BOUNDARY PROBLEM FOR THE PARABOLIC POISSON KERNEL

## 3.1 Introduction

In this paper we prove a parabolic analogue of Kenig and Toro's Poisson kernel characterization of vanishing chord arc domains, (KT03). This continues a program started by Hofmann, Lewis and Nyström, (HLN04), who introduced the concept of parabolic chord arc domains and proved parabolic versions of results in (KT99) and (KT97) (see below for more details). Precisely, we show that if $\Omega$ is a $\delta$-Reifenberg flat parabolic chord arc domain, with $\delta > 0$ small enough, and the logarithm of the Poisson kernel has vanishing mean oscillation, then $\Omega$ actually satisfies a vanishing Carleson measure condition (see (3.1.6)). The key step in this proof is a classification of "flat" blowups (see Theorem 3.1.10 below), which itself was an open problem of interest (see, e.g., the remark at the end of Section 5 in (HLN04)).

Let us recall the defintions and concepts needed to state our main results. In these we mostly follow the conventions established in (HLN04). We then briefly sketch the contents of the paper, taking special care to highlight when the difficulties introduced by the parabolic setting require substantially new ideas. Throughout, we work in two or more spacial dimensions ($n \geq 2$); the case of one spacial dimension is addressed in (Eng16). Finally, for more historical background on free boundary problems involving harmonic or caloric measure we suggest the introduction of (HLN04).

We denote points $(x_1, ..., x_n, t) = (X, t) \in \mathbb{R}^{n+1}$ and the parabolic distance between them is $d((X,t),(Y,s)) := |X - Y| + |t - s|^{1/2}$. For $r > 0$, the parabolic cylinder $C_r(X,t) := \{(Y,s) \mid |s - t| < r^2, |X - Y| < r\}$. Our main object of study will be $\Omega$, an unbounded,

0. The contents of this chapter are taken from a preprint with the same title which has been submitted for publication. While writing that paper I was partially supported by the National Science Foundation's Graduate Research Fellowship, Grant No. (DGE-1144082). I also thank Abdalla Nimer for helpful comments regarding Section 3.5 and Professor Tatiana Toro for helping me overcome a technical difficulty in Section 3.6.

connected open set in $\mathbb{R}^{n+1}$ such that $\Omega^c$ is also unbounded and connected. As the time variable has a "direction" we will often consider $\Omega^{t_0} := \Omega \cap \{(X,s) \mid s < t_0\}$. Finally, for any Borel set $F$ we will define $\sigma(F) = \int_F d\sigma_t dt$ where $d\sigma_t := \mathcal{H}^{n-1}|_{\{s=t\}}$, the $(n-1)$-dimension Hausdorff measure restricted to the time-slice $t$. We normalize $\mathcal{H}^{n-1}$ so that $\sigma(C_1(0,0) \cap V) = 1$ for any $n$-plane through the origin containing a direction parallel to the time axis (we also normalize the Lebesgue measure, $dXdt$, by the same multiplicative factor).

**Definition 3.1.1.** *We say that $\Omega$ is $\delta$-***Reifenberg flat***, $\delta > 0$, if for $R > 0$ and $(Q,\tau) \in \partial\Omega$ there exists a $n$-plane $L(Q,\tau,R)$, containing a direction parallel to the time axis and passing through $(Q,\tau)$, such that*

$$\begin{aligned} &\{(Y,s) + r\hat{n} \in C_R(Q,\tau) \mid r > \delta R, (Y,s) \in L(Q,\tau,R)\} \subset \Omega \\ &\{(Y,s) - r\hat{n} \in C_R(Q,\tau) \mid r > \delta R, (Y,s) \in L(Q,\tau,R)\} \subset \mathbb{R}^{n+1} \backslash \overline{\Omega}. \end{aligned} \tag{3.1.1}$$

*Where $\hat{n}$ is the normal vector to $L(Q,\tau,R)$ pointing into $\Omega$ at $(Q,\tau)$.*

The reader may be more familiar with a definition of Reifenberg flatness involving the Hausdorff distance between two sets (recall that the Hausdorff distance between $A$ and $B$ is defined as $D(A,B) = \sup_{a\in A} d(a,B) + \sup_{b \in B} d(b,A))$. These two notions are essentially equivalent as can be seen in the following remark (which follows from the triangle inequality).

**Remark 3.1.2.** *If $\Omega$ is a $\delta$-Reifenberg flat domain, then for any $R > 0$ and $(Q,\tau) \in \partial\Omega$ there exists a plane $L(Q,\tau,R)$, containing a line parallel to the time axis and through $(Q,\tau)$, such that $D[C_R(Q,\tau) \cap L(Q,\tau,R), C_R(Q,\tau) \cap \partial\Omega] \leq 4\delta R$.*

*Similarly, if* (3.1.1) *holds for some $\delta_0$ and there always exists an $L(Q,\tau,R)$ such that $D[C_R(Q,\tau) \cap L(Q,\tau,R), C_R(Q,\tau) \cap \partial\Omega] \leq \delta R$ then* (3.1.1) *holds for $2\delta$.*

Let $\theta((Q,\tau),R) := \inf_P \frac{1}{R} D[C_R(Q,\tau) \cap P, C_R(Q,\tau) \cap \partial\Omega]$ where the infimun is taken over all planes containing a line parallel to the time axis and through $(Q,\tau)$.

**Definition 3.1.3.** *We say that* $\Omega$ *(or* $\Omega^{t_0}$*) is* **vanishing Reifenberg flat** *if* $\Omega$ *is* $\delta$*-Reifenberg flat for some* $\delta > 0$ *and for any compact set* $K$ *(alternatively* $K \subset\subset \{t < t_0\}$*),*

$$\lim_{r \downarrow 0} \sup_{(Q,\tau) \in K \cap \partial\Omega} \theta((Q,\tau), r) = 0. \tag{3.1.2}$$

**Definition 3.1.4.** $\partial\Omega$ *is* **Ahflors regular** *if there exists an* $M \geq 1$ *such that for all* $(Q,\tau) \in \partial\Omega$ *and* $R > 0$ *we have*

$$\left(\frac{R}{2}\right)^{n+1} \leq \sigma(C_R(Q,\tau) \cap \partial\Omega) \leq MR^{n+1}.$$

Note that left hand inequality follows immediately in a $\delta$-Reifenberg flat domain for $\delta > 0$ small enough (as Hausdorff measure decreases under projection, and $C_{R/2}(Q,\tau) \cap L(Q,\tau,R) \subset \mathrm{proj}_{L(Q,\tau,R)}(C_R(Q,\tau) \cap \partial\Omega)$).

Following (HLN04), define, for $r > 0$ and $(Q,\tau) \in \partial\Omega$,

$$\gamma(Q,\tau,r) = \inf_P \left( r^{-n-3} \int_{\partial\Omega \cap C_r(Q,\tau)} d((X,t),P)^2 d\sigma(X,t) \right) \tag{3.1.3}$$

where the infimum is taken over all $n$-planes containing a line parallel to the $t$-axis and going through $(Q,\tau)$. This is an $L^2$ analogue of Jones' $\beta$-numbers ((Jon90)). We want to measure how $\gamma$, "on average", grows in $r$ and to that end introduce

$$d\nu(Q,\tau,r) = \gamma(Q,\tau,r) d\sigma(Q,\tau) r^{-1} dr. \tag{3.1.4}$$

Recall that $\mu$ is a Carleson measure with norm $\|\mu\|_+$ if

$$\sup_{R>0} \sup_{(Q,\tau) \in \partial\Omega} \mu((C_R(Q,\tau) \cap \partial\Omega) \times [0,R]) \leq \|\mu\|_+ R^{n+1}. \tag{3.1.5}$$

In analogy to David and Semmes (DS93) (who defined uniformly rectifiable domains in the isotropic setting) we define a parabolic uniformly rectifiable domain;

**Definition 3.1.5.** *If $\Omega \subset \mathbb{R}^{n+1}$ is such that $\partial\Omega$ is Ahlfors regular and $\nu$ is a Carleson measure then we say that $\Omega$ is a* **(parabolic) uniformly rectifiable domain.**

*As in (HLN04), if $\Omega$ is a parabolic uniformly rectifiable domain which is also $\delta$-Reifenberg flat for some $\delta > 0$ we say that $\Omega$ is a* **parabolic regular domain**. *We may also refer to them as* **parabolic chord arc domains**.

*Finally, if $\Omega$ is a parabolic regular domain and satisfies a vanishing Carleson measure condition,*

$$\lim_{R\downarrow 0} \sup_{0<\rho<R} \sup_{(Q,\tau)\in\partial\Omega} \rho^{-n-1}\nu((C_\rho(Q,\tau)\cap\partial\Omega)\times[0,\rho]) = 0 \tag{3.1.6}$$

*we call $\Omega$ a* **vanishing chord arc domain**. *Alternatively, if* (3.1.6) *holds when the $\partial\Omega$ is replaced by $K$ for any $K \subset\subset \{s < t_0\}$ then we say that $\Omega^{t_0}$ is a vanishing chord arc domain.*

Readers familiar with the elliptic theory will note that these definitions differ from, e.g. Definition 1.5 in (KT97). It was observed in (HLN03) that these definitions are equivalent in the time independent setting whereas the elliptic definition is weaker when $\Omega$ changes with time. Indeed, in the time independent setting, uniform recitifiability with small Carleson norm and being a chord arc domain with small constant are both equivalent to the existence of big pieces of Lipschitz graphs, in the sense of Semmes (Sem91), at every scale (see Theorem 2.2 in (KT97) and Theorem 1.3 in Part IV of (DS93)). On the other hand, in the time dependent case, even $\sigma(\Delta_R(Q,\tau)) \equiv R^{n+1}$ does not imply the Carleson measure condition (see the example at the end of (HLN03)).

The role of this Carleson measure condition becomes clearer when we consider domains of the form $\Omega = \{(X,t) \mid x_n > f(x,t)\}$ for some function $f$. Dahlberg (Dah77) proved that surface measure and harmonic measure are mutually absolutely continuous in a Lipschitz domain. However, Kaufman and Wu (KW88) proved that surface measure and caloric measure are not necessarily mutually absolutely continuous when $f \in \mathrm{Lip}(1,1/2)$. To ensure mutual absolute continuity one must also assume that the $1/2$ time derivative of $f$ is in BMO (see (LM95)). In (HLN04) it is shown that the BMO norm of the $1/2$ time derivative of $f$ can be controlled by the Carleson norm of $\nu$. Morally, the growth of $\sigma(\Delta_R(Q,\tau))$ controls the Lip$(1,1/2)$ norm of $f$ but cannot detect the BMO norm of the 1/2-time derivative of $f$ (for $n=1$ this is made precise in (Eng16)).

For $(X,t)\in\Omega$, the caloric measure with a pole at $(X,t)$, denoted $\omega^{(X,t)}(-)$, is the measure associated to the map $f\mapsto U_f(X,t)$ where $U_f$ solves the heat equation with Dirichlet data $f\in C_0(\partial\Omega)$. If $\Omega$ is Reifenberg flat the Dirichlet problem has a unique solution and this measure is well defined (in fact, weaker conditions on $\Omega$ suffice to show $\omega^{(X,t)}$ is well defined c.f. the discussion at the bottom of page 283 in (HLN04)). Associated to $\omega^{(X,t)}$ is the parabolic Green function $G(X,t,-,-)\in C(\Omega\backslash\{(X,t)\})$, which satisfies

$$\begin{cases} G(X,t,Y,s) \geq 0, \ \forall (Y,s)\in \Omega\backslash\{(X,t)\}, \\ G(X,t,Y,s) \equiv 0, \ \forall (Y,s)\in\partial\Omega, \\ -(\partial_s+\Delta_Y)G(X,t,Y,s) = \delta_0((X,t)-(Y,s)), \\ \int_{\partial\Omega}\varphi d\omega^{(X,t)} = \int_\Omega G(X,t,Y,s)(\Delta_Y-\partial_s)\varphi dY ds, \ \forall\varphi\in C_c^\infty(\mathbb{R}^{n+1}). \end{cases} \tag{FP}$$

(Of course there are analogous objects for the adjoint equation; $G(-,-,Y,s)$ and $\hat{\omega}^{(Y,s)}$.) We are interested in what the regularity of $\omega^{(X,t)}$ can tell us about the regularity of $\partial\Omega$. Observe that by the parabolic maximum theorem the caloric measure with a pole at $(X,t)$ can only "see" points $(Q,\tau)$ with $\tau<t$. Thus, any regularity of $\omega^{(X,t)}$ will only give information about $\Omega^t$ (recall $\Omega^t := \Omega\cap\{(X,s)\mid s<t\}$). Hence, our results and proofs will often be clearer when we work with $\omega$, the caloric measure with a pole at infinity, and $u\in C(\Omega)$, the associated Green function, which satisfy

$$\begin{cases} u(Y,s) \geq 0, \ \forall (Y,s)\in\Omega, \\ u(Y,s) \equiv 0, \ \forall (Y,s)\in\partial\Omega, \\ -(\partial_s+\Delta_Y)u(Y,s) = 0, \ \forall (Y,s)\in\Omega \\ \int_{\partial\Omega}\varphi d\omega = \int_\Omega u(Y,s)(\Delta_Y-\partial_s)\varphi dY ds, \ \forall\varphi\in C_c^\infty(\mathbb{R}^{n+1}). \end{cases} \tag{IP}$$

(For the existence, uniqueness and some properties of this measure/function, see Appendix B.3). However, when substantial modifications are needed, we will also state and prove our theorems in the finite pole setting.

Let us now recall some salient concepts of "regularity" for $\omega^{(X,t)}$. Denote the *surface ball* at a point $(Q,\tau) \in \partial\Omega$ and for radius $r$ by $\Delta_r(Q,\tau) := C_r(Q,\tau) \cap \partial\Omega$.

**Definition 3.1.6.** *Let $(X_0,t_0) \in \Omega$. We say $\omega^{(X,t)}$ is a* **doubling measure** *if, for every $A \geq 2$, there exists an $c(A) > 0$ such that for any $(Q,\tau) \in \partial\Omega$ and $r > 0$, where $|X_0 - Q|^2 < A(t_0 - \tau)$ and $t_0 - \tau \geq 8r^2$, we have*

$$\omega^{(X_0,t_0)}(\Delta_{2r}(Q,\tau)) \leq c(A)\omega^{(X_0,t_0)}(\Delta_r(Q,\tau)). \tag{3.1.7}$$

*Alternatively, we say $\omega$ is a doubling measure if there exists a $c > 0$ such that $\omega(\Delta_{2r}(Q,\tau)) \leq c\omega(\Delta_r(Q,\tau))$ for all $r > 0$ and $(Q,\tau) \in \partial\Omega$.*

**Definition 3.1.7.** *Let $(X_0,t_0) \in \Omega$, such that $\omega^{(X_0,t_0)}$ is a doubling measure, $\omega^{(X_0,t_0)} << \sigma$ on $\partial\Omega$ and $k^{(X_0,t_0)}(Q,\tau) := \frac{d\omega^{(X_0,t_0)}}{d\sigma}(Q,\tau)$. We say that $\omega^{(X_0,t_0)} \in A_\infty(d\sigma)$ (***is an $A_\infty$-weight***) if it satisfies a "reverse Hölder inequality." That is, if there exists a $p > 1$ such that if $A \geq 2, (Q,\tau) \in \partial\Omega, r > 0$ are as in Definition 3.1.6 then there exists a $c \equiv c(p,A) > 0$ where*

$$⨍_{\Delta_{2r}(Q,\tau)} k^{(X_0,t_0)}(Q,\tau)^p d\sigma(Q,\tau) \leq c\left(⨍_{\Delta_r(Q,\tau)} k^{(X_0,t_0)}(Q,\tau) d\sigma(Q,\tau)\right)^p. \tag{3.1.8}$$

*We can similarly say $\omega \in A_\infty(d\sigma)$ if $\omega << \sigma$ on $\partial\Omega, h(Q,\tau) := \frac{d\omega}{d\sigma}$, and there exists a $c > 0$ such that*

$$⨍_{\Delta_{2r}(Q,\tau)} h(Q,\tau)^p d\sigma(Q,\tau) \leq c\left(⨍_{\Delta_r(Q,\tau)} h(Q,\tau) d\sigma(Q,\tau)\right)^p. \tag{3.1.9}$$

In analogy to the results of David and Jerison (DJ90), it was shown in (HLN04) that if $\Omega$ is a "flat enough" parabolic regular domain, then the caloric measure is an $A_\infty$ weight (note that (HLN04) only mentions the finite pole case but the proof works unchanged for a pole at infinity, see Proposition B.3.5).

**Theorem** (Theorem 1 in (HLN04) ). *If $\Omega$ is a parabolic regular domain with Reifenberg constant $\delta_0 > 0$ sufficiently small (depending on $M, \|\nu\|_+$) then $\omega^{(X_0,t_0)}$ is an $A_\infty$-weight.*

Closely related to $A_\infty$ weights are the BMO and VMO function classes.

**Definition 3.1.8.** *We say that $f \in \mathrm{BMO}(\partial\Omega)$ with norm $\|f\|_*$ if*

$$\sup_{r>0} \sup_{(Q,\tau)\in\partial\Omega} \fint_{C_r(Q,\tau)} |f(P,\eta) - f_{C_r(Q,\tau)}| d\sigma(P,\eta) \leq \|f\|_*,$$

*where $f_{C_r(Q,\tau)} := \fint_{C_r(Q,\tau)} f(P,\eta)d\sigma(P,\eta)$, the average value of $f$ on $C_r(Q,\tau)$.*

*Define $\mathrm{VMO}(\partial\Omega)$ to be the closure of uniformly continuous functions vanishing at infinity in $\mathrm{BMO}(\partial\Omega)$ (analogously we say that $k^{(X_0,t_0)} \in \mathrm{VMO}(\partial\Omega^{t_0})$ if $k^{(X_0,t_0)} \in \mathrm{VMO}(\Delta_r(Q,\tau))$ for any $(Q,\tau) \in \partial\Omega, r > 0$ which satisfies the hypothesis of Definition 3.1.6 for some $A \geq 2$).*

This definition looks slightly different than the one given by equation 1.11 in (HLN04). In the infinite pole setting it gives control over the behavior of $f$ on large scales. In the finite pole setting it is actually equivalent to the definition given in (HLN04) as can be seen by a covering argument.

In analogy with the elliptic case, if $\Omega$ is a vanishing chord arc domain then we expect control on the small scale oscillation of $\log(k^{(X_0,t_0)})$.

**Theorem** (Theorem 2 in (HLN04)). *If $\Omega$ is chord arc domain with vanishing constant and $(X_0, t_0) \in \Omega$ then $\log(k^{(X_0,t_0)}) \in \mathrm{VMO}(\partial\Omega^{t_0})$.*

Our main result is the converse to the above theorem and the parabolic analogue of the Main Theorem in (KT03).

**Theorem 3.1.9.** *[Main Theorem] Let $\Omega \subset \mathbb{R}^{n+1}$ be a $\delta$-Reifenberg flat parabolic regular domain with $\log(h) \in \mathrm{VMO}(\partial\Omega)$ (or $\log(k^{(X_0,t_0)}) \in \mathrm{VMO}(\partial\Omega^{t_0})$). There is a $\delta_n > 0$ such* *that if $\delta < \delta_n$, then $\Omega$ is a parabolic vanishing chord arc domain (alternatively, $\Omega^{t_0}$ is a vanishing chord arc domain).*

Contrast this result to Theorem 3 in (HLN04);

**Theorem** (Theorem 3 in (HLN04)). *Let $\Omega$ be a $\delta$-Reifenberg flat parabolic regular domain with $(\hat{X}, \hat{t}) \in \Omega$. Assume that*

*(i)* $\omega^{(\hat{X},\hat{t})}(-)$ *asymptotically optimally doubling,*

*(ii)* $\log k^{(\hat{X},\hat{t})} \in \mathrm{VMO}(\partial\Omega^{\hat{t}})$,

*(iii)* $\|\nu\|_+$ *small enough.*

*Then $\Omega^{\hat{t}}$ is a vanishing chord arc domain.*

Our main theorem removes the asymptotically optimally doubling and small Carleson measure hypotheses. As mentioned above, this requires a classification of the "flat" limits of pseudo-blowups (Definition 3.4.1 below), which was heretofore open in the parabolic setting.

**Theorem 3.1.10.** *[Classification of "flat" Blowups] Let $\Omega_\infty$ be a $\delta$-Reifenberg flat parabolic regular domain with Green function at infinity, $u_\infty$, and associated parabolic Poisson kernel, $h_\infty$ (i.e. $h_\infty = \frac{d\omega_\infty}{d\sigma}$). Furthermore, assume that $|\nabla u_\infty| \leq 1$ in $\Omega_\infty$ and $|h_\infty| \geq 1$ for $\sigma$-almost every point on $\partial\Omega_\infty$. There exists a $\delta_n > 0$ such that if $\delta_n \geq \delta > 0$ we may conclude that, after a potential rotation and translation, $\Omega_\infty = \{(X,t) \mid x_n > 0\}$.*

Nyström (Nys06a) proved a version of Theorem 3.1.10 under the additional assumptions that $\Omega$ is a graph domain and that the Green function is comparable with the distance function from the boundary. Furthermore, under the additional assumption that $\Omega$ is a graph domain, Nyström (Nys12) also proved that Theorem 3.1.10 implies Theorem 3.1.9. Our proof of Theorem 3.1.10 (given in Appendix B.1) is heavily inspired by the work of Andersson and Weiss (AW09), who studied a related free boundary problem arising in combustion. However, we are unable to apply their results directly as they consider solutions in the sense of "domain variations" and it is not clear if the parabolic Green function is a solution in this sense. For example, solutions in the sense of domain variations satisfy the bound

$$\int_{C_r(X,t)} |\partial_t u|^2 \leq C_1 r^n, \ \forall C_r(X,t) \subset \Omega,$$

and it is unknown if this inequality holds in a parabolic regular domain (see, e.g., the remark at the end of Section 1 in (Nys12)). Furthermore, the results in (AW09) are local, whereas Theorem 3.1.10 is a global result. Nevertheless, we were able to adapt the ideas in (AW09) to our setting. For further discussion of exactly how our work fits in with that of (AW09) see the beginning of Appendix B.1 below.

Let us now briefly outline this paper and sketch the contents of each section. The paper follows closely the structure, and often the arguments, of (KT03). In Section 3.2 we prove some technical estimates which will be used in Sections 3.3 and 3.4. Section 3.3 is devoted to proving an integral bound for the gradient of the Green function. The arguments in this section are much like those in the elliptic case. However, we were not able to find the necessary results on non-tangential convergence in parabolic Reifenberg flat domains (e.g. Fatou's theorem) in the literature. Therefore, we prove them in Appendix B.4. Of particular interest may be Proposition B.4.3 which constructs interior "sawtooth" domains (the elliptic construction does not seem to generalize to the parabolic setting). Section 3.4 introduces the blowup procedure and uses estimates from Sections 3.2 and 3.3 to show that the limit of this blowup satisfies the hypothesis of Theorem 3.1.10. This allows us to conclude that $\Omega$ is vanishing Reifenberg flat and, after an additional argument, gives the weak convergence of surface measure under pseudo-blowup.

By combining the weak convergence of $\sigma$ with the weak convergence of $\hat{n}\sigma$ under pseudo-blowups (the latter follows from the theory of sets with finite perimeter) we can conclude easily that $\hat{n} \in \text{VMO}$ (morally, bounds on the growth of $\sigma(\Delta_r(Q,\tau))$ give bounds on the BMO norm of $\hat{n}$, see Theorem 2.1 in (KT97)). Therefore, in the elliptic setting, the weak convergence of surface measure is essentially enough to prove that $\Omega$ is a vanishing chord arc domain. On the other hand, to show that $\Omega$ is a parabolic vanishing chord arc domain one must establish a vanishing Carleson measure condition (equation (3.1.6)). Furthermore, the aforementioned example at the end of (HLN03) shows that control of the growth of $\sigma(\Delta_r(Q,\tau))$ does not necessarily give us control on $\|\nu\|_+$. In Section 3.5 we use purely geometric measure theoretical arguments to prove Theorem 3.5.1; that a vanishing Reifenberg flat parabolic chord arc domain whose surface measure converges weakly under pseudo-blowups must be a parabolic vanishing chord arc domain. To establish this (and thus finish the proof of Theorem 3.1.9), we adapt approximation theorems of Hofmann, Lewis and Nyström, (HLN03), and employ a compactness argument.

The remainder of our paper is devoted to free boundary problems with conditions above the continuous threshold. In particular, we prove (stated here in the infinite pole setting),

**Theorem 3.1.11.** *Let $\Omega \subset \mathbb{R}^{n+1}$ be a parabolic regular domain and $k \in \mathbb{N}, \alpha \in (0,1)$ such that $\log(h) \in \mathbb{C}^{k+\alpha,(k+\alpha)/2}(\mathbb{R}^{n+1})$. There is a $\delta_n > 0$ such that if $\delta_n \geq \delta > 0$ and $\Omega$ is $\delta$-Reifenberg flat, then $\Omega$ is a $\mathbb{C}^{k+1+\alpha,(k+1+\alpha)/2}(\mathbb{R}^{n+1})$ domain.*

*Furthermore, if $\log(h)$ is analytic in $X$ and in the second Gevrey class (see Definition 3.7.7) in $t$ then, under the assumptions above, we can conclude that $\Omega$ is the region above the graph of a function which is analytic in the spatial variables and in the second Gevrey class in $t$. Similarly, if $\log(h) \in C^\infty$ then $\partial\Omega$ is locally the graph of a $C^\infty$ function.*

The case of $k = 0$ follows in much the same manner as the proof of Theorem 3.1.10 but nevertheless is done in full detail in Section 3.6. For larger values of $k$, we use the techniques of Kinderlehrer and Nirenberg (see e.g. (KN78)), parabolic Schauder-type estimates (see e.g. (Lie86)) and an iterative argument inspired by Jerison (Jer90). These arguments are presented in Section 3.7.

Finally, let us comment on the hypothesis of Theorem 3.1.11. For $n \geq 3$, this theorem is sharp. In particular, Alt and Caffarelli, (AC81), constructed an Ahlfors regular domain $\Omega \subset \mathbb{R}^3$ with $\log(h) = 0$ but which is not a $C^1$ domain (it has a cone point at the origin). A cylinder over this domain shows that the flatness condition is necessary. On the other hand, Keldysh and Lavrentiev (see (KL37)) constructed a domain in $\mathbb{R}^2$ which is rectifiable but not Ahlfors regular, where $h \equiv 1$ but the domain is not a $C^1$ domain. A cylinder over this domain shows that the Parabolic regular assumption is necessary. In one spatial dimension, our upcoming preprint (Eng16) shows that the the flatness condition is not necessary (as topology implies that a parabolic NTA domain is a graph domain). When $n = 2$ it is not known if the flatness assumption is necessary and we have no intuition as to what the correct answer should be.

## 3.2 Notation and Preliminary Estimates

As mentioned above, all our theorems will concern a $\delta$-Reifenberg flat, parabolic regular domain $\Omega$. Throughout, $\delta > 0$ will be small enough such that $\Omega$ is a *non-tangentially accesible* (NTA) domain (for the definition see (LM95), Chapter 3, Sec 6, and Lemma 3.3 in (HLN04)). In particular, for each $(Q,\tau) \in \partial\Omega$ and $r > 0$ there exists two "corkscrew" points, $A_r^{\pm}(Q,\tau) := (X_r^{\pm}(Q,\tau), t_r^{\pm}(Q,\tau)) \in C_r(Q,\tau) \cap \Omega$ such that $d(A_r^{\pm}(Q,\tau), \partial\Omega) \geq r/100$ and $\min\{t_r^{+}(Q,\tau) - \tau, \tau - t_r^{-}(Q,\tau)\} \geq r^2/100$.

Our theorems apply both to finite and infinite pole settings. Unfortunately, we will often have to treat these instances seperately. Fix (for the remainder of the paper) $(X_0, t_0) \in \Omega$ and define $u^{(X_0,t_0)}(-,-) = G(X_0, t_0, -, -)$, the Green function (which is adjoint-caloric), with a pole at $(X_0, t_0)$. As above, $\omega^{(X_0,t_0)}$ is the associated caloric measure and $k^{(X_0,t_0)}$ the corresponding Poisson kernel (which exists by (HLN04), Theorem 1). In addition, $u$ is the Green function with a pole at $\infty$, $\omega$ the associated caloric measure and $h$ the corresponding Poisson kernel. We will always assume (unless stated otherwise) that $\log(h) \in \mathrm{VMO}(\partial\Omega)$ or $\log(k^{(X_0,t_0)}) \in \mathrm{VMO}(\partial\Omega^{t_0})$.

Finally, define, for convenience, the distance from $(X,t) \in \Omega$ to the boundary

$$\delta(X,t) = \inf_{(Q,\tau)\in\partial\Omega} \|(X,t) - (Q,\tau)\|.$$

### *3.2.1 Estimates for Green Functions in Parabolic Reifenberg Flat Domains*

Here we will state some estimates on the Green function of a parabolic Reifenberg flat domain that will be essential for the gradient bounds of Section 3.3. Corresponding estimates for the Green function with a pole at infinity are discussed in Appendix B.3.

We begin by bounding the growth of caloric functions which vanish on surface balls. The reader should note this result appears in different forms elsewhere in the literature (e.g. (LM95), Lemma 6.1 and (HLN04) Lemma 3.6), so we present the proof here for the sake of completeness.

**Lemma 3.2.1.** *Let $\Omega$ be a $\delta$-Reifenberg flat domain and $(Q,\tau) \in \partial\Omega$. Let $w$ be a continuous non-negative solution to the (adjoint)-heat equation in $C_{2r}(Q,\tau) \cap \Omega$ such that $w = 0$ on $C_{2r}(Q,\tau) \cap \partial\Omega$. Then for any $\varepsilon > 0$ there exists a $\delta_0 = \delta_0(\varepsilon) > 0$ such that if $\delta < \delta_0$ there exists a $c = c(\delta_0) > 0$ such that*

$$w(X,t) \leq c \left( \frac{d((X,t),(Q,\tau))}{r} \right)^{1-\varepsilon} \sup_{(Y,s) \in C_{2r}(Q,\tau)} w(Y,s) \tag{3.2.1}$$

*whenever $(X,t) \in C_r(Q,\tau) \cap \Omega$.*

*Proof.* We argue is in the proof of Lemma 2.1 in (KT03). Let $(Q,\tau) \in \partial\Omega$ and $r > 0$. Let $v_0$ be adjoint caloric in $C_{2r}(Q,\tau) \cap \Omega$ such that $v_0 = 0$ on $\Delta_{2r}(Q,\tau)$ and $v_0 \equiv 1$ on $\partial_p C_{2r}(Q,\tau) \cap \Omega$. By the maximum principle,

$$w(X,t) \leq [\sup_{(Y,s) \in C_{2r}(Q,\tau)} w(Y,s)] v_0(X,t). \tag{3.2.2}$$

We will now attempt to bound $v_0$ from above.

Assume, without loss of generality, that the plane of best fit at $(Q,\tau)$ for scale $2r$ is $\{x_n = 0\}$ and that $(Q,\tau) = (0,0)$. Define $\Lambda = \{(X,t) = (x, x_n, t) \mid x_n \geq -4r\delta\}$. It is a consequence of Reifenberg flatness that $C_r(0,0) \cap \Omega \subset C_r(0,0) \cap \Lambda$. Define $h_0$ to be an adjoint-caloric function in $\Lambda \cap C_{2r}(0,0)$ such that $h_0 = 0$ on $\partial\Lambda \cap C_{2r}(0,0)$ and $h_0 = 1$ on

$\partial_p C_{2r}(0,0) \cap \Lambda$. By the maximum principle $h_0(X,t) \geq v_0(X,t)$ for all $(X,t) \in C_{2r}(0,0) \cap \Omega$.

Finally, consider the function $g_0$ defined by $g_0(x, x_n, t) = x_n + 4\delta r$. It is clear that $g_0$ is an adjoint caloric function on $\Lambda \cap C_{2r}(0,0)$. Furthermore, $h_0, g_0$ both vanish on $\partial\Lambda \cap C_{2r}(Q,\tau)$. Recall that (adjoint-)caloric functions in a cylinder satsify a comparison principle (see Theorem 1.6 in (FGS86)). Hence, there is a constant $C > 0$ such that

$$\frac{h_0(X,t)}{g_0(X,t)} \leq C\frac{h_0(0, r/2, 0)}{r}, \ \forall (X,t) \in C_{r/4}(0,0) \cap \Lambda. \tag{3.2.3}$$

Let $(X,t) = (x, a, t)$. Then equation (3.2.3) becomes

$$h_0(X,t) \leq C\frac{a + 4\delta r}{r}. \tag{3.2.4}$$

It is then easy to see, for any $\theta < 1$ and $(X,t) \in C_{\theta r}(0,0)$, that $v_0(X,t) \leq h_0(X,t) \leq C(\theta+\delta)$. Let $\theta = \delta$ and iterate this process. The desired result follows. □

Using the parabolic Harnack inequality, we can say more about $\sup_{(Y,s)\in C_{2r}(Q,\tau)} w(Y,s)$.

**Lemma 3.2.2.** *[Lemma 3.7 in (HLN04)] Let $\Omega, w, (Q,\tau), \delta_0$ be as in Lemma 3.2.1. There is a universal constant $c(\delta_0) \geq 1$ such that if $(Y,s) \in \Omega \cap C_{r/2}(Q,\tau)$ then*

$$w(Y,s) \leq cw(A_r^{\pm}(Q,\tau)),$$

*where we choose $A^-$ if $w$ is a solution to the adjoint-heat equation and $A^+$ otherwise.*

As the heat equation is anisotropic, given a boundary point $(Q,\tau)$ it will behoove us to distinguish the points in $\Omega$ which are not much closer to $(Q,\tau)$ in time than in space.

**Definition 3.2.3.** *For $(Q,\tau) \in \partial\Omega$ and $A \geq 100$ define the* **time-space cone at scale** $r$ **with constant** $A$*, $T^{\pm}_{A,r}(Q,\tau)$, by*

$$T^{\pm}_{A,r}(Q,\tau) := \{(X,t) \in \Omega \mid |X - Q|^2 \leq A|t-\tau|, \pm(t-\tau) \geq 4r^2\}.$$

The next four estimates are presented, and proven, in (HLN04). We will simply state them here. The first compares the value of the Green function at a corkscrew point with the caloric or adjoint caloric measure of a surface ball.

**Lemma 3.2.4.** *[Lemma 3.10 in (HLN04)] Let $\Omega, (Q,\tau), \delta_0$ be as in Lemma 3.2.1. Additionally suppose from some $A \geq 100, r > 0$ that $(X,t) \in T^+_{A,r}(Q,\tau)$. There exists some $c = c(A) \geq 1$ (independent of $(Q,\tau)$) such that*

$$c^{-1} r^n G(X,t,A^+_r(Q,\tau)) \leq \omega^{(X,t)}(\Delta_{r/2}(Q,\tau)) \leq c r^n G(X,t,A^-_r(Q,\tau)).$$

*Similarly if $(X,t) \in T^-_{A,r}(Q,\tau)$ we have*

$$c^{-1} r^n G(A^-_r(Q,\tau),X,t) \leq \hat{\omega}^{(X,t)}(\Delta_{r/2}(Q,\tau)) \leq c r^n G(A^+_r(Q,\tau),X,t).$$

We now recall what it means for an (adjoint-)caloric function to satisfy a backwards in time Harnack inequality (see e.g. (FGS86)).

**Definition 3.2.5.** *If $(Q,\tau) \in \partial\Omega$ and $\rho > 0$ we say that $w > 0$ satisfies a backwards Harnack inequality in $C_\rho(Q,\tau) \cap \Omega$ provided $w$ is a solution to the (adjoint-)heat equation in $C_\rho(Q,\tau) \cap \Omega$ and there exists $1 \leq \lambda < \infty$ such that*

$$w(X,t) \leq \lambda w(\tilde{X},\tilde{t}),\ \forall (X,t),(\tilde{X},\tilde{t}) \in C_r(Z,s),$$

*where $(Z,s), r$ are such that $C_{2r}(Z,s) \subset C_\rho(Q,\tau) \cap \Omega$.*

In Reifenberg flat domains, the Green function satisfies a backwards Harnack inequality.

**Lemma 3.2.6.** *[Lemma 3.11 in (HLN04)] Let $\Omega, (Q,\tau), \delta_0$ be as in Lemma 3.2.1. Additionally suppose from some $A \geq 100, r > 0$ that $(X,t) \in T^+_{A,r}(Q,\tau)$. There exists some $c = c(A) \geq 1$ such that*

$$G(X,t,A^-_r(Q,\tau)) \leq c G(X,t,A^+_r(Q,\tau)).$$

*On the other hand, if* $(X,t) \in T^-_{A,r}(Q,\tau)$ *we conclude*

$$G(A^+_r(Q,\tau),X,t) \leq cG(A^-_r(Q,\tau),X,t).$$

Lemmas 3.2.4 and 3.2.6, imply that (adjoint-)caloric measure is doubling.

**Lemma 3.2.7.** *[Lemma 3.17 in (HLN04)] Let* $\Omega, (Q,\tau), \delta_0$ *be as in Lemma 3.2.1. Additionally suppose from some* $A \geq 100, r > 0$ *that* $(X,t) \in T^+_{A,r}(Q,\tau)$*. Then there exists a constant* $c = c(A) \geq 1$ *such that*

$$\omega^{(X,t)}(\Delta_r(Q,\tau)) \leq c\omega^{(X,t)}(\Delta_{r/2}(Q,\tau)).$$

*If* $(X,t) \in T^-_{A,r}(Q,\tau)$ *a similar statement holds for* $\hat{\omega}$*.*

In analogy to Lemma 4.10 in (JK82), there is a boundary comparison theorem for (adjoint-)caloric functions in Reifenberg flat domains (see also Theorem 1.6 in (FGS86), which gives a comparison theorem for caloric functions in cylinders).

**Lemma 3.2.8.** *[Lemma 3.18 in (HLN04)] Let* $\Omega, (Q,\tau), \delta_0$ *be as in Lemma 3.2.1. Let* $w, v \geq 0$ *be continuous solutions to the (adjoint)-heat equations in* $\overline{C}_{2r}(Q,\tau) \cap \overline{\Omega}$ *with* $w, v > 0$ *in* $\Omega \cap C_{2r}(Q,\tau)$ *and* $w = v = 0$ *on* $C_{2r}(Q,\tau) \cap \partial\Omega$*. If* $w, v$ *satisfy a backwards Harnack inequality in* $C_{2r}(Q,\tau) \cap \Omega$ *for some* $\lambda \geq 1$ *then*

$$\frac{w(Y,s)}{v(Y,s)} \leq c(\lambda)\frac{w(A^{\pm}_r(Q,\tau))}{v(A^{\pm}_r(Q,\tau))},\ \forall (Y,s) \in C_{r/2}(Q,\tau) \cap \overline{\partial\Omega}.$$

*Where we choose* $A^-$ *if* $w, v$ *are solutions to the adjoint heat equation and* $A^+$ *otherwise.*

As in the elliptic setting, a boundary comparison theorem leads to a growth estimate.

**Lemma 3.2.9.** *[Lemma 3.19 in (HLN04)] Let* $\Omega, (Q,\tau), \delta_0, w, v$ *be as in Lemma 3.2.8. There* *exists a* $0 < \gamma \equiv \gamma(\lambda) \leq 1/2$ *and a* $c \equiv c(\lambda) \geq 1$ *such that*

$$\left|\frac{w(X,t)v(Y,s)}{w(Y,s)v(X,t)} - 1\right| \leq c\left(\frac{\rho}{r}\right)^{\gamma},\ \forall (X,t),(Y,s) \in C_\rho(Q,\tau)\cap\Omega$$

*whenever* $0 < \rho \leq r/2$.

### *3.2.2 VMO functions on Parabolic Chord Arc Domains*

Here we state some consequences of the condition $\log(h) \in \mathrm{VMO}(\partial\Omega)$ or $\log(k^{(X_0,t_0)}) \in \mathrm{VMO}(\partial\Omega^{t_0})$. Our first theorem is a reverse Hölder inequality for every exponent. This is a consequence of the John-Nirenberg inequality (JN16), in the Euclidean case (see Garnett and Jones, (GJ78)). However, per a remark in (GJ78), the result remains true in our setting as $\partial\Omega$ is a "space of homogenous type". For further remarks and justification, see Theorem 2.1 in (KT03), which is the analogous result for the elliptic problem.

**Lemma 3.2.10.** *Let* $\Omega \subset \mathbb{R}^{n+1}$ *be a parabolic chord arc domain and* $\log(f) \in \mathrm{VMO}(\partial\Omega)$. *Then for all* $(Q,\tau) \in \partial\Omega$ *and* $r > 0$ *and* $1 < q < \infty$ *we have*

$$\left(\fint_{\Delta_r(Q,\tau)} f^q d\sigma\right)^{1/q} \leq C \fint_{\Delta_r(Q,\tau)} f d\sigma. \tag{3.2.5}$$

*Here* $C$ *depends only on the* VMO *character of* $f$, *the chord arc constants of* $\Omega$, $n$ *and* $q$.

For the Poisson kernel with finite pole a localized analogue of the above Lemma holds (and is proved in much the same way):

**Lemma 3.2.11.** *Let* $(X_0,t_0) \in \Omega$ *with* $(Q,\tau) \in \partial\Omega, A \geq 100, r > 0$ *such that* $(X_0,t_0) \in T^+_{A,r}(Q,\tau)$. *If* $\log(k^{(X_0,t_0)}) \in \mathrm{VMO}(\Omega^{t_0})$ *then, for any* $1 < q < \infty$

$$\left(\fint_{\Delta_r(Q,\tau)} (k^{X_0,t_0})^q d\sigma\right)^{1/q} \leq C \fint_{\Delta_r(Q,\tau)} k^{(X_0,t_0)} d\sigma. \tag{3.2.6}$$

*Here $C > 0$ depends on $n, q, A$ the* VMO *character of $k^{(X_0,t_0)}$ and the chord arc constants of $\Omega$.*

Substitute the poisson kernel, $h := \frac{d\omega}{d\sigma}$, for $f$ in Lemma 3.2.10 to glean information on the concentration of harmonic measure in balls. This is the parabolic analogue of Corollary 2.4 in (KT03).

**Corollary 3.2.12.** *If $\Omega, h$ are as above, then for all $\varepsilon > 0, (Q,\tau) \in \partial\Omega, r > 0$ and $E \subset \Delta_r(Q,\tau)$*

$$C^{-1}\left(\frac{\sigma(E)}{\sigma(\Delta_r(Q,\tau))}\right)^{1+\varepsilon} \le \frac{\omega(E)}{\omega(\Delta_r(Q,\tau))} \le C\left(\frac{\sigma(E)}{\sigma(\Delta_r(Q,\tau))}\right)^{1-\varepsilon}. \quad (3.2.7)$$

*Here $C$ depends on $n, \varepsilon$, the chord arc constants of $\Omega$ and the* VMO *character of $h$.*

Similarly, in the finite pole case we can conclude:

**Corollary 3.2.13.** *Let $(X_0, t_0) \in \Omega, \log(k^{(X_0,t_0)}) \in \mathrm{VMO}(\partial\Omega^{t_0})$, and $A \ge 100, r > 0$ and $(Q,\tau) \in \partial\Omega$ such that $(X_0,t_0) \in T^+_{A,r}(Q,\tau)$. Then for all $\varepsilon > 0$ and $E \subset \Delta_r(Q,\tau)$*

$$C^{-1}\left(\frac{\sigma(E)}{\sigma(\Delta_r(Q,\tau))}\right)^{1+\varepsilon} \le \frac{\omega^{(X_0,t_0)}(E)}{\omega^{(X_0,t_0)}(\Delta_r(Q,\tau))} \le C\left(\frac{\sigma(E)}{\sigma(\Delta_r(Q,\tau))}\right)^{1-\varepsilon}. \quad (3.2.8)$$

*Here $C$ depends on $n, \varepsilon, A$, the chord arc constants of $\Omega$ and the* VMO *character of $k^{(X_0,t_0)}$.*

Finally, the John-Nirenberg inequality and the definition of VMO lead to the following decomposition (see the discussion in the proof of Lemma 4.3 in (KT03) for more detail– specifically equations 4.95 and 4.96).

**Lemma 3.2.14.** *Let $\Omega, h$ be as above. Given $\varepsilon > 0$ and $(Q_0,\tau_0) \in \partial\Omega$ there exists an $r(\varepsilon) > 0$ such that for $\rho \in (0, r(\varepsilon))$ and $(Q,\tau) \in \Delta_1(Q_0,\tau_0)$ there exists a $G(Q,\tau,\rho) \subset \Delta_\rho(Q,\tau)$ such that $\sigma(\Delta_\rho(Q,\tau)) \le (1+\varepsilon)\sigma(G(Q,\tau,\rho))$ and, for all $(P,\eta) \in G(Q,\tau,\rho)$,*

$$(1+\varepsilon)^{-1} \fint_{\Delta_\rho(Q,\tau)} h d\sigma \le h(P,\eta) \le (1+\varepsilon) \fint_{\Delta_\rho(Q,\tau)} h d\sigma. \quad (3.2.9)$$

And in the finite pole setting:

**Lemma 3.2.15.** *Let* $\Omega, k^{(X_0,t_0)}, (Q,\tau) \in \partial\Omega, r > 0, A \geq 100$ *be as above. Given* $\varepsilon > 0$ *exists an* $r(\varepsilon) > 0$ *such that for* $\rho \in (0, r(\varepsilon))$ *and* $(\tilde{Q}, \tilde{\tau}) \in C_r(Q,\tau)$ *there exists a* $G(\tilde{Q}, \tilde{\tau}, \rho) \subset \Delta_\rho(\tilde{Q}, \tilde{\tau})$ *such that* $\sigma(\Delta_\rho(\tilde{Q}, \tilde{\tau})) \leq (1+\varepsilon)\sigma(G(\tilde{Q}, \tilde{\tau}, \rho))$ *and, for all* $(P,\eta) \in G(\tilde{Q}, \tilde{\tau}, \rho)$,

$$(1+\varepsilon)^{-1} \fint_{\Delta_\rho(\tilde{Q},\tilde{\tau})} k^{(X_0,t_0)} d\sigma \leq k^{(X_0,t_0)}(P,\eta) \leq (1+\varepsilon) \fint_{\Delta_\rho(\tilde{Q},\tilde{\tau})} k^{(X_0,t_0)} d\sigma. \quad (3.2.10)$$

## 3.3 Bounding the Gradient of the Green Function

As mentioned in the introduction, the first step in our proof is to establish an integral bound for $\nabla u$ (and $\nabla G(X_0, t_0, -, -)$). Later, this will aid in demonstrating that our blowup satisfies the hypothesis of the classification result, Theorem 3.1.10.

We begin by estimating the non-tagential maximal function of the gradient. Recall the definition of a non-tangetial region:

**Definition 3.3.1.** *For* $\alpha > 0, (Q,\tau) \in \partial\Omega$ *define,* $\Gamma_\alpha(Q,\tau)$*, the non-tangential region at* $(Q,\tau)$ *with aperture* $\alpha$*, as*

$$\Gamma_\alpha(Q,\tau) = \{(X,t) \in \Omega \mid \|(X,t) - (Q,\tau)\| \leq (1+\alpha)\delta(X,t)\}.$$

*For* $R > 0$ *let* $\Gamma_\alpha^R(Q,\tau) := \Gamma_\alpha(Q,\tau) \cap C_R(Q,\tau)$ *denote the truncated non-tangential region.*

*Associated with these non-tangential regions are maximal functions*

$$N_\alpha(f)(Q,\tau) := \sup_{(X,t)\in\Gamma_\alpha(Q,\tau)} |f(X,t)|$$

$$N_\alpha^R(f)(Q,\tau) := \sup_{(X,t)\in\Gamma_\alpha^R(Q,\tau)} |f(X,t)|.$$

*Finally, we say that $f$ has a non-tangential limit, $L$, at $(Q,\tau) \in \partial\Omega$ if for any $\alpha > 0$*

$$\lim_{\substack{(X,t)\to(Q,\tau)\\(X,t)\in\Gamma_\alpha(Q,\tau)}} f(X,t) = L.$$

In order to apply Fatou's theorem and Martin's repesentation theorem (see Appendix B.4) we must bound the non-tagential maximal function by a function in $L^2$. We argue as in the proof of Lemma 3.1 in (KT03) (which proves the analogous result in the elliptic setting).

**Lemma 3.3.2.** *For any $\alpha > 0, R > 0, N_\alpha^R(|\nabla u|) \in L^2_{loc}(d\sigma)$.*

*Proof.* Let $K \subset \mathbb{R}^{n+1}$ be a compact set and $\hat{K}$ be the compact set of all points parabolic distance $\leq 4R$ away from $K$. Pick $(X,t) \in \Gamma_\alpha^R(Q,\tau)$. Standard estimates for adjoint-caloric functions, followed by Lemmas 3.2.2 and B.3.4 yield

$$|\nabla u(X,t)| \leq C\frac{u(X,t)}{\delta(X,t)} \leq C\frac{u(A^-_{4\|(Q,\tau)-(X,t)\|}(Q,\tau))}{\delta(X,t)} \leq C\frac{\omega(\Delta_{2\|(Q,\tau)-(X,t)\|}(Q,\tau))}{\delta(X,t)\|(Q,\tau)-(X,t)\|^n}.$$

In the non-tangential region, $\delta(X,t) \sim_\alpha \|(Q,\tau)-(X,t)\|$, which, as $\sigma$ is Ahlfors regular and $\omega$ is doubling, implies

$$|\nabla u(X,t)| \leq C_\alpha \fint_{\Delta_{\|(Q,\tau)-(X,t)\|}(Q,\tau)} h d\sigma \leq C_\alpha M_R(h)(Q,\tau).$$

$M_R(h)(Q,\tau) := \sup_{0<r\leq R} \fint_{C_r(Q,\tau)} |h(P,\eta)| d\sigma(P,\eta)$ is the truncated Hardy-Littlewood maximal operator at scale R. $\partial\Omega$ is a space of homogenous type and $h \in L^2_{\mathrm{loc}}(d\sigma)$, so we may apply the Hardy-Littlewood maximal theorem to conclude

$$\int_K M_R(h)^2 d\sigma \leq C\int_{\hat{K}} h^2 d\sigma < \infty.$$

□

The result in the finite pole setting follows in the same way;

**Lemma 3.3.3.** *For $(X_0,t_0)\in\Omega$ let $(Q,\tau)\in\partial\Omega$, $R>0$ and $A\geq 100$ be such that $(X_0,t_0)\in T^+_{A,R}(Q,\tau)$. Then for any $\alpha>0$, $N^{R/8}_\alpha(|\nabla u^{(X_0,t_0)}|)|_{\Delta_{R/2}(Q,\tau)}\in L^2(d\sigma)$.*

*Proof.* Let $(P,\eta)\in\Delta_{R/2}(Q,\tau)$ and pick $(X,t)\in\Gamma^{R/8}_\alpha(P,\eta)$. Standard estimates for adjoint-caloric functions, followed by Lemma 3.2.2 yield

$$|\nabla u^{(X_0,t_0)}(X,t)|\leq C\frac{u^{(X_0,t_0)}(X,t)}{\delta(X,t)}\leq C\frac{u^{(X_0,t_0)}(A^-_{4\|(P,\eta)-(X,t)\|}(P,\eta))}{\delta(X,t)}.$$

Note $(X,t)\in\Gamma^{R/8}_\alpha(P,\eta)$ hence $4\|(P,\eta)-(X,t)\|\leq R/2$. By our assumption on $(Q,\tau)$ and $(P,\eta)\in\Delta_{R/2}(Q,\tau)$ we can compute that $R/2,(P,\eta),(X_0,t_0)$ satisfy the hypothesis of Lemmas 3.2.6 and 3.2.4 for some $A\geq 100$ which can be taken uniformly over $(P,\eta)\in\Delta_{R/2}(Q,\tau)$. Therefore,

$$|\nabla u^{(X_0,t_0)}(X,t)|\leq C(A)\frac{\omega^{(X_0,t_0)}(\Delta_{2\|(P,\eta)-(X,t)\|}(P,\eta))}{\delta(X,t)\|(P,\eta)-(X,t)\|^n}. \tag{3.3.1}$$

In the non-tangential region, $\delta(X,t)\sim_\alpha\|(P,\eta)-(X,t)\|$, which, as $\sigma$ is Ahlfors regular and $\|(P,\eta)-(X,t)\|\leq R/8$ implies

$$|\nabla u^{(X_0,t_0)}(X,t)|\leq C_{\alpha,A}\fint_{\Delta_{2\|(P,\eta)-(X,t)\|}(P,\eta)}k^{(X_0,t_0)}d\sigma\leq C_{\alpha,A}M_{R/2}(k^{(X_0,t_0)})(P,\eta).$$

The result then follows as in Lemma 3.3.2. □

Unfortunately, the above argument only bounds the *truncated* non-tangential maximal operators. We need a cutoff argument to transfer this estimate to the untruncated non-tangential maximal operator. We will do this argument first for the infinite pole case and then in the finite pole setting. The following lemma is a parabolic version of Lemma 3.3 in (KT03) or Lemma 3.5 in (KT06), whose exposition we will follow quite closely.

**Lemma 3.3.4.** *Assume that $(0,0)\in\partial\Omega$ and fix $R>1$ large. Let $\varphi_R\in C^\infty_c(\mathbb{R}^{n+1}),\varphi_R\equiv 1$ on $C_R(0,0)$, $0\leq\varphi_R\leq 1$ and assume $\mathrm{spt}(\varphi_R)\subset C_{2R}(0,0)$. It is possible to ensure that*

*$|\nabla \varphi_R| \leq C/R$ and $|\partial_t \varphi_R|, |\Delta \varphi_R| \leq C/R^2$. For $(X,t) \in \Omega$ define*

$$w_R(X,t) = \int_\Omega G(Y,s,X,t)(\partial_s + \Delta_Y)[\varphi_R(Y,s)\nabla u(Y,s)]dYds, \tag{3.3.2}$$

*here, as before $G(Y,s,X,t)$ is the Green's function for the heat equation with a pole at $(X,t)$ (i.e. $(\partial_s - \Delta_Y)(G(Y,s,X,t)) = \delta_{(X,t),(Y,s)}$ and $G(Y,s,-,-) \equiv 0$ on $\partial\Omega$). Then, $w_R|_{\partial\Omega} \equiv 0$ and $w_R \in C(\overline{\Omega})$. Furthermore, if $\|(X,t)\| \leq \frac{R}{2}$, then*

$$|w_R(X,t)| \leq C\frac{\delta(X,t)^{3/4}}{R^{1/2}}. \tag{3.3.3}$$

*Additionally, if $\|(X,t)\| \geq 4R$, there is a constant $C \equiv C(R) > 0$ such that*

$$|w_R(X,t)| \leq C \tag{3.3.4}$$

*Proof.* That $w_R|_{\partial\Omega} \equiv 0$ and $w_R \in C(\overline{\Omega})$ follows immediately from the definition of $w_R$. For ease of notation let $V(x,t) := \nabla u(X,t)$. By the product rule

$$(\partial_s + \Delta_Y)[\varphi_R(Y,s)V(Y,s)] = V(Y,s)(\Delta_Y + \partial_s)\varphi_R(Y,s) + 2\left\langle \nabla\varphi_R(Y,s), \nabla V(Y,s)\right\rangle.$$

Split $w_R(X,t) = w_R^1(X,t) + w_R^2(X,t)$ where,

$$\begin{aligned} w_R^1(X,t) &:= \int_\Omega G(Y,s,X,t)V(Y,s)(\Delta_Y + \partial_s)\varphi_R(Y,s)dYds \\ w_R^2(X,t) &:= 2\int_\Omega G(Y,s,X,t)\left\langle \nabla\varphi_R(Y,s), \nabla V(Y,s)\right\rangle dYds. \end{aligned}$$

Regularity theory gives $|\nabla u(Y,s)| \leq C\frac{u(Y,s)}{\delta(Y,s)}$ and $|\nabla\nabla u(Y,s)| \leq C\frac{u(Y,s)}{\delta^2(Y,s)}$. This, along with our bounds on $\varphi_R$, yields

$$\begin{aligned}|w_R^1(X,t)| &\leq \frac{C}{R^2}\int_{\{(Y,s)\in\Omega | R<\|(Y,s)\|\leq 2R\}} \frac{u(Y,s)}{\delta(Y,s)}G(Y,s,X,t)dY\,ds \\ |w_R^2(X,t)| &\leq \frac{C}{R}\int_{\{(Y,s)\in\Omega | R<\|(Y,s)\|\leq 2R\}} \frac{u(Y,s)}{\delta^2(Y,s)}G(Y,s,X,t)dY\,ds.\end{aligned} \tag{3.3.5}$$

Any upper bound on $w_R^2$ will also be an upper bound for $w_R^1$ (as $R \gtrsim \delta(Y,s)$). Assume first, in order to prove (3.3.3), that $\|(X,t)\| < R/2$. We start by showing that there is a universal constant, $C > 1$ such that

$$G(Y,s,X,t) \leq C\left(\frac{\delta(Y,s)}{R}\right)^{3/4} G(A_{3R}^+(0,0),X,t) \tag{3.3.6}$$

for all $(Y,s) \in \Omega \cap (C_{2R}(0,0)\backslash C_R(0,0))$ and $(X,t) \in \Omega \cap C_{R/2}(0,0)$. To prove this, first assume that $\delta(Y,s) \geq R/10$. We would like to construct a Harnack chain between $A_{3R}^+(0,0)$ and $(Y,s)$. To do so, we need to verify that the parabolic distance between the two points is less than 100 times the square root of the distance between the two points along the time axis. As we are in a $\delta$-Reifenberg flat domain the $t$ coordinate of $A_{3R}^+(0,0)$ is equal to $(3R)^2$ and so $A_{3R}^+(0,0)$ and $(Y,s)$ are seperated in the $t$-direction by a distance of $5R^2$. On the other hand $\|A_{3R}^+(0,0)-(Y,s)\| \leq 20R < 100(5R^2)^{1/2}$. So there is a Harnack chain connecting $(Y,s)$ and $A_{3R}^+(0,0)$. In a $\delta$-Reifenberg flat domain the chain can be constructed to stay outside of $C_{R/2}(0,0)$ (see the proof of Lemma 3.3 in (HLN04)). Furthermore, as $\delta(Y,s)$ is comparable to $R$, the length of this chain is bounded by some constant. Therefore, by the Harnack inequality, we have equation (3.3.6) (note in this case $\left(\frac{\delta(Y,s)}{R}\right)^{3/4}$ is greater than some constant, and so can be included on the right hand side).

If $\delta(Y,s) < R/10$, there is a point $(Q,\tau) \in \partial\Omega$ such that $C_{R/5}(Q,\tau)\cap C_{R/2}(0,0) = \emptyset$, and $(Y,s) \in C_{R/10}(Q,\tau)$. Lemma 3.2.1 yields $G(Y,s,X,t) \leq \left(\frac{\delta(Y,s)}{R}\right)^{3/4} G(A_{R/5}^+(Q,\tau),X,t)$. We can then create a Harnack chain, as above, connecting $A_{R/5}^+(Q,\tau)$ and $A_{3R}^+(0,0)$ to obtain equation (3.3.6).

$G(A^+_{3R}(0,0),-,-)$ is an adjoint caloric function in $\Omega \cap C_R(0,0)$, whence,

$$\begin{aligned} G(Y,s,X,t) \leq & C\left(\frac{\delta(Y,s)}{R}\right)^{3/4} G(A^+_{3R}(0,0),X,t) \\ \leq & C\left(\frac{\delta(Y,s)}{R}\right)^{3/4}\left(\frac{\delta(X,t)}{R}\right)^{3/4} G(A^+_{3R}(0,0),A^-_R(0,0)) \\ \leq & C\left(\frac{\delta(Y,s)}{R}\right)^{3/4}\left(\frac{\delta(X,t)}{R}\right)^{3/4} R^{-n}, \end{aligned} \tag{3.3.7}$$

where the penultimate inequality follows from Lemma 3.2.1 applied in $(X,t)$. The bound on $G(A^+_{3R}(0,0),A^-_R(0,0))$ and, therefore, the last inequality above, is a consequence of Lemmas 3.2.6 and 3.2.4:

$$G(A^+_{3R}(0,0),A^-_R(0,0)) \leq G(A^+_{3R}(0,0),A^+_R(0,0)) \leq cR^{-n}\omega^{A^+_{3R}(0,0)}(C_{R/2}(0,0)) \leq cR^{-n}.$$

Lemma 3.2.1 applied to $u(Y,s)$ and (3.3.7) allow us to bound

$$\frac{C}{R}\int_{C_{2R}(0,0)\setminus C_R(0,0)} \frac{u(Y,s)G(Y,s,X,t)}{\delta^2(Y,s)}dY\,ds$$

by

$$\frac{C}{R^{n+5/2}}\left(\frac{\delta(X,t)}{R}\right)^{3/4}\int_{\Omega\cap C_{2R}(0,0)} \frac{u(A^-_{2R}(0,0))}{\delta(Y,s)^{1/2}}dY\,ds.$$

Ahlfors regularity implies, for any $\beta > (1/(2R))^{1/2}$, that

$$|\{(Y,s)\in\Omega\cap C_{2R}(0,0) \mid \delta(Y,s)^{-1/2} > \beta\}| \lesssim \frac{R^{n+1}}{\beta^2}.$$

Therefore,

$$\int_{\Omega\cap C_{2R}(0,0)} \frac{1}{\delta(Y,s)^{1/2}}dY\,ds \lesssim R^{n+1}\int_{(1/(2R))^{1/2}}^{\infty}\frac{1}{\beta^2}d\beta \simeq R^{n+1+1/2}.$$

Putting everything together we get that

$$|w(X,t)| \leq C\frac{u(A^-_{2R}(0,0))}{R}\left(\frac{\delta(X,t)}{R}\right)^{3/4} \leq C\frac{\delta(X,t)^{3/4}}{R^{1/2}}, \tag{3.3.8}$$

where the last inequality follows from the fact that $u(A^-_{2R}(0,0))$ cannot grow faster than $R^{1+\alpha}$ for any $\alpha > 0$. This can be established by arguing as in proof of Lemma 3.3.2 and invoking Corollary 3.2.12.

We turn to proving equation (3.3.4), and assume that $\|(X,t)\| \geq 4R$. Following the proof of equation (3.3.6) we can show

$$G(Y,s,X,t) \leq CG(Y,s,A^-_{2^jR}(0,0)). \tag{3.3.9}$$

Above, $j$ is such that $2^{j-2}R \leq \|(X,t)\| \leq 2^{j-1}R$. Note that $G(Y,s,A^-_{2^jR})$ is a caloric function in $C_{2^{j-1}R}(0,0)$ and apply Lemma 3.2.1 to obtain

$$G(Y,s,X,t) \leq C\left(\frac{\delta(Y,s)}{2^{j-1}R}\right)^{3/4} G(A^+_{2^{j-1}R}(0,0), A^-_{2^jR}(0,0)) \leq C\delta(Y,s)^{3/4}(2^jR)^{-n-3/4}.$$

The last inequality follows from estimating the Green's function as we did in the proof of equation (3.3.7). Proceeding as in the proof of equation (3.3.3) we write

$$\frac{C}{R}\int_{\Omega\cap(C_{2R}(0,0)\setminus C_R(0,0))} \frac{u(Y,s)}{\delta^2(Y,s)}G(Y,s,X,t)dYds \leq (2^jR)^{-n-3/4}\frac{R^{n+1+1/2}}{R^{1+3/4}}u(A^-_{4R}(0,0)).$$

Putting everything together,

$$|w_R(X,t)| \leq C\frac{u(A^-_{4R}(0,0))}{R}2^{-nj} \leq C(R). \tag{3.3.10}$$

□

**Corollary 3.3.5.** *For any $(Y,s) \in \Omega$, $\nabla u$ has a non-tangential limit, $F(Q,\tau)$, for $d\hat{\omega}^{(Y,s)}$-almost every $(Q,\tau) \in \Omega$. In particular, the non-tangential limit exists for $\sigma$-almost every*

$(Q,\tau) \in \partial\Omega$. *Furthermore,* $F(Q,\tau) \in L^1_{\text{loc}}(d\hat{\omega}^{(Y,s)})$ *and* $F(Q,\tau) \in L^2_{\text{loc}}(d\sigma)$.

*Proof.* Theorem 1 in (HLN04) implies that for any compact set $K \subset \partial\Omega$ there exists $(Y,s) \in \Omega$ such that $\hat{\omega}^{(Y,s)}|_K \in A_\infty(\sigma|_K)$. Therefore, if the non-tangential limit exists $d\hat{\omega}^{(Y,s)}$-almost everywhere for any $(Y,s) \in \Omega$ we can conclude that it exists $\sigma$-almost everywhere. Additionally, Lemma 3.3.2 implies that if $\nabla u$ has a non-tangential limit, that limit is in $L^2_{\text{loc}}(d\sigma)$ and therefore $L^1_{\text{loc}}(d\hat{\omega}^{(Y,s)})$ for any $(Y,s) \in \Omega$.

Thus it suffices to prove, for any $(Y,s) \in \Omega$, that $\nabla u$ has a non-tangential limit $d\hat{\omega}^{(Y,s)}$-almost everywhere. Let $R > 0$ and define, for $(X,t) \in \Omega$, $H_R(X,t) = \varphi_R(X,t)\nabla u(X,t) - w_R(X,t)$, where $w_R, \varphi_R$ were introduced in Lemma 3.3.4. Equation (3.3.4) and $w_R(X,t) \in C(\overline{\Omega})$ imply that $w_R(X,t) \in L^\infty(\Omega)$, which, with Lemma 3.3.2, gives that $N(H_R)(X,t) \in L^1(d\hat{\omega}^{(Y,s)})$ for any $(Y,s) \in \Omega$. By construction, $H_R$ is a solution to the adjoint heat equation in $\Omega$, hence, by Lemma B.4.1, $H_R(X,t)$ has a non-tangential limit $\hat{\omega}^{(Y,s)}$-almost everywhere. Finally, because $w_R, \varphi_R \in C(\overline{\Omega})$ we can conclude that $\nabla u$ has a non-tangential limit for $\hat{\omega}^{(Y,s)}$-almost every point in $C_R(0,0)$. As $R$ is arbitrary the result follows. □

If we assume higher regularity in $\partial\Omega$, it is easy to conclude $\nabla u(Q,\tau) = h(Q,\tau)\hat{n}(Q,\tau)$ for every $(Q,\tau) \in \partial\Omega$. The following lemma, proved in Appendix B.2, says that this remains true in our (low regularity) setting.

**Lemma 3.3.6.** *For* $\sigma$*-a.e.* $(Q,\tau) \in \partial\Omega$ *we have* $F(Q,\tau) = h(Q,\tau)\hat{n}(Q,\tau)$

Finally, we can prove the integral estimate.

**Lemma 3.3.7.** *Let* $\Omega$ *be a* $\delta$*-Reifenberg flat parabolic regular domain. Let* $u, h$ *be the Green function and parabolic Poisson kernel with poles at infinity respectively. Fix* $R >> 1$*, then for any* $(X,t) \in \Omega$ *with* $\|(X,t)\| \leq R/2$

$$|\nabla u(X,t)| \leq \int_{\Delta_{2R}(0,0)} h(Q,\tau) d\hat{\omega}^{(X,t)}(Q,\tau) + C\frac{\|(X,t)\|^{3/4}}{R^{1/2}}. \tag{3.3.11}$$

*Proof.* For $(X,t) \in \Omega$ define $H_R(X,t) = \varphi_R(X,t)\nabla u(X,t) - w_R(X,t)$ where $w_R, \varphi_R$ were introduced in Lemma 3.3.4. $H_R(X,t)$ is a solution to the adjoint heat equation (by construction) and $N(H_R) \in L^1(d\hat{\omega}^{(Y,s)})$ for every $(Y,s) \in \Omega$ (as shown in the proof of Corollary 3.3.5). Hence, by Proposition B.4.4 we have $H_R(X,t) = \int_{\partial\Omega} g(Q,\tau) d\hat{\omega}^{(X,t)}(Q,\tau)$, where $g(Q,\tau)$ is the non-tangential limit of $H_R$.

Lemma 3.3.6 and $w_R|_{\partial\Omega} \equiv 0$ imply that $g(Q,\tau) = \varphi_R(Q,\tau)h(Q,\tau)\hat{n}(Q,\tau)$. Estimate (3.3.3) on the growth of $w_R$ allows us to conclude

$$|\nabla u(X,t)| \leq |H_R(X,t)| + |w_R(X,t)| \leq \int_{\partial\Omega \cap C_{2R}(0,0)} h(Q,\tau) d\hat{\omega}^{(X,t)}(Q,\tau) + C\frac{\|(X,t)\|^{3/4}}{R^{1/2}}.$$

□

The finite pole case begins similarly; we start with a cut-off argument much in the style of Lemma 3.3.4.

**Lemma 3.3.8.** *Let $(X_0,t_0) \in \Omega$ and fix any $(Q,\tau) \in \partial\Omega, R > 0, A \geq 100$ such that $(X_0,t_0) \in T^+_{A,R}(Q,\tau)$. Let $\varphi \in C^\infty_c(C_{R/2}(Q,\tau))$. Furthermore, it is possible to ensure that $\varphi \equiv 1$ on $C_{R/4}(Q,\tau)$, $0 \leq \varphi \leq 1, |\nabla\varphi| \leq C/R$ and $|\partial_t\varphi|, |\Delta\varphi| \leq C/R^2$.*

*For $(X,t) \in \Omega$ define*

$$W(X,t) = \int_\Omega G(Y,s,X,t)(\partial_s + \Delta_Y)[\varphi(Y,s)\nabla u^{(X_0,t_0)}(Y,s)]dY ds. \tag{3.3.12}$$

*Then, $W|_{\partial\Omega} \equiv 0$ and $W \in C(\overline{\Omega})$. Additionally, if $\|(X,t) - (Q,\tau)\| \leq R/8$ then*

$$|W(X,t)| \leq C(A)\left(\frac{\delta(X,t)}{R}\right)^{3/4} \frac{\omega^{(X_0,t_0)}(\Delta_R(Q,\tau))}{R^{n+1}}. \tag{3.3.13}$$

*Finally, if $\|(X,t) - (Q,\tau)\| \geq 32R$ there is a constant $C > 0$ (which might depend on $(X_0,t_0), (Q,\tau)$ but is independent of $(X,t)$) such that*

$$|W(X,t)| \leq C \tag{3.3.14}$$

*Proof.* Using the notation from Lemma 3.3.8, observe that $\varphi(X,t) \equiv \varphi_{R/4}((X,t)-(Q,\tau))$. Therefore, the continuity and boundary values of $W$ follows as in the infinite pole case. Furthermore, arguing exactly as in the proof of equation (3.3.13) (taking into account our modifications on $\varphi$) we establish an analogue to equation (3.3.8) in the finite pole setting;

$$|W(X,t)| \leq C \frac{u^{(X_0,t_0)}(A^-_{R/4}(Q,\tau))}{R} \left(\frac{\delta(X,t)}{R}\right)^{3/4}. \tag{3.3.15}$$

By assumption, $(X_0,t_0),(Q,\tau)$ and $R$ satisfy the hypothesis of Lemmas 3.2.4 and 3.2.6. As such, we may apply these lemmas to obtain the desired inequality (3.3.13).

To prove equation (3.3.14) we follow the proof of equation (3.3.4) to obtain an analogue of (3.3.10);

$$|W(X,t)| \leq C \frac{u^{(X_0,t_0)}(A^-_{R/4}(Q,\tau))}{R} \frac{R}{\|(X,t)-(Q,\tau)\|^n} \leq C. \tag{3.3.16}$$

□

**Corollary 3.3.9.** *For $(X_0,t_0) \in \Omega$, let $(Q,\tau) \in \partial\Omega, R > 0, A \geq 100$ be as in Lemma 3.3.8. For any $(Y,s) \in \Omega$, $\nabla u^{(X_0,t_0)}(-,-)$ has a non-tangential limit, $F^{(X_0,t_0)}(P,\eta)$, for $d\hat{\omega}^{(Y,s)}$-almost every $(P,\eta) \in \Delta_{R/4}(Q,\tau)$. In particular, the non-tangential limit exists for $\sigma$-almost every $(P,\eta) \in \Delta_{R/4}(Q,\tau)$. Furthermore, $F^{(X_0,t_0)}|_{\Delta_{R/4}(Q,\tau)} \in L^1(d\hat{\omega}^{(Y,s)})$ and $F^{(X_0,t_0)}|_{\Delta_{R/4}(Q,\tau)} \in L^2(d\sigma)$.*

*Proof.* Theorem 1 in (HLN04) implies that $\exists (Y,s) \in \Omega$ such that $\hat{\omega}^{(Y,s)}|_{\Delta_{\delta(X_0,t_0)/4}(Q,\tau)} \in A_\infty(\sigma|_{\Delta_R(Q,\tau)})$. Therefore, if the non-tangential limit exists $d\hat{\omega}^{(Y,s)}$-almost everywhere on $\Delta_{R/4}(Q,\tau)$ for any $(Y,s) \in \Omega$ we can conclude that it exists $\sigma$-almost everywhere on $\Delta_{R/4}(Q,\tau)$. Additionally, Lemma 3.3.3 implies that if $\nabla u^{(X_0,t_0)}(-,-)$ has a non-tangential limit on $\Delta_{R/4}(Q,\tau)$, that limit is in $L^2(d\sigma)$-integrable on $\Delta_{R/4}(Q,\tau)$ and therefore $L^1(d\hat{\omega}^{(Y,s)})$-integrable on $\Delta_{R/4}(Q,\tau)$ for any $(Y,s) \in \Omega$.

Thus it suffices to prove, for any $(Y,s) \in \Omega$, that $\nabla u^{(X_0,t_0)}(-,-)$ has a non-tangential limit $d\hat{\omega}^{(Y,s)}$-almost everywhere on $\Delta_{R/4}(Q,\tau)$. Let $\varphi, W$ be as in Lemma 3.3.8 and define, for $(X,t) \in \Omega$, $H(X,t) = \varphi(X,t)\nabla u^{(X_0,t_0)}(X,t) - W(X,t)$. Equation (3.3.14) and $W(X,t) \in C(\overline{\Omega})$ imply that $W(X,t) \in L^\infty(\Omega)$. Lemma 3.3.3 implies that $N(H)(P,\eta) \in L^1(d\hat{\omega}^{(Y,s)})$ for any $(Y,s) \in \Omega$ (as outside of $C_{R/2}(Q,\tau)$ we have $H = -W$, which is bounded). By construction, $H$ is a solution to the adjoint heat equation in $\Omega$, hence, by Lemma B.4.1, $H(X,t)$ has a non-tangential limit $d\hat{\omega}^{(Y,s)}$-almost everywhere. Finally, because $W, \varphi \in C(\overline{\Omega})$ we can conclude that $\nabla u^{(X_0,t_0)}(-,-)$ has a non-tangential limit for $d\hat{\omega}^{(Y,s)}$-almost every point in $\Delta_{R/4}(Q,\tau)$. □

As in the infinite pole case, if we assume higher regularity in $\partial\Omega$, it is easy to conclude that $\nabla u^{(X_0,t_0)}(-,-)(P,\eta) = k^{(X_0,t_0)}(P,\eta)\hat{n}(P,\eta)$ for every $(P,\eta) \in \partial\Omega$. The following lemma, proved in Appendix B.2, says that this remains true in our (low regularity) setting.

**Lemma 3.3.10.** *For $(X_0,t_0) \in \Omega$ let $(Q,\tau) \in \partial\Omega, R > 0, A \geq 100$ be as in Lemma 3.3.8. Then for $\sigma$-a.e. $(P,\eta) \in \Delta_{R/4}(Q,\tau)$ we have $F^{(X_0,t_0)}(P,\eta) = k^{(X_0,t_0)}(P,\eta)\hat{n}(P,\eta)$*

Finally, we have the integral estimate (the proof follows as in the infinite pole case and so we omit it).

**Lemma 3.3.11.** *For $(X_0,t_0) \in \Omega$ let $(Q,\tau) \in \partial\Omega, R > 0, A \geq 100$ be as in Lemma 3.3.8. Then for any $(X,t) \in \Omega$ with $\|(X,t) - (Q,\tau)\| \leq \delta R/8$*

$$|\nabla u^{(X_0,t_0)}(X,t)| \leq \int_{\Delta_{R/2}(Q,\tau)} k^{(X_0,t_0)} d\hat{\omega}^{(X,t)} + C\left(\frac{\delta(X,t)}{R}\right)^{3/4} \frac{\omega^{(X_0,t_0)}(\Delta_R(Q,\tau))}{R^{n+1}}. \tag{3.3.17}$$

*Here $C \equiv C(A) < \infty$.*

## 3.4 Ω is Vanishing Reifenberg Flat

In this section we use a blowup argument to prove Proposition 3.4.6, that $\Omega$ is vanishing Reifenberg flat, and Lemma 3.4.7, that $\lim_{r\downarrow 0}\sup_{(Q,\tau)\in K\cap\partial\Omega} \frac{\sigma(C_r(Q,\tau))}{r^{n+1}} = 1$. To do this, we invoke Theorem 3.1.10, the classification of "flat blow-ups".

We now describe the blowup process,

**Definition 3.4.1.** *Let $K$ be a compact set (in the finite pole case we require $K = \Delta_R(Q,\tau)$ where $(Q,\tau) \in \partial\Omega, R > 0$ satisfy $(X_0, t_0) \in T^+_{A,4R}(Q,\tau)$ for some $A \geq 100$), $(Q_i, \tau_i) \in K \cap \partial\Omega$ and $r_i \downarrow 0$. Then we define*

$$\Omega_i := \{(X,t) \mid (r_i X + Q_i, r_i^2 t + \tau_i) \in \Omega\} \tag{3.4.1a}$$

$$u_i(X,t) := \frac{u(r_i X + Q_i, r_i^2 t + \tau_i)}{r_i \fint_{C_{r_i}(Q_i,\tau_i)\cap\partial\Omega} h d\sigma} \tag{3.4.1b}$$

$$\omega_i(E) := \frac{\sigma(C_{r_i}(Q_i,\tau_i))}{r_i^{n+1}} \frac{\omega(\{(P,\eta) \in \Omega \mid ((P-Q_i)/r_i, (\eta-\tau_i)/r_i^2) \in E\})}{\omega(C_{r_i}(Q_i,\tau_i))} \tag{3.4.1c}$$

$$h_i(P,\eta) := \frac{h(r_i P + Q_i, r_i^2 \eta + \tau_i)}{\fint_{C_{r_i}(Q_i,\tau_i)\cap\partial\Omega} h d\sigma} \tag{3.4.1d}$$

$$\sigma_i := \sigma|_{\partial\Omega_i}. \tag{3.4.1e}$$

*Similarly we can define $u_i^{(X_0,t_0)}, \omega_i^{(X_0,t_0)}$ and $k_i^{(X_0,t_0)}$.*

**Remark 3.4.2.** *Using the uniqueness of the Green function and caloric measure it follows by a change of variables that $u_i$ is the adjoint-caloric Green's function for $\Omega_i$ with caloric measure $\omega_i$ and*

$$d\omega_i = h_i d\sigma_i.$$

*Similarly, $u_i^{(X_0,t_0)}$ is the Green function for $\Omega_i$ with a pole at $(\frac{X_0-Q_i}{r_i}, \frac{t_0-\tau_i}{r_i^2})$ with associated caloric measure $\omega_i^{(X_0,t_0)}$ and Poisson kernel $k_i^{(X_0,t_0)}$.*

We first need to show that (perhaps passing to a subsequence) the blowup process limits to a parabolic chord arc domain. In the elliptic setting this is Theorem 4.1 in (KT03). Additionally, in (Nys06b), Nyström considered a related parabolic blowup to the one above and proved similar convergence results.

**Lemma 3.4.3.** *Let $\Omega_i, u_i, h_i, \omega_i$ (or $u_i^{(X_0,t_0)}, k_i^{(X_0,t_0)}$ and $\omega_i^{(X_0,t_0)}$) be as in Definition 3.4.1. Then there exists a subsequence (which we can relabel for convenience) such that for any* *compact $K$*

$$D[K \cap \Omega_i, K \cap \Omega_\infty] \to 0 \tag{3.4.2}$$

*where $\Omega_\infty$ is a parabolic chord arc domain which is $4\delta$-Reifenberg flat.*

*Moreover there is a $u_\infty \in C(\overline{\Omega}_\infty)$ such that*

$$u_i \to u_\infty \tag{3.4.3}$$

*uniformly on compacta. Additionally, there is a Radon measure $\omega_\infty$ supported on $\partial\Omega_\infty$ such that*

$$\omega_i \rightharpoonup \omega_\infty. \tag{3.4.4}$$

*Finally, $u_\infty, \omega_\infty$ are the Green function and caloric measure with poles at infinity for $\Omega_\infty$ (i.e. they satisfy equation* (IP)*).*

*Proof.* Lemma 16 in (Nys06b) proves that $\Omega_j \to \Omega_\infty$ and that $\Omega_\infty$ is $4\delta$-Reifenberg flat. In the same paper, Lemma 17 proves that $u_j \to u_\infty, \omega_j \to \omega_\infty$ and that $u_\infty, \omega_\infty$ satisfy equation (IP). A concerned reader may point out that their blowup differs slightly from ours (as their $\Omega$ is not necessarily a chord arc domain). However, using Ahlfors regularity their argument works virtually unchanged in our setting (see also the proof of Theorem 4.1 in (KT03)).

Therefore, to finish the proof it suffices to show that $\Omega_\infty$ is a parabolic regular domain. That is, $\sigma_\infty \equiv \sigma|_{\partial\Omega_\infty}$ is Ahlfors regular and $\partial\Omega$ is uniformly parabolic rectifiable. Let us first concentrate on $\sigma_\infty$. Note that for each $t_0 \in \mathbb{R}$ we have that $(\Omega_j)_{t_0} \equiv \Omega_j \cap \{(Y,s) \mid s = t_0\}$ is $\delta$-Reifenberg flat (and thus the topological boundary coincides with measure theoretic boundary). Furthermore, we claim that $(\Omega_j)_{t_0} \to (\Omega_\infty)_{t_0}$ in the Hausdorff distance sense. This follows from the observation that, in a Reifenberg flat domain $\Omega$, the closest point on $\partial\Omega$ to $(X,t) \in \Omega$ is also at time $t$ (see Remark B.4.2).

For almost every $s_0$ we know that $\Omega_{s_0}$ is a set of locally finite perimeter in $X \in \mathbb{R}^n$ and thus $(\Omega_j)_{t_0}$ is a set of locally finite perimeter for almost every $t_0$. We claim, for those $t_0$, that $\chi_{\Omega_j}(X,t_0) \to \chi_{\Omega_\infty}(X,t_0)$ in $L^1_{\text{loc}}(\mathbb{R}^n)$. Indeed, by compactness, there exists an $E_{t_0}$ such that $\chi_{\Omega_j}(X,t_0) \to \chi_{E_{t_0}}$. That $E_{t_0} = (\Omega_\infty)_{t_0}$ is a consequence of $(\Omega_j)_{t_0} \to (\Omega_\infty)_{t_0}$ in the Hausdorff distance sense (for more details, see the bottom of page 351 in (KT03)). Hence, for almost every $t_0$, $(\Omega_\infty)_{t_0}$ is a set of locally finite perimeter. In addition, lower semicontinuity and Fatou's lemma imply

$$\begin{aligned}
\sigma_\infty(\Delta_R(P,\eta)) &= \int_{\eta-R^2}^{\eta+R^2} \mathcal{H}^{n-1}(\{(X,s) \mid (X,s)\in\partial\Omega_\infty, |X-P|\le R\})ds \\
&\le \int_{\eta-R^2}^{\eta+R^2} \liminf_{i\to\infty} \mathcal{H}^{n-1}(\{(X,s) \mid (X,s)\in\partial\Omega_i, |X-P|\le R\})ds \\
&\le \liminf_{i\to\infty} \int_{\eta-R^2}^{\eta+R^2} \mathcal{H}^{n-1}(\{(X,s) \mid (X,s)\in\partial\Omega_i, |X-P|\le R\})ds \\
&\le MR^{n+1}.
\end{aligned} \tag{3.4.5}$$

(The last inequality above follows from the fact that $\sigma$ is Ahflors regular and the definition of the blowup). The lower Ahlfors regularity is given immediately by the $\delta$-Reifenberg flatness of $\Omega_\infty$.

It remains to show that $\nu_\infty$ (defined as in (3.1.4) but with respect to $\Omega_\infty$) is a Carleson measure. Define $\gamma^{(\infty)}(Q,\tau,r) := \inf_P r^{-n-3}\int_{C_r(Q,\tau)\cap\partial\Omega_\infty} d((Y,s),P)^2 d\sigma_\infty(Y,s)$ where the infinum is taken over all $n$-planes $P$ containing a line parallel to the $t$-axis. Similarly define $\gamma^{(i)}(Q,\tau,r)$. We claim that

$$\gamma^{(\infty)}(P,\eta,r) \le \liminf_{i\to\infty} \gamma^{(i)}(P_i,\eta_i,r+\varepsilon_i), \forall (P,\eta)\in\partial\Omega_\infty, \tag{3.4.6}$$

where $(P_i,\eta_i)\in\partial\Omega_i$ is the closest point in $\partial\Omega_i$ to $(P,\eta)$ and $\varepsilon_i \downarrow 0$ is any sequence such that $\varepsilon_i \ge 2D[\partial\Omega_i \cap C_{2r}(P,\eta), \partial\Omega_\infty\cap C_{2r}(P,\eta)]$.

Let $V_i$ be a plane which achieves the infinum in $\gamma^{(i)}(P_i,\eta_i,r+\varepsilon_i)$. Passing to a subsequence, the $V_i$ converge in the Hausdorff distance to some $V_\infty$. As such, there exists $\delta_i \downarrow 0$

with $D[V_i \cap C_1(0,0), V_\infty \cap C_1(0,0)] < \delta_i$. Estimate,

$$\begin{aligned}
\gamma^{(i)}(P_i, \eta_i, r+\varepsilon_i) &= (r+\varepsilon_i)^{-n-3} \int_0^\infty 2\lambda \sigma_i(\{(Y,s) \in C_{r+\varepsilon_i}(P_i,\eta_i) \mid d((Y,s),V_i) > \lambda\}) d\lambda \\
&\geq (r+\varepsilon_i)^{-n-3} \int_0^\infty 2\lambda \sigma_i(\{(Y,s) \in C_r(P,\eta) \mid d((Y,s),V_i) > \lambda\}) d\lambda \\
&\geq (r+\varepsilon_i)^{-n-3} \int_0^\infty 2\lambda \sigma_i(\{(Y,s) \in C_r(P,\eta) \mid d((Y,s),V_\infty) > \lambda + r\delta_i\}) d\lambda \\
&\overset{\gamma=\lambda+r\delta_i}{\geq} (r+\varepsilon_i)^{-n-3} \int_{o(1)}^\infty 2(\gamma - o(1)) \sigma_i(\{(Y,s) \in C_r(P,\eta) \mid d((Y,s),V_\infty) > \gamma\}) d\gamma.
\end{aligned}$$

Take lim infs of both sides and recall, as argued above, that for all open $U$, $\sigma_\infty(U) \leq \liminf_{i\to\infty} \sigma_i(U)$. Equation (3.4.6) then follows from dominated convergence theorem, Fatou's lemma and Ahlfors regularity.

We claim, for any $\rho > 0$,

$$\int_{C_\rho(P,\eta)} \gamma^{(\infty)}(Y,s,r) d\sigma_\infty(Y,s) \leq \liminf_i \int_{C_{\rho+\varepsilon_i}(P_i,\eta_i)} \gamma^{(i)}(Y,s,r+\varepsilon_i) d\sigma_i(Y,s) =: F^i(r) \tag{3.4.7}$$

where $\rho > r > 0$, $(P,\eta) \in \partial\Omega_\infty$, the $(P_i,\eta_i)$ are as above and $\varepsilon_i \downarrow 0$ with $\varepsilon_i \geq 3D[\partial\Omega_i \cap C_{3\rho}(P,\eta), \partial\Omega_\infty \cap C_{3\rho}(P,\eta)] \to 0$. The proof of equation (3.4.7) is in the same vein as that of equation (3.4.6), and thus we will omit it. Observing that the $\|\nu_i\|_+$ are bounded uniformly in $i$, Fatou's lemma implies

$$\begin{aligned}
\nu_\infty(C_\rho(P,\eta) \times [0,\rho)) = \int_0^\rho F^\infty(r) \frac{dr}{r} &\overset{\text{eq (3.4.7)}}{\leq} \int_0^\rho \liminf_{i\to\infty} F^i(r) \frac{dr}{r} \\
&\leq \liminf_{i\to\infty} \nu_i(C_{\rho+\varepsilon_i}(P_i,\eta_i) \times [0,\rho+\varepsilon_i)) \leq C\rho^{n+1} \limsup_i \|\nu_i\|_+.
\end{aligned} \tag{3.4.8}$$

□

We now want to show a bound on $\nabla u_\infty$ (in hopes of applying Proposition 3.1.10). Here we follow (Nys12) (see, specifically, the proof of Lemma 3.3 there).

**Proposition 3.4.4.** $|\nabla u_\infty(X,t)| \leq 1$ *for all* $(X,t) \in \Omega_\infty$.

*Proof: Infinite Pole case.* Let $(X,t) \in \Omega_\infty$ and define $d((X,t), \partial\Omega_\infty) =: \delta_\infty$. There exists an $i_0 > 0$ such that if $i > i_0$ then $C_{3\delta_\infty/4}(X,t) \subset \Omega_i$. By standard parabolic regularity theory, $\nabla u_i \to \nabla u_\infty$ uniformly on $C_{\delta_\infty/2}(X,t)$.

$\Omega_i$ is a parabolic regular domain with $(0,0) \in \partial\Omega_i$ so by Lemma 3.3.7 we can write

$$|\nabla u_i(X,t)| \leq \int_{\partial\Omega_i \cap C_M(0,0)} h_i(Q,\tau) d\hat{\omega}_i^{(X,t)}(Q,\tau) + C\frac{\|(X,t)\|^{3/4}}{M^{1/2}} \tag{3.4.9}$$

as long as $M/2 \geq \max\{2\|(X,t)\|, 1\}$. For $\varepsilon > 0$, let $M \geq 2$ be such that $C\frac{\|(X,t)\|^{3/4}}{M^{1/2}} < \varepsilon/2$ and let $\varepsilon' = \varepsilon'(M, \delta_\infty, \varepsilon) > 0$ be a small constant to be chosen later. By Lemma 3.2.14 there exists an $r(\varepsilon') > 0$ such that if $(Q_i, \tau_i) \in K \cap \partial\Omega$ and $r_i M < r(\varepsilon')$ there is a set $G_i := G((Q_i,\tau_i), Mr_i) \subset C_{Mr_i}(Q_i,\tau_i) \cap \partial\Omega$ with the properties that $(1+\varepsilon')\sigma(G_i) \geq \sigma(C_{Mr_i}(Q_i,\tau_i))$ and, for all $(P,\eta) \in G_i$,

$$(1+\varepsilon')^{-1} \fint_{C_{Mr_i}(Q_i,\tau_i)} h d\sigma \leq h(P,\eta) \leq (1+\varepsilon') \fint_{C_{Mr_i}(Q_i,\tau_i)} h d\sigma.$$

Throughout we will assume that $i$ is large enough such that $Mr_i < r(\varepsilon')$.

Define $\tilde{G}_i := \{(P,\eta) \in \partial\Omega_i \mid (r_i P + Q_i, r_i^2\eta + \tau_i) \in G_i\}$, the image of $G_i$ under the blowup. Then

$$h_i(P,\eta) = \frac{h(r_iP + Q_i, r_i^2\eta + \tau_i)}{\fint_{\Delta_{r_i}(Q_i,\tau_i)} h d\sigma} \simeq_{\varepsilon'} \frac{\fint_{\Delta_{Mr_i}(Q_i,\tau_i)} h d\sigma}{\fint_{\Delta_{r_i}(Q_i,\tau_i)} h d\sigma}, \ \forall (P,\eta) \in \tilde{G}_i \tag{3.4.10}$$

where, as in (KT03), we write $a \simeq_{\varepsilon'} b$ if $\frac{1}{1+\varepsilon'} \leq \frac{a}{b} \leq (1+\varepsilon')$.

Observe

$$\begin{aligned} \fint_{\Delta_{r_i}(Q_i,\tau_i)} h d\sigma &\geq \frac{1}{\sigma(\Delta_{r_i}(Q_i,\tau_i))} \int_{G_i \cap \Delta_{r_i}(Q_i,\tau_i)} h d\sigma \\ &\geq \frac{\sigma(G_i \cap \Delta_{r_i}(Q_i,\tau_i))}{(1+\varepsilon')\sigma(\Delta_{r_i}(Q_i,\tau_i))} \fint_{\Delta_{Mr_i}(Q_i,\tau_i)} h d\sigma. \end{aligned} \tag{3.4.11}$$

Combining this with equation (3.4.10) we can conclude

$$h_i(P,\eta) \leq (1+\varepsilon')^2 \frac{\sigma(\Delta_{r_i}(Q_i,\tau_i))}{\sigma(G_i \cap \Delta_{r_i}(Q_i,\tau_i))},\ \forall (P,\eta) \in \tilde{G}_i.$$

Ahlfors regularity implies

$$\begin{aligned} \sigma(G_i \cap \Delta_{r_i}(Q_i,\tau_i)) &= \sigma(\Delta_{r_i}(Q_i,\tau_i)) - \sigma(\Delta_{r_i}(Q_i,\tau_i)\backslash G_i) \\ &\geq \sigma(\Delta_{r_i}(Q_i,\tau_i)) - \sigma(\Delta_{Mr_i}(Q_i,\tau_i)\backslash G_i) \\ &\geq \sigma(\Delta_{r_i}(Q_i,\tau_i)) - \varepsilon'\sigma(\Delta_{Mr_i}(Q_i,\tau_i)) \\ &\geq \sigma(\Delta_{r_i}(Q_i,\tau_i))(1 - CM^{n+1}\varepsilon'). \end{aligned} \tag{3.4.12}$$

Putting everything together $h_i(P,\eta) \leq (1+\varepsilon')^2(1-CM^{n+1}\varepsilon')^{-1},\ \forall (P,\eta) \in \tilde{G}_i$. Hence,

$$\int_{\tilde{G}_i} h_i d\hat{\omega}_i^{(X,t)} \leq \frac{(1+\varepsilon')^2}{(1-CM^{n+1}\varepsilon')}\hat{\omega}_i^{(X,t)}(\tilde{G}_i) \leq \frac{(1+\varepsilon')^2}{(1-CM^{n+1}\varepsilon')}, \tag{3.4.13}$$

as $\hat{\omega}_i^{(X,t)}$ is a probability measure.

Define $F_i := (C_{Mr_i}(Q_i,\tau_i) \cap \partial\Omega)\backslash G_i$ and $\tilde{F}_i$ analogously to $\tilde{G}_i$. Let $A_i \in \Omega_i$ be the backwards non-tangential point at $(0,0)$ and scale $30M$. We want to connect $A_i$ with $(X,t)$ by a Harnack chain in $\Omega_i$. Thus we need to show that that the square root of the difference in the $t$-coordinates of $A_i$ and $(X,t)$ is greater than $\frac{d(A_i,(X,t))}{100}$. This follows after observing that the $t$-coordinate of $A_i$ is $\leq -9M^2$. The Harnack inequality then tells us that there is a $C = C(n, M, \delta_\infty) > 0$ such that $d\hat{\omega}^{(X,t)} \leq Cd\hat{\omega}^{A_i}$ on $C_{2M}(0,0) \cap \partial\Omega_i$. Furthermore, $A_i \in T^-_{100,M}(0,0)$ which implies, by Theorem 1 in (HLN04), that there is a $p > 1$ such that $\hat{k}^{A_i} := \frac{d\hat{\omega}^{A_i}}{d\sigma}$ satisfies a reverse Hölder inequality with exponent $p$ and constant $C$ (as the $\Omega_i$ are uniformly parabolic regular and $\delta$-Reifenberg flat, the arguments in (HLN04) ensure that $p, C$ can be taken independent of $i$). Let $q$ be the dual exponent; then, by Hölder's inequality,

$$\int_{\tilde{F}_i} h_i d\hat{\omega}_i^{(X,t)} \leq C \int_{\tilde{F}_i} h_i d\hat{\omega}_i^{A_i} \leq C \left(\int_{\tilde{F}_i} h_i^q d\sigma_i\right)^{1/q} \left(\int_{\tilde{F}_i} (\hat{k}^{A_i})^p d\sigma_i\right)^{1/p}. \tag{3.4.14}$$

To bound the first term in the product note,

$$\int_{\tilde{F}_i} h_i^q d\sigma_i = r_i^{-n-1} \frac{\int_{F_i} h^q d\sigma}{\left(\fint_{\Delta_{r_i}(Q_i,\tau_i)} h d\sigma\right)^q}. \tag{3.4.15}$$

Per Lemma 3.2.10, $h^2 \in A_2(d\sigma)$ (as $\log(h^2) \in \mathrm{VMO}(\partial\Omega)$). Apply Hölder's inequality and then the reverse Hölder inequality with exponent 2 to obtain

$$\begin{aligned} \int_{F_i} h^q d\sigma &\leq \sigma(\Delta_{Mr_i}(Q_i,\tau_i))^{1/2} \sigma(F_i)^{1/2} \left(\fint_{\Delta_{Mr_i}(Q_i,\tau_i)} h^{2q} d\sigma\right)^{1/2} \\ &\leq C \left(\frac{\sigma(F_i)}{\Delta_{Mr_i}(Q_i,\tau_i)}\right)^{1/2} \int_{\Delta_{Mr_i}(Q_i,\tau_i)} h^q d\sigma \\ &\leq C\sqrt{\varepsilon'}\sigma(\Delta_{Mr_i}(Q_i,\tau_i)) \fint_{\Delta_{Mr_i}(Q_i,\tau_i)} h^q d\sigma, \end{aligned} \tag{3.4.16}$$

where that last inequality comes from the fact that $F_i$ is small in $\Delta_{Mr_i}(Q_i,\tau_i)$. Invoking Lemma 3.2.10 again, $h$ satisfies a reverse Hölder inequality with exponent $q$. This fact, combined with equations (3.4.15), (3.4.16), implies

$$\begin{aligned} \int_{\tilde{F}_i} h_i^q d\sigma_i \leq & C\sqrt{\varepsilon'} \frac{\sigma(\Delta_{Mr_i}(Q_i,\tau_i))}{r_i^{n+1}} \left(\frac{\fint_{\Delta_{Mr_i}(Q_i,\tau_i)} h d\sigma}{\fint_{\Delta_{r_i}(Q_i,\tau_i)} h d\sigma}\right)^q \\ \overset{\text{eq (3.4.11)}}{\leq} & C\sqrt{\varepsilon'} M^{n+1} \left(\frac{(1+\varepsilon')\sigma(\Delta_{r_i}(Q_i,\tau_i))}{\sigma(G_i \cap \Delta_{r_i}(Q_i,\tau_i))}\right)^q \\ \overset{\text{eq (3.4.12)}}{\leq} & C\sqrt{\varepsilon'} M^{n+1} (1 - CM^{n+1}\varepsilon')^{-q}. \end{aligned} \tag{3.4.17}$$

To bound the second term of the product in equation (3.4.14) we recall that $\hat{k}^{A_i}$ satisfies a reverse Hölder inequality with exponent $p$ at scale $M$,

$$\begin{aligned}\left(\int_{\tilde{F}_i}(\hat{k}^{A_i})^p d\sigma_i\right)^{1/p} \leq&\left(\int_{C_M(0,0)\cap\partial\Omega_i}(\hat{k}^{A_i})^p d\sigma_i\right)^{1/p}\\ \leq& C\sigma_i(C_M(0,0)\cap\partial\Omega_i)^{1/p}\left(\fint_{C_M(0,0)\cap\partial\Omega_i}\hat{k}^{A_i}d\sigma_i\right)\\ \leq& C\sigma_i(C_M(0,0)\cap\partial\Omega_i)^{1/p-1}\hat{\omega}^{A_i}(C_M(0,0))\\ \leq& C\left(\frac{\sigma(\Delta_{Mr_i}(Q_i,\tau_i))}{r_i^{n+1}}\right)^{1/p-1}\leq CM^{-(n+1)/q}.\end{aligned} \tag{3.4.18}$$

Together, equations (3.4.14), (3.4.17) and (3.4.18), say

$$\int_{\tilde{F}_i} h_i d\omega_i^{(X,t)} \leq CM^{-(n+1)/q}\left(C\sqrt{\varepsilon'}M^{n+1}(1-CM^{n+1}\varepsilon')^{-q}\right)^{1/q} = C\frac{(\varepsilon')^{1/(2q)}}{1-CM^{n+1}\varepsilon'}. \tag{3.4.19}$$

Having estimated the integral over $\tilde{G}_i$ in equation (3.4.13) and the integral over $\tilde{F}_i$ in equation (3.4.19) we can invoke equation (3.4.9) to conclude

$$\begin{aligned}|\nabla u_\infty(X,t)| \leq& \limsup_i \int_{\partial\Omega_i\cap C_M(0,0)} h_i(Q,\tau)d\hat{\omega}_i^{(X,t)}(Q,\tau) + C\frac{\|(X,t)\|^{3/4}}{M^{1/2}}\\ \leq&\frac{(1+\varepsilon')^2}{(1-CM^{n+1}\varepsilon')} + C\frac{(\varepsilon')^{1/(2q)}}{1-CM^{n+1}\varepsilon'}+\varepsilon/2\\ \leq& 1+\varepsilon.\end{aligned} \tag{3.4.20}$$

The last inequality follows by picking $\varepsilon' > 0$ small so that $\frac{(1+\varepsilon')^2}{(1-CM^{n+1}\varepsilon')} + C\frac{(\varepsilon')^{1/(2q)}}{1-CM^{n+1}\varepsilon'} < 1+\varepsilon/2$. □

*Proof: Finite Pole Case.* Let $(X,t) \in \Omega_\infty$ and define $d((X,t),\partial\Omega_\infty) =: \delta_\infty$. There exists an $i_0 > 0$ such that if $i > i_0$ then $C_{3\delta_\infty/4}(X,t) \subset \Omega_i$ but $C_{3\delta_\infty/2}(X,t)\cap\partial\Omega_i \neq \emptyset$. By standard parabolic regularity theory, $\nabla u_i^{(X_0,t_0)} \to \nabla u_\infty$ uniformly on $C_{\delta_\infty/2}(X,t)$. Let $(\hat{X}_i,\hat{t}_i)\in\partial\Omega_i$ be a point on $\partial\Omega_i$ closest to $(X,t)$. Note that $u_i^{(X_0,t_0)}$ is the Green function of $\Omega_i$ with a pole at $\left(\frac{X_0-Q_i}{r_i}, \frac{t_0-\tau_i}{r_i^2}\right)$. Let $M \geq \max\{4R, 100\delta_\infty\}$, be arbitrarily large to be chosen later. We want to check that $\left(\frac{X_0-Q_i}{r_i}, \frac{t_0-\tau_i}{r_i^2}\right) \in T^+_{2A,M}(\hat{X}_i, \hat{t}_i)$ for $i$ large enough (recall our assumption that $(Q_i, \tau_i) \in \Delta_R(Q, \tau)$ where $(X_0, t_0) \in T^+_{A,4R}(Q, \tau)$).

Observe

$$\frac{t_0 - \tau_i}{r_i^2} - \hat{t}_i > 4M^2 \Leftrightarrow t_0 - \tau_i - r_i^2 \hat{t}_i > 4r_i^2 M^2$$

But $t_0 - \tau_i > 0$ and $|\hat{t}_i| < C(|t| + \delta_\infty) < C$. So for $i$ large enough $\frac{t_0-\tau_i}{r_i^2} - \hat{t}_i > 4M^2$. Similarly,

$$\left|\frac{X_0 - Q_i}{r_i} - \hat{X}_i\right|^2 \leq 2A\left|\frac{t_0 - \tau_i}{r_i^2} - \hat{t}_i\right| \Leftrightarrow |X_0 - Q_i - r_i\hat{X}_i|^2 \leq 2A|t_0 - \tau_i - r_i^2\hat{t}_i|.$$

As $|X_0 - Q_i|^2 \leq \frac{3}{2}A|t_0 - \tau_i|$ we may conclude, for large $i$, $\left(\frac{X_0-Q_i}{r_i}, \frac{t_0-\tau_i}{r_i^2}\right) \in T^+_{2A,M}(\hat{X}_i, \hat{t}_i)$. Invoking Lemma 3.3.11,

$$|\nabla u_i^{(X_0,t_0)}(X,t)| \leq \int_{C_M(\hat{X}_i,\hat{t}_i)\cap\partial\Omega_i} k_i^{(X_0,t_0)} d\hat{\omega}_i^{(X,t)} + C\left(\frac{\delta_\infty}{M}\right)^{3/4} \frac{\omega_i^{(X_0,t_0)}(C_M(Q,\tau)\cap\partial\Omega_i)}{M^{n+1}}. \tag{3.4.21}$$

For any $\varepsilon > 0$, pick an $M \equiv M(\varepsilon) > 0$ large such that

$$C\left(\frac{\delta_\infty}{M}\right)^{3/4} \frac{\omega_i^{(X_0,t_0)}(C_M(Q,\tau)\cap\partial\Omega_i)}{M^{n+1}} \leq \varepsilon/2.$$

For large enough $i$,

$$\begin{gathered}(P,\eta) \in C_M(\hat{X}_i,\hat{t}_i)\cap\partial\Omega_i \Rightarrow \\ (r_iP + Q_i, r_i^2\eta + \tau_i) \in \Delta_{r_iM}(r_i\hat{X}_i + Q_i, r_i^2\hat{t}_i + \tau_i) \subset \Delta_{2r_iM}(Q_i,\tau_i) \subset \Delta_{2R}(Q,\tau).\end{gathered} \tag{3.4.22}$$

Therefore, we can apply Lemmas 3.2.11, 3.2.13 and 3.2.15 to $k^{(X_0,t_0)}$ on $\Delta_{2r_iM}(Q_i,\tau_i)$ for large enough $i$. Let $\varepsilon' \equiv \varepsilon'(M,\varepsilon) > 0$ be small and chosen later. There exists an $i_0 \in \mathbb{N}$ such that $i \geq i_0$ implies that equations (3.4.21) and (3.4.22) hold and that $2r_iM \leq r(\varepsilon')$ (where $r(\varepsilon')$ is given by Lemma 3.2.15). We may now proceed as in the infinite pole case to get the desired conclusion. □

To invoke Theorem 3.1.10 we must also show that $h_\infty \geq 1$ almost everywhere. Here we follow closely the method of (KT03).

**Lemma 3.4.5.** *Let $\Omega_\infty, u_\infty, \omega_\infty$ be as above. Then $h_\infty = \frac{d\omega_\infty}{d\sigma_\infty}$ exists and*

$$h_\infty(Q,\tau) \geq 1$$

*for $\sigma_\infty$-a.e. $(Q,\tau) \in \partial\Omega_\infty$.*

*Proof.* In Lemma 3.4.3 we prove that $\Omega_\infty$ is a $\delta$-Reifenberg flat parabolic regular domain. By Theorem 1 in (HLN04) (see Proposition B.3.5 for remarks when the pole is at infinity) $\omega_\infty \in A_\infty(d\sigma_\infty)$; thus $h_\infty$ exists.

By the divergence theorem, the limiting process described in Lemma 3.4.3, and $\langle e, \vec{n}_\infty\rangle = 1 - \frac{1}{2}|\vec{n}_\infty - e|^2$ we have, for any positive $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$ and any $e \in \mathbb{S}^{n-1}$,

$$\begin{aligned}\int_{\partial\Omega_i} \varphi d\sigma_i &\geq \int_{\partial\Omega_i} \varphi \langle e, \vec{n}_i\rangle\, d\sigma_i = -\int_{\Omega_i} \operatorname{div}(\varphi e) dX dt \\ &\overset{i\to\infty}{\to} -\int_{\Omega_\infty} \operatorname{div}(\varphi e) dX dt = \int_{\partial\Omega_\infty} \varphi \langle e, \vec{n}_\infty\rangle\, d\sigma_\infty \\ &\geq \int_{\partial\Omega_\infty} \varphi d\sigma_\infty - \frac{1}{2}\int_{\partial\Omega_\infty} \varphi|\vec{n}_\infty - e|^2 d\sigma_\infty.\end{aligned} \tag{3.4.23}$$

We claim, for any positive $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$,

$$\int_{\partial\Omega_\infty} \varphi h_\infty d\sigma_\infty \geq \limsup_{i\to\infty} \int_{\partial\Omega_i} \varphi d\sigma_i. \tag{3.4.24}$$

If our claim is true then $\int_{\partial\Omega_\infty} \varphi h_\infty d\sigma_\infty \geq \int_{\partial\Omega_\infty} \varphi d\sigma_\infty - \frac{1}{2}\int_{\partial\Omega_\infty} \varphi|\vec{n}_\infty - e|^2 d\sigma_\infty$. For $(Q_0,\tau_0) \in \partial\Omega_\infty$, let $e = \vec{n}_\infty(Q_0,\tau_0)$ and $\varphi \to \chi_{C_r(Q_0,\tau_0)}$ to obtain

$$\begin{aligned}&\int_{C_r(Q_0,\tau_0)} h_\infty d\sigma_\infty \geq \sigma_\infty(C_r(Q_0,\tau_0)) - \frac{1}{2}\int_{C_r(Q_0,\tau_0)} |\vec{n}_\infty(P,\eta) - \vec{n}_\infty(Q_0,\tau_0)|^2 d\sigma_\infty(P,\eta) \Rightarrow \\ &⨍_{C_r(Q_0,\tau_0)} h_\infty d\sigma_\infty \geq 1 - \frac{1}{2}⨍_{C_r(Q_0,\tau_0)} |\vec{n}_\infty(P,\eta) - \vec{n}_\infty(Q_0,\tau_0)|^2 d\sigma_\infty(P,\eta).\end{aligned}$$

If $(Q_0,\tau_0)$ is a point of density for $\vec{n}_\infty, h_\infty$ letting $r\downarrow 0$ gives $h_\infty(Q_0,\tau_0)\geq 1$ for $\sigma_\infty$-a.e. $(Q_0,\tau_0)$.

Thus we need only to establish equation (3.4.24). Pick any positive $\varphi\in C_c^\infty(\mathbb{R}^{n+1}), \varepsilon>0$ and let $M\geq 0$ be large enough such that $\varphi\in C_c^\infty(C_M(0,0))$ and $\|\varphi\|_{L^\infty}\leq M$. Let $\varepsilon'>0$ to be choosen later (depending on $M,\varepsilon$). We will prove equation (3.4.24) for the infinite pole blowup. However, the arguments in the finite pole setting are completely unchanged; for large enough $i$ we have $C_{Mr_i}(Q_i,\tau_i)\subset C_{2R}(Q,\tau)$ and hence can apply Lemmas 3.2.11, 3.2.13 and 3.2.15 to $k^{(X_0,t_0)}$ on $\Delta_{Mr_i}(Q_i,\tau_i)$.

$\log(h)\in \mathrm{VMO}(\partial\Omega)$, so Lemma 3.2.14 gives an $R=R(\varepsilon')>0$ such that for $r_iM\leq R$ we can split $C_{Mr_i}(Q_i,\tau_i)\cap\partial\Omega$ into $G_i,F_i$ with $\sigma(\Delta_{Mr_i}(Q_i,\tau_i))\leq(1+\varepsilon')\sigma(G_i)$ and $⨍_{\Delta_{Mr_i}(Q_i,\tau_i)} hd\sigma\sim_{\varepsilon'} h(P,\eta)$ for $(P,\eta)\in G_i$. Define $\tilde{G}_i$ and $\tilde{F}_i$ as in the proof of Proposition 3.4.4.

For $(P,\eta)\in\tilde{G}_i$ we have $h_i(P,\eta)\sim_{\varepsilon'}\frac{⨍_{\Delta_{Mr_i}(Q_i,\tau_i)} hd\sigma}{⨍_{\Delta_{r_i}(Q_i,\tau_i)} hd\sigma}$, and consequently

$$\int_{\tilde{G}_i} h_i\varphi d\sigma_i\sim_{\varepsilon'}\frac{⨍_{\Delta_{Mr_i}(Q_i,\tau_i)} hd\sigma}{⨍_{\Delta_{r_i}(Q_i,\tau_i)} hd\sigma}\int_{\tilde{G}_i}\varphi d\sigma_i. \tag{3.4.25}$$

We can then estimate

$$\int_{\tilde{G}_i}\varphi d\sigma_i=\int_{\partial\Omega_i}\varphi d\sigma_i-\int_{\tilde{F}_i}\varphi d\sigma_i\geq\int_{\partial\Omega_i}\varphi d\sigma_i-C\varepsilon' M^{n+2}, \tag{3.4.26}$$

using the Ahflors regularity of $\partial\Omega_i$ and the definition of $\sigma_i,\tilde{F}_i$.

Therefore,

$$\int_{\partial\Omega_\infty} \varphi h_\infty d\sigma_\infty = \lim_{i\to\infty} \int_{\partial\Omega_i} \varphi h_i d\sigma_i \geq \limsup_{i\to\infty} \int_{\tilde{G}_i} h_i \varphi d\sigma_i$$
$$\overset{(3.4.25)}{\geq} \limsup_{i\to\infty} (1+\varepsilon')^{-1} \frac{⨍_{\Delta_{Mr_i}(Q_i,\tau_i)} h d\sigma}{⨍_{\Delta_{r_i}(Q_i,\tau_i)} h d\sigma} \int_{\tilde{G}_i} \varphi d\sigma_i$$
$$\overset{(3.4.26)}{\geq} \limsup_{i\to\infty} (1+\varepsilon')^{-1} \frac{⨍_{\Delta_{Mr_i}(Q_i,\tau_i)} h d\sigma}{⨍_{\Delta_{r_i}(Q_i,\tau_i)} h d\sigma} \left( \int_{\partial\Omega_i} \varphi d\sigma_i - CM^{n+2}\varepsilon' \right). \tag{3.4.27}$$

To estimate $\frac{⨍_{\Delta_{Mr_i}(Q_i,\tau_i)} h d\sigma}{⨍_{\Delta_{r_i}(Q_i,\tau_i)} h d\sigma}$ from below, we write

$$\begin{aligned} ⨍_{\Delta_{r_i}(Q_i,\tau_i)} h d\sigma =& \frac{1}{\sigma(\Delta_{r_i}(Q_i,\tau_i))} \left( \int_{\Delta_{r_i}(Q_i,\tau_i)\cap G_i} h d\sigma + \int_{\Delta_{r_i}(Q_i,\tau_i)\cap F_i} h d\sigma \right) \\ \leq& (1+\varepsilon') \frac{\sigma(\Delta_{r_i}(Q_i,\tau_i)\cap G_i)}{\sigma(\Delta_{r_i}(Q_i,\tau_i))} ⨍_{\Delta_{Mr_i}(Q_i,\tau_i)} h d\sigma + \frac{\omega(F_i\cap\Delta_{r_i}(Q_i,\tau_i))}{\sigma(\Delta_{r_i}(Q_i,\tau_i))} \\ \leq& (1+\varepsilon') ⨍_{\Delta_{Mr_i}(Q_i,\tau_i)} h d\sigma + \left( \frac{\sigma(F_i)}{\sigma(\Delta_{r_i}(Q_i,\tau_i))} \right)^{1/2} \left( ⨍_{\Delta_{r_i}(Q_i,\tau_i)} h^2 d\sigma \right)^{1/2} \\ \leq& (1+\varepsilon') ⨍_{\Delta_{Mr_i}(Q_i,\tau_i)} h d\sigma + (C\varepsilon' M^{n+1})^{1/2} ⨍_{\Delta_{r_i}(Q_i,\tau_i)} h d\sigma. \end{aligned} \tag{3.4.28}$$

To justify the penultimate inequality above note, for any set $E \subset \Delta_{r_i}(Q_i,\tau_i)$, Hölder's inequality gives

$$\omega(E) \leq \sigma(E)^{1/2} \left( \int_{\Delta_{r_i}(Q_i,\tau_i)} h^2 d\sigma \right)^{1/2}.$$

The last inequality in equation (3.4.28) follows from the fact that $F_i$ has small volume and $h$ satisfies a reverse Hölder inequality with exponent 2 (Lemma 3.2.10).

After some algebraic manipulation, equation (3.4.28) implies

$$\frac{⨍_{\Delta_{Mr_i}(Q_i,\tau_i)} h d\sigma}{⨍_{\Delta_{r_i}(Q_i,\tau_i)} h d\sigma} \geq (1+\varepsilon')^{-1}(1-(C\varepsilon' M^{n+1})^{1/2}).$$

Hence, in light of equation (3.4.27), and choosing $\varepsilon'$ wisely,

$$\int_{\partial\Omega_\infty} \varphi h_\infty d\sigma_\infty \geq (1-\varepsilon)\limsup_{i\to\infty}\int_{\partial\Omega_i}\varphi d\sigma_i - \varepsilon.$$

Let $\varepsilon \downarrow 0$ to prove equation (3.4.24). □

We have shown that our blowup satisfies the hypothesis of Proposition 3.1.10 (the classification of flat blowups).

**Proposition 3.4.6.** *After a rotation (which may depend on on the sequences $(Q_i, \tau_i), r_i$), $\Omega_\infty = \{x_n > 0\}, u_\infty = x_n^+$ and $d\omega_\infty = d\sigma_\infty = \mathcal{H}^{n-1}|_{\{x_n=0\}\cap\{s=t\}}dt$. In particular, $\Omega$ is vanishing Reifenberg flat.*

In the above we have shown that any pseudo-blowup (i.e. a blowup described in Definition 3.4.1) of $\Omega$ is a half space. However, we will need a slightly stronger result, namely that under this blowup $\sigma_i \rightharpoonup \sigma_\infty$. In the elliptic setting this is Theorem 4.4 in (KT03).

**Proposition 3.4.7.** *For any blowup described in Definition 3.4.1, $\sigma_i \rightharpoonup \sigma_\infty$. In particular, for any compact set $K$ (in the finite pole case $K$ is as in Definition 3.4.1), we have*

$$\lim_{r\downarrow 0}\sup_{(Q,\tau)\in K\cap\partial\Omega}\frac{\sigma(C_r(Q,\tau)\cap\partial\Omega)}{r^{n+1}} = 1. \tag{3.4.29}$$

*Proof.* Observe that $\sigma_i \rightharpoonup \sigma_\infty$ implies equation (3.4.29): let $(Q_i,\tau_i)\in K\cap\partial\Omega$ and $r_i\downarrow 0$ be such that

$$\lim_{i\to\infty}\frac{\sigma(C_{r_i}(Q_i,\tau_i)\cap\partial\Omega)}{r_i^{n+1}} = \lim_{r\downarrow 0}\sup_{(Q,\tau)\in K\cap\partial\Omega}\frac{\sigma(C_r(Q,\tau)\cap\partial\Omega)}{r^{n+1}}.$$

Blowing up along this sequence (possibly passing to subsequences) we get $\Omega_i \to \Omega_\infty$ and, by Proposition 3.4.6, we have that $\Omega_\infty = \{x_n > 0\}$ (after a rotation). Since $\sigma_\infty(\partial C_1(0,0)) = 0$, if $\sigma_i \rightharpoonup \sigma_\infty$ we have $\lim_i \sigma_i(C_1(0,0)) = \sigma_\infty(C_1(0,0)) = 1$ (recall our normalization from the introduction). By definition, $\sigma_i(C_1(0,0)) = \frac{1}{r_i^{n+1}}\sigma(C_{r_i}(Q_i,\tau_i))$ which implies equation (3.4.29).

Proposition 3.4.6 proved that $\sigma_\infty = \omega_\infty$ so to show $\sigma_i \rightharpoonup \sigma_\infty$ it suffices to prove, for any positive $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$,

$$\lim_{i\to\infty} \int_{\partial\Omega_i} \varphi d\sigma_i = \int_{\partial\Omega} \varphi d\omega_\infty.$$

Equation (3.4.24) says that the right hand side is larger than the left. Hence, it is enough to show

$$\liminf_{i\to\infty} \int_{\partial\Omega_i} \varphi d\sigma_i \geq \int_{\partial\Omega_\infty} \varphi d\omega_\infty.$$

We will work in the infinite pole setting. The finite pole setting follows similarly (i.e. for large $i$ we may assume $C_{Mr_i}(Q_i,\tau_i) \subset C_R(Q,\tau)$, where $(X_0,t_0) \in T^+_{A,2R}(Q,\tau)$, and then adapt the arguments below).

Keeping the notation from the proof of Lemma 3.4.5, it is true that, for large $i$,

$$\begin{aligned}\int_{\partial\Omega_i} \varphi d\sigma_i \geq \int_{\tilde{G}_i} \varphi d\sigma_i &\overset{(3.4.25)}{\geq} (1+\varepsilon')^{-1} \frac{\fint_{\Delta_{r_i}(Q_i,\tau_i)} h d\sigma}{\fint_{\Delta_{Mr_i}(Q_i,\tau_i)} h d\sigma} \int_{\tilde{G}_i} h_i \varphi d\sigma_i \\ &= (1+\varepsilon')^{-1} \frac{\fint_{\Delta_{r_i}(Q_i,\tau_i)} h d\sigma}{\fint_{\Delta_{Mr_i}(Q_i,\tau_i)} h d\sigma} \left(\int_{\partial\Omega_i} \varphi d\omega_i - \int_{\tilde{F}_i} h_i \varphi d\sigma_i\right) \\ &\overset{(3.4.11)+(3.4.12)}{\geq} (1+\varepsilon')^{-2}(1-CM^{n+1}\varepsilon') \left(\int_{\partial\Omega_i} \varphi d\omega_i - \int_{\tilde{F}_i} h_i \varphi d\sigma_i\right).\end{aligned} \quad (3.4.30)$$

We need to bound from above the integral of $h_i\varphi$ on $\tilde{F}_i$,

$$\begin{aligned}\int_{\tilde{F}_i} h_i \varphi d\sigma_i \leq& M\omega_i(\tilde{F}_i) = M\frac{\sigma(\Delta_{r_i}(Q_i,\tau_i))}{r_i^{n+1}} \frac{\omega(F_i)}{\omega(\Delta_{r_i}(Q_i,\tau_i))} \\ \overset{\text{Hölder's Inequality}}{\leq}& M\frac{\sigma(\Delta_{r_i}(Q_i,\tau_i))\sigma(\Delta_{Mr_i}(Q_i,\tau_i))^{1/2}}{\omega(\Delta_{r_i}(Q_i,\tau_i))r_i^{n+1}} \sigma(F_i)^{1/2} \left(\fint_{\Delta_{Mr_i}(Q_i,\tau_i)} h^2 d\sigma\right)^{1/2} \\ \overset{h\in A_2(d\sigma)}{\leq}& CM\frac{\sigma(\Delta_{r_i}(Q_i,\tau_i))\sigma(\Delta_{Mr_i}(Q_i,\tau_i))^{1/2}}{\omega(\Delta_{r_i}(Q_i,\tau_i))r_i^{n+1}} \sigma(F_i)^{1/2} \fint_{\Delta_{Mr_i}(Q_i,\tau_i)} h d\sigma \\ \overset{\sigma(F_i)\leq C\varepsilon(r_iM)^{n+1}}{\leq}& C(\varepsilon')^{1/2} M^{n+2} \frac{\fint_{\Delta_{Mr_i}(Q_i,\tau_i)} h d\sigma}{\fint_{\Delta_{r_i}(Q_i,\tau_i)} h d\sigma} \\ \overset{(3.4.11)+(3.4.12)}{\leq}& C(\varepsilon')^{1/2} M^{n+2} (1+\varepsilon')(1-CM^{n+1}\varepsilon')^{-1}.\end{aligned} \quad (3.4.31)$$

From equations (3.4.30) and (3.4.31) we can conclude, for large $i$, that

$$\int_{\partial\Omega_i} \varphi d\sigma_i \geq (1+\varepsilon')^{-2}(1-CM^{n+1}\varepsilon') \int_{\partial\Omega_i} \varphi d\omega_i - CM^{n+2}(\varepsilon')^{1/2}(1+\varepsilon')^{-1}.$$

$\omega_i \rightharpoonup \omega_\infty$, consequently, let $i \to \infty$ and then $\varepsilon' \downarrow 0$ to obtain the desired result. □

## 3.5 The Vanishing Carleson Condition

In this section we prove the following geometric measure theory proposition to finish our proof of Theorem 3.1.9.

**Proposition 3.5.1.** *Let $\Omega$ be a parabolic uniformly rectifiable domain which is also vanishing Reifenberg flat. Furthermore, assume that*

$$\lim_{r\downarrow 0} \sup_{(Q,\tau)\in K\cap\partial\Omega} \frac{\sigma(\Delta_r(Q,\tau))}{r^{n+1}} = 1$$

*holds for all compact sets $K$. Then $\Omega$ is actually a vanishing chord arc domain.*

Propositions 3.4.6 and 3.4.7 show that the assumptions of Proposition 3.5.1 are satisfied and therefore Proposition 3.5.1 implies Theorem 3.1.9 (restricting to $K \subset\subset \{t < t_0\}$ in Proposition 3.5.1 implies Theorem 3.1.9 in the finite pole setting).

In the elliptic case, Proposition 3.5.1 is also true but the proof is substantially simpler (see the proof beginning on page 366 in (KT03)). This is due to the fact (mentioned in the introduction) that the growth of the ratio $\frac{\sigma(\Delta_r(Q,\tau))}{r^{n+1}}$ controls the oscillation of $\hat{n}$ (see, e.g. Theorem 2.1 in (KT97)). However, as we also alluded to before, the behaviour of $\frac{\sigma(\Delta_r(Q,\tau))}{r^{n+1}}$ does not give information about the Carleson measure $\nu$; see the example at the end of (HLN03) in which $\sigma(\Delta_r(Q,\tau)) \equiv r^{n+1}$ but $\nu$ is not a Carleson measure. So we cannot hope that the methods in (KT03) can be adapted to prove Proposition 3.5.1 above. We also mention that the previous example in (HLN03) shows that Proposition 3.5.1 is not true without the *a priori* assumption that $\nu$ is a Carleson measure (i.e. that the domain is parabolic uniformly rectifiable).

When $\Omega = \{(x, x_n, t) \mid x_n \geq \psi(x,t)\}$ and $\psi \in \mathrm{Lip}(1,1/2)$ with $D_t^{1/2}\psi \in \mathrm{BMO}(\mathbb{R}^{n+1})$ (see the introduction of (HLN04) for precise definitions), Nyström, in (Nys12), showed that vanishing Reifenberg flatness implies the vanishing Carleson condition. To summarize his argument, for any $r_i \downarrow 0, (Q_i, \tau_i) \in \partial\Omega \cap K$ we can write

$$r_i^{-n-1}\nu(C_{r_i}(Q_i,\tau_i) \times [0, r_i]) \lesssim \int_0^1 \int_{\{(x,t)||x|\leq 1, |t|\leq 1\}} \frac{\gamma_i((y, \psi_i(y,s), s), r)}{r} dydsdr, \quad (3.5.1)$$

where $\psi_i(x,t) := \frac{\psi(r_i x + q_i, r_i^2 t + \tau_i)}{r_i}$ and $\gamma_i$ is defined as in equation (3.1.4) but with respect to the graph of $\psi_i$. By vanishing Reifenberg flatness, $\gamma_i(-,-,r) \downarrow 0$ pointwise and the initial assumptions on $\psi$ imply that $\{\gamma_i/r\}$ is uniformly integrable. Hence, we can apply the dominated convergence theorem to get the desired result.

The argument above relies on the fact that, for a graph domain, $\sigma_i = \sqrt{1 + \nabla\psi_i} dyds \lesssim dyds$, where the implicit constant in $\lesssim$ is independent of $i$. In general, $\Omega$ need not be a graph domain and, although $\sigma_i \rightharpoonup \sigma$ and $\gamma_i \to 0$ pointwise, we cannot, *a priori*, control the integral of $\gamma_i d\sigma_i$. Instead, for each $i$, we will approximate $\Omega$, near $(Q_i, \tau_i)$ and at scale $r_i$, by a graph domain and then adapt the preceeding argument. Our first step is to approximate $\Omega_i$ by graphs whose $\mathrm{Lip}(1,1/2)$ and Carleson measure norms are bounded independently of $i$. The proof follows closely that of Theorem 1 in (HLN03), which shows that parabolic chord arc domains contain big pieces of graphs of $f \in \mathrm{Lip}(1,1/2)$ with $D_t^{1/2} f \in \mathrm{BMO}$. However, we don't need to bound the BMO norm of $D_t^{1/2}\psi_i$ so the quantities we focus on are different.

**Lemma 3.5.2.** *Let $\Omega$ satisfy the conditions of Proposition 3.5.1. Also let $r_i \downarrow 0$ and $(Q_i, \tau_i) \in K \cap \partial\Omega$. Then, for every $\varepsilon > 0$, there exists an $i_0 \equiv i_0(\varepsilon, K) > 1$ where $i > i_0$ implies the existence of a $\psi_i \in \mathrm{Lip}(1,1/2)(\mathbb{R}^{n-1} \times \mathbb{R})$ such that:*

1. $\sup_i \|\psi_i\|_{\mathrm{Lip}(1,1/2)} \leq C \equiv C(n, \varepsilon) < \infty$.

2. *Let $P_i \equiv P((Q_i, \tau_i), r_i)$ be the plane which best approximates $\Omega$ at scale $r_i$ around*

*$(Q_i, \tau_i)$. Define $\tilde{\Omega}_i$ be the domain above the graph of $\psi_i$ over $P((Q_i, \tau_i), r_i)$. Then*

$$\sigma(\Delta_{r_i}(Q_i, \tau_i) \backslash \partial\tilde{\Omega}_i) < \varepsilon r_i^{n+1} \tag{3.5.2}$$

3. *$D[C_{r_i}(Q_i, \tau_i) \cap \partial\tilde{\Omega}_i, P_i \cap C_{r_i}(Q_i, \tau_i)] \leq C(n) D[C_{r_i}(Q_i, \tau_i) \cap \partial\Omega, C_{r_i}(Q_i, \tau_i) \cap P_i]$.*

4. *If $\tilde{\nu}_i$ is the Carleson measure defined as in equation (3.1.4) but with respect to $\tilde{\Omega}_i$ then*

$$\tilde{\nu}_i(C_{r_i}(Q_i, \tau_i) \times [0, r_i]) \leq K(n, \varepsilon, \|\nu\|) r_i^{n+1} \tag{3.5.3}$$

*Proof.* Let $\varepsilon > 0$ and $r_i \downarrow 0, (Q_i, \tau_i) \in K \cap \partial\Omega$. By the condition on $\sigma$ there exists an $i_1 > 0$ such that $(1-\varepsilon^2)\rho^{n+1} < \sigma(\Delta_\rho(P, \eta)) < (1+\varepsilon^2)\rho^{n+1}$ for all $\rho < 2r_{i_1}, (P, \eta) \in \partial\Omega \cap K$. There is also an $i_2$ such $\rho < r_{i_2}$ implies that $\Delta_\rho(P, \eta)$ is contained in a $\varepsilon^2\rho$ neighborhood of some $n$-plane which contains a line parallel to the $t$-axis. Let $i_0(\varepsilon) = \max\{i_1, i_2\}$ and $i > i_0$.

Henceforth, we will work at scale $r_i$ and so, for ease of notation, let $r_i \equiv R, (Q_i, \tau_i) \equiv (0, 0)$ and $P \equiv P((0, 0), R) \equiv \{x_n = 0\}$. If $\tilde{D} \equiv \frac{1}{R} D[C_R(0, 0) \cap P, C_R(0, 0) \cap \partial\Omega]$, then, by assumption, $\tilde{D} \leq \varepsilon^2$. Let $p : \mathbb{R}^{n+1} \to P$ be the orthogonal projection, i.e. $p(Y, s) := (y, 0, s)$. Fix a $\theta \in (0, 1)$ to be choosen later (depending on $n, \varepsilon$) and define

$$E = \{(P, \eta) \in \Delta_R(0, 0) \mid \exists \rho < R, \text{ s.t., } \mathcal{H}^n(p(\Delta_\rho(P, \eta))) \leq \theta\rho^{n+1}\}. \tag{3.5.4}$$

The Vitali covering lemma gives $(P_i, \eta_i) \in E$, such that $E \subset \bigcup_i \Delta_{\rho_i}(P_i, \eta_i) \subset \Delta_{2R}(0, 0)$, the $C_{\rho_i/5}(P_i, \eta_i)$ are pairwise disjoint and $\mathcal{H}^n(p(\Delta_{\rho_i}(P_i, \eta_i))) \leq \theta\rho_i^{n+1}$. Then

$$\mathcal{H}^n(p(E)) \leq \sum_i \mathcal{H}^n(p(\Delta_{\rho_i}(P_i, \eta_i))) \leq \theta \sum_i \rho_i^{n+1}$$

$$\leq 5^{n+1}(1-\varepsilon^2)^{-1}\theta \sum_i \sigma(\Delta_{\rho_i/5}(P_i, \eta_i)) \leq 10^{n+1}(1-\varepsilon^2)^{-1}(1+\varepsilon^2)\theta R^{n+1}.$$

If $F := \Delta_R(0, 0) \backslash E$ then $\sigma(F) \geq \mathcal{H}^n(p(F)) \geq \mathcal{H}^n(p(\Delta_R(0, 0)) \backslash p(E)) \geq (1-\varepsilon^2)R^{n+1} -$

$\sigma(p(E))$ by Reifenberg flatness. This implies $\sigma(E) \leq (1+\varepsilon^2)R^{n+1} - \sigma(F) \leq 2\varepsilon^2 R^{n+1} + 10^{n+1}(1-\varepsilon^2)^{-1}(1+\varepsilon^2)\theta R^{n+1}$. So if $\theta = 10^{-(n+1)}\varepsilon/100$ then $\sigma(E) \leq \varepsilon R^{n+1}/2$ (as long is $\varepsilon > 0$ is sufficiently small).

We want to show that $F$ is the graph of a Lipschitz function over $P$. Namely, that if $(Y,s),(Z,t) \in F$ then $\|(Y,s)-(Z,t)\| \leq C(|y-z|+|s-t|^{1/2})$. Let $\rho := 2(|y-z|+|s-t|^{1/2})$ and note that if $\rho \geq R$ then the flatness of $\Omega$ at scale $R$ implies the desired estimate. Write $Z = Z' + Z''$ and $Y = Y' + Y''$ where $(Y',s),(Z',t)$ are the projections of $(Y,s)$ and $(Z,t)$ on $P((Y,s),\rho)$. By vanishing Reifenberg flatness $|Y''|,|Z''| \leq \varepsilon^2\rho$. We can write

$$Y'-Z'-((Y'-Z')\cdot e_n)e_n = p(Y'-Z') \Rightarrow |Y'-Z'| \leq \frac{|p(Z'-Y')|}{\min_{\hat{v}\in P((Y,s),\rho)} |\hat{v} - (\hat{v}\cdot e_n)e_n|}. \quad (3.5.5)$$

Define $\gamma = \min_{\hat{v}\in P((Y,s),\rho)} |\hat{v} - (\hat{v}\cdot e_n)e_n|$. Combine the above estimates to obtain

$$\|(Y,s)-(Z,t)\| \leq |s-t|^{1/2} + |Y''-Z''| + |Y'-Z'| \leq \rho + 2\varepsilon^2\rho + \frac{|p(Z'-Y')|}{\gamma} \leq (1+2\varepsilon^2+\gamma^{-1})\rho.$$

It remains only to bound $\gamma$ from below. As $p(C_\rho(Y,s)\cap P((Y,s),\rho))$ is a convex body in $P$, equation (3.5.5) implies $\mathcal{H}^n(p(C_\rho(Y,s)\cap P((Y,s),\rho))) \leq c\gamma\rho^{n+1}$ for some constant $c$ (depending only on dimension). As $C_\rho(Y,s)\cap\partial\Omega$ is well approximated by $C_\rho(Y,s)\cap P((Y,s),\rho)$ it must be the case that $\mathcal{H}^n(p(C_\rho(Y,s)\cap\partial\Omega)) \leq c(\gamma+\varepsilon^2)\rho^{n+1}$. As $(Y,s)\in F$ we know $(c\gamma+\varepsilon^2) \geq \theta = c(n)\varepsilon \Rightarrow \gamma \geq \tilde{c}\varepsilon$. In particular we have shown that for $(Y,s),(Z,t)\in F$ that

$$\|(Y,s)-(Z,t)\| \leq C(n,\varepsilon)(|y-z|+|s-t|^{1/2}). \quad (3.5.6)$$

In order to eventually get the bound, (3.5.3), on the Carleson norm we need to shrink $F$ slightly, to $F_1$. $\Omega$ is a parabolic regular domain so if

$$f(P,\eta) := \int_0^{2R} \gamma(P,\eta,r)r^{-1}dr,\ (P,\eta)\in C_R(0,0)$$

then

$$\int_{C_{2R}(0,0)} f(P,\eta)d\sigma(P,\eta) \leq \|\nu\|(2R)^{n+1}.$$

Hence, by Markov's inequality,

$$\sigma(\{(P,\eta) \in C_{2R}(0,0) \mid f(P,\eta) \geq (2\varepsilon^{-1})^{n+1}\|\nu\|\}) \leq \varepsilon^{n+1}R^{n+1}.$$

Let $F_1 = F\backslash\{(P,\eta) \in C_{2R}(0,0) \mid f(P,\eta) \geq (2\varepsilon^{-1})^{n+1}\|\nu\|\}$. It is clear that equation (3.5.6) holds for $(Y,s),(Z,t) \in F_1$ and that

$$f(P,\eta) \leq 2^{n+1}\varepsilon^{-(n+1)}\|\nu\|, \forall (P,\eta) \in F_1. \tag{3.5.7}$$

Finally, we have the estimate

$$\sigma(C_R(0,0) \cap \partial\Omega\backslash F_1) \leq \varepsilon R^{n+1}. \tag{3.5.8}$$

At this point we are ready to construct $\psi$ using a Whitney decomposition of $\{x_n = 0\}$. Let $\psi^*$ be such that if $(y,0,s) \in p(F_1)$ then $(y,\psi^*(y,s),s) \in F_1$. Let $Q_i := Q_{\rho_i}(\hat{x}_i,\hat{t}_i) \subset \{x_n = 0\}$ be such that

1. $\{x_n = 0\}\backslash p(F_1) = \bigcup \overline{Q}_i$.
2. Each $Q_i$ is centered at $(\hat{x}_i,\hat{t}_i)$ with side length $2\rho_i$ in the spacial directions and $2\rho_i^2$ in the time direction.
3. $Q_i \cap Q_j = \emptyset$ for all $i \neq j$
4. $10^{-10n}d(Q_i,p(F_1)) \leq \rho_i \leq 10^{-8n}d(Q_i,p(F_1))$.

Then let $v_i$ be a partition of unity subordinate to $Q_i$. Namely,

(I) $\sum_i v_i \equiv 1$ on $\mathbb{R}^n\backslash p(F_1)$.

(II) $v_i \equiv 1$ on $\frac{1}{2}Q_i$ and $v_i$ is supported on the double of $Q_i$.

(III) $v_i \in C^\infty(\mathbb{R}^n)$ and $\rho_i^\ell |\partial_x^\ell v_i| + \rho_i^{2\ell}|\partial_t^\ell v_i| \leq c(\ell, n)$ for $\ell = 1, 2, ...$

For each $i$ there exists $(x_i, t_i) \in p(F_1)$ such that

$$d_i := d(p(F_1), Q_i) = d((x_i, t_i), Q_i).$$

Finally let $\Lambda = \{i \mid \overline{Q}_i \cap C_{2R}(0,0) \neq \emptyset\}$ and define

$$\psi(y,s) := \begin{cases} \psi^*(y,s), \ (y,s) \in p(F_1) \\ \sum_{i \in \Lambda}(\psi^*(x_i,t_i) + \tilde{D}d_i)v_i(y,s), \ (y,s) \in \mathbb{R}^n \backslash p(F_1) \end{cases} \tag{3.5.9}$$

where, as before, $\tilde{D} = \frac{1}{R}D[C_R(0,0) \cap P, C_R(0,0) \cap \partial\Omega] \leq \varepsilon^2$.

Let $\widetilde{\Omega}$ be the graph of $\psi$ over $\{x_n = 0\}$ and recall the conditions we want $\psi$ and $\widetilde{\Omega}$ to satisfy. Condition (2) is a consequence of equation (3.5.8). Condition (3) follows as $|\psi| \leq C(n)\tilde{D}$.

It remains to show Condition (1): $|\psi(y,s) - \psi(z,t)| \leq C(n, M, \varepsilon)(|y-z| + |s-t|^{1/2})$. Equation (3.5.6) says this is true when $(y,s), (z,t) \in p(F_1)$. When $(y,s) \in p(F_1)$ and $(z,t) \notin p(F_1)$ we can estimate

$$|\psi(y,s) - \psi(z,t)| \leq \sum_{\{i \in \Lambda | (z,t) \in 2Q_i\}} v_i(z,t)|\psi^*(y,s) - \psi^*(x_i,t_i)| + C(n)\varepsilon^2 d((z,t), p(F_1))$$

as $v_i(z,t) \neq 0$ implies $d_i \leq 10d((z,t), p(F_1))$. Apply the triangle inequality to conclude

$$\begin{aligned}
|\psi(y,s) - \psi(z,t)| &\leq C(n)\varepsilon^2 \left(|y-z| + |s-t|^{1/2}\right) + \sum_{\{i \in \Lambda | (z,t) \in 2Q_i\}} |y - x_i| + |s - t_i|^{1/2} \\
&\leq C(n,\varepsilon) \left(|y-z| + |s-t|^{1/2}\right) + \sum_{\{i \in \Lambda | (z,t) \in 2Q_i\}} |z - x_i| + |t - t_i|^{1/2} \\
&\leq C(n,\varepsilon) \left(|y-z| + |s-t|^{1/2} + \sum_{\{i \in \Lambda | (z,t) \in 2Q_i\}} d_i\right) \\
&\leq C(n,\varepsilon)|y-z| + |s-t|^{1/2}.
\end{aligned} \tag{3.5.10}$$

In the above, we used that $|\{i \in \Lambda \mid (z,t) \in 2Q_i\}| \leq C$ and, if $(z,t) \in Q_i$ that $d_i \leq C(n)d((z,t), p(F_1))$. From now on we write, $a \lesssim b$ if there is a constant $C$, (which can depend on $\varepsilon$, the dimension and the parabolic uniform regularity constants of $\Omega$) such that $a \leq Cb$.

The last case is if $(y,s), (z,t) \notin p(F_1)$. When $\max\{d((y,s), p(F_1)), d((z,t), p(F_1))\} \leq \|(y,s) - (z,t)\|$ estimate $|\psi(y,s) - \psi(z,t)| \leq |\psi(y,s) - \psi(\tilde{y}, \tilde{s})| + |\psi(\tilde{y}, \tilde{s}) - \psi(\tilde{z}, \tilde{t})| + |\psi(\tilde{z}, \tilde{t}) - \psi(z,t)|$ where $(\tilde{y}, \tilde{s})$ is the closest point in $p(F_1)$ to $(y,s)$ and similarly $(\tilde{z}, \tilde{t})$. The Lipschitz bound is then a trivial consequence of the fact that $d((\tilde{y}, \tilde{s}), (\tilde{z}, \tilde{t})) \leq d((y,s), p(F_1)) + d((z,t), p(F_1)) + \|(y,s) - (z,t)\| \leq 3\|(y,s) - (z,t)\|$ and the above analysis.

Now assume $\min\{d((y,s), p(F_1)), d((z,t), p(F_1))\} \geq \|(y,s) - (z,t)\|$. Recall that if $(y,s) \in 2Q_i$ then $d_i \leq C(n)d((y,s), p(F_1)) \leq \|(y,s) - (z,t)\|$. Similarly if $i,j$ are such that $(y,s) \in$

$2Q_i$ and $(z,t) \in 2Q_j$ then $\|(x_i,t_i)-(x_j,t_j)\| \leq C(n)\|(y,s)-(z,t)\|$. Then we can write

$$\begin{aligned}|\psi(y,s)-\psi(z,t)| = &\left|\sum_{i\in\Lambda}(\psi^*(x_i,t_i)+\tilde{D}d_i)v_i(y,s) - \sum_{j\in\Lambda}(\psi^*(x_j,t_j)+\tilde{D}d_j)v_j(z,t)\right| \\ &\leq \sum_{i,j\in\Lambda} |\psi^*(x_i,t_i)-\psi^*(x_j,t_j)| + \tilde{D}|d_i-d_j|)v_i(y,s)v_j(z,t) \\ &\leq \sum_{\{i,j\in\Lambda|(y,s)\in 2Q_i,(z,t)\in 2Q_j\}} C(n,\varepsilon)\|(x_i,t_i)-(x_j,t_j)\| + C(n)\|(y,s)-(z,t)\| \\ &\leq C(n,\varepsilon)\|(y,s)-(z,t)\|.\end{aligned} \tag{3.5.11}$$

In the above we use that $|\{i,j\in\Lambda \mid (y,s)\in 2Q_i, (z,t)\in 2Q_j\}| \leq C(n)$ and equation (3.5.6).

Finally, we may assume, without loss of generality, that $d((y,s),p(F_1)) \leq \|(y,s)-(z,t)\| \leq d((z,t),p(F_1))$. Then

$$|\psi(y,s)-\psi(z,t)| \leq \sum_{i\in\Lambda}((\psi^*(x_i,t_i)-\psi(y,s))+\tilde{D}d_i)|v_i(y,s)-v_i(z,t)| \tag{3.5.12}$$

as $\sum\psi(y,s)(v_i(y,s)-v_i(z,t)) = 0$. Arguing as in equation (3.5.10), $|\psi^*(x_i,t_i)-\psi(y,s)| \leq C(n,\varepsilon)\|(y,s)-(x_i,t_i)\|$. As before, if $(y,s)\in 4Q_i$ then $\|(y,s)-(x_i,t_i)\| \leq \tilde{C}(n)d_i \leq c(n)d((y,s),p(F_1)) \leq c(n)\|(y,s)-(z,t)\|$. If $(y,s)\notin 4Q_i$ then we may assume $(z,t)\in 2Q_i$ (or else $v_i(y,s)-v_i(z,t)=0$). So $\|(y,s)-(x_i,t_i)\| \leq \|(y,s)-(z,t)\| + \|(z,t)-(x_i,t_i)\| \leq \|(y,s)-(z,t)\| + c(n)\rho_i \leq \tilde{c}(n)\|(y,s)-(z,t)\|$ (as $\|(y,s)-(z,t)\| \geq c'(n)\rho_i$). Either way, $\|(y,s)-(x_i,t_i)\|, d_i \leq C(n)\|(y,s)-(z,t)\|$. Hence,

$$|\psi(y,s)-\psi(z,t)| \leq C(n,\varepsilon)\|(y,s)-(z,t)\|\sum_{i\in\Lambda}|v_i(y,s)-v_i(z,t)| \leq C(n,\varepsilon)\|(y,s)-(z,t)\|.$$

It remains only to estimate the Carleson norm of $\tilde{\nu}$. Our first claim in this direction is that if $(Y,s)\in\partial\tilde{\Omega}\cap C_{2R}(0,0)$ then

$$d((Y,s),\partial\Omega) \leq C(n,M,\varepsilon)d((y,s),p(F_1)). \tag{3.5.13}$$

Indeed, if $(\tilde{y}, \tilde{s}) \in p(F_1)$ be the point in $p(F_1)$ closest to $(y, s)$ then,

$$d((Y,s), \partial\Omega) \leq d((y,s), p(F_1)) + |\psi(y,s) - \psi(\tilde{y}, \tilde{s})| \leq C(n, M, \varepsilon) d((y,s), p(F_1))$$

by the boundedness of $\psi$'s Lip$(1, 1/2)$ norm.

Define $\Gamma_i$ to be the graph of $\psi$ over $Q_i$, and, for $r > 0, (X,t) \in F_1$, define $\xi(X,t,r) := \{i \mid \overline{\Gamma}_i \cap C_r(X,t) \neq \emptyset\}$. By equation (3.5.13) and standard covering theory there are constants $k \equiv k(n, M, \varepsilon), \tilde{k} \equiv \tilde{k}(n, M, \varepsilon)$ such that $\overline{\Gamma}_i \subset \bigcup_j C_{kd_i}(Z_{i,j}, \tau_{i,j}) \subset C_{\tilde{k}r}(X,t)$ where $(Z_{i,j}, \tau_{i,j}) \in \partial\Omega$ and the $C_{kd_i/5}(Z_{i,j}, \tau_{i,j})$ is disjoint from $C_{kd_i/5}(Z_{i,\ell}, \tau_{i,\ell})$ for $j \neq \ell$. For any $(Z, \tau) \in \Gamma_i \cap C_{kd_i}(Z_{i,j}, \tau_{i,j})$ and any $n$-plane $\hat{P}$ containing a line parallel to the $t$-axis we have

$$d((Z,\tau), \hat{P})^2 \leq C(n) \left( \min_{(Y,s) \in C_{kd_i}(Z_{i,j}, \tau_{i,j}) \cap \partial\Omega} d((Y,s), \hat{P})^2 + k^2 d_i^2 \right). \tag{3.5.14}$$

Define $\tilde{\gamma}$ as in equation (3.1.3) but with respect to $\tilde{\Omega}$. For any $(X,t) \in F_1$, equation (3.5.14) gives:

$$\tilde{\gamma}(X,t,r) \leq \frac{1}{r^{n+3}} \left( \int_{C_r(X,t) \cap F_1} d((Z,\tau), P)^2 d\sigma + \sum_{\substack{i \in \xi \\ j}} \int_{C_{kd_i}(Z_{i,j}, \tau_{i,j}) \cap \partial\tilde{\Omega}} d((Z,\tau), P)^2 d\tilde{\sigma} \right)$$

$$\overset{Eq.(3.5.14)}{\leq} \gamma(X,t,r) + C(n,k,\varepsilon) \sum_{i \in \xi} \left(\frac{d_i}{r}\right)^{n+3} + \frac{C(n)}{r^{n+3}} \sum_{i \in \xi} \sum_j \int_{\Delta_{kd_i/5}(Z_{i,j}, \tau_{i,j})} d((Z,\tau), P)^2 d\sigma.$$

As $Q_i$ can be adjacent to at most $c(n)$ many other $Q_k$ we can be sure that $C_{kd_i/5}(Z_{i,j}, \tau_{i,j})$ intersects at most $\tilde{c}(n)$ other $C_{kd_\ell/5}(Z_{\ell,j}, \tau_{\ell,j})$. Additionally, the $C_{kd_i/5}(Z_{i,j}, \tau_{i,j}) \subset C_{\tilde{k}r}(X,t)$ for all $i, j$. Hence, we can control $\tilde{\gamma}$ on $F_1$:

$$\tilde{\gamma}(X,t,r) \leq c(M,n,\varepsilon) \left( \sum_{i \in \xi} (d_i/r)^{n+3} + \gamma(X,t,\tilde{k}r) \right), \ \forall (X,t) \in F_1. \tag{3.5.15}$$

Integrating equation (3.5.15) over $F_1$ and then in $r$ from $[0, R]$, allows us to conclude

$$\tilde{\nu}(F_1 \times [0, R]) \leq \|\nu\|(\tilde{k}R)^{n+1} + \int_{(x,t)\in p(F_1)} \int_0^R r^{-1} \sum_{i \in \xi(X,t,r)} \left(\frac{d_i}{r}\right)^{n+3} dr dx dt. \quad (3.5.16)$$

Note that $i \in \xi(X, t, r)$ implies that $Q_i \cap C_r(X, t) \neq \emptyset \Rightarrow r \geq d((x, t), Q_i) \geq d_i$. Therefore,

$$\int_{(x,t)\in p(F_1)} \int_0^R r^{-1} \sum_{i\in\xi(X,t,r)} \left(\frac{d_i}{r}\right)^{n+3} dr dx dt \leq \int_{(x,t)\in p(F_1)} \sum_{i\in\xi(X,t,2R)} 1 dx dt \leq C(n)R^{n+1},$$

as $F_1 \subset C_R(0, 0)$. Putting this together with equation (3.5.16) gives us

$$\tilde{\nu}(F_1 \times [0, R]) \leq C(\|\nu\|, n, \varepsilon)R^{n+1} \quad (3.5.17)$$

If $(x, t) \in Q_i$ (i.e. $(X, t) \subset C_R(0, 0) \cap \partial\tilde{\Omega}\backslash F_1$) then approximation by affine functions and a Taylor series expansion yields

$$\tilde{\gamma}(X, t, r) \leq c(n, M, \varepsilon)r^2 d_i^{-2},\ (x, t) \in Q_i\ \forall r \leq 8d_i, \quad (3.5.18)$$

(see (HLN03) pp 367, for more details). When $r \geq 8d_i$ we can lazily estimate $\tilde{\gamma}(X, t, r) \lesssim \tilde{\gamma}(X_i, t_i, \tilde{k}r)$ where $(X_i, t_i) = (x_i, \psi(x_i, t_i), t_i)$ and $(x_i, t_i) \in p(F_1)$ such that $d(p(F_1), Q_i) = d((x_i, t_i), Q_i)$ (as in the definition of $\psi$). This is because $C_r(X, t) \subset C_{\tilde{k}r}(X_i, t_i)$ (where $\tilde{k}$ is as above). Hence,

$$\begin{aligned} \int_{8d_i}^R \tilde{\gamma}(X, t, r)r^{-1} dr &\lesssim \int_{8d_i}^R \tilde{\gamma}(X_i, t_i, \tilde{k}r)r^{-1} dr \\ &\overset{Eq.(3.5.15)}{\lesssim} \int_{8d_i}^R \gamma(X_i, t_i, \tilde{k}^2 r)r^{-1} dr + \int_{8d_i}^R r^{-1} \sum_{j\in\xi(X_i,t_i,r)} (d_j/r)^{n+3} dr,\ \forall (X, t) \in \Gamma_i. \end{aligned} \quad (3.5.19)$$

Note that $(X_i, t_i) \in F_1$, thus $\int_0^R \gamma(X_i, t_i, \tilde{k}^2 r)r^{-1} dr \lesssim 2^{n+1}\varepsilon^{-n-1}\|\nu\|$. As before, $j \in$

$\xi(X_i, t_i, r) \Rightarrow r \geq d_j$. So we can bound

$$\int_{8d_i}^{R} r^{-1} \sum_{j \in \xi(X_i,t_i,r)} (d_j/r)^{n+3} dr \leq c(n) \sum_{j \in \xi(X_i,t_i,2R)} 1.$$

If we combine these estimates and integrate over $\Gamma_i$ we get

$$\int_{\Gamma_i} \int_{8d_i}^{R} \tilde{\gamma}(X,t,r) r^{-1} dr d\tilde{\sigma} \leq C(\|\nu\|, \varepsilon, n) \left( \tilde{\sigma}(\Gamma_i) + \int_{\Gamma_i} |\{j \in \xi(X_i, t_i, 2R)\}| d\tilde{\sigma} \right).$$

Use equation (3.5.18) to bound the integral for small $r$ and sum over all $Q_i$s to obtain:

$$\tilde{\nu}((C_R(0,0) \backslash F_1) \times [0,R]) \lesssim R^{n+1} + \sum_{j \in \xi(0,0,2R)} \tilde{\sigma}(\Gamma_i) \lesssim R^{n+1}. \tag{3.5.20}$$

Combine equations (3.5.17) and (3.5.20) to obtain

$$\tilde{\nu}(C_R(0,0) \times [0,R]) \leq C(n, \varepsilon, \|\nu\|) R^{n+1}.$$

□

We now want to control the Carleson norm of $\Omega$ by that of the graph domain.

**Lemma 3.5.3.** *Let $\Omega$ be a parabolic uniformly rectifiable domain and let $\Psi$ be a* $\mathrm{Lip}(1,1/2)$ *function such that*

$$\sigma(C_1(0,0) \cap \partial\Omega \backslash \partial\tilde{\Omega}) < \varepsilon, \tag{3.5.21}$$

*where $\tilde{\Omega}$ is the domain above the graph of $\Psi$ over some $n$-plane $\overline{P}$ which contains a line parallel to the $t$-axis.*

*Then, if $\tilde{\nu}$ is defined as in* (3.1.5) *but associated $\partial\tilde{\Omega}$ we have*

$$\nu(C_{1/2}(0,0) \times [0,1/2]) \leq c(\tilde{\nu}(C_{3/4}(0,0) \times [0,3/4]) + \varepsilon^{1/2}), \tag{3.5.22}$$

*where $c > 0$ depends on $n, \|\nu\|$ and the Ahlfors regularity constant of $\Omega$.*

*Proof.* This proof follows closely the last several pages of (HLN04). For ease of notation let $\overline{P} = \{x_n = 0\}$, so that $\partial\tilde{\Omega} = \{(y, \Psi(y,s), s)\}$. Let $\tilde{\chi}$ be the characteristic function of $\Delta_1(0,0)\backslash\partial\tilde{\Omega}$. The Hardy-Littlewood maximal function of $\tilde{\chi}$ with respect to $\sigma$ is

$$M_\sigma(\tilde{\chi})(Y,s) = \sup_{\rho>0} \frac{\sigma(C_\rho(Y,s) \cap \partial\Omega \cap C_1(0,0)\backslash\partial\tilde{\Omega})}{\sigma(\partial\Omega \cap C_\rho(Y,s))},\ (Y,s) \in \partial\Omega.$$

The Hardy-Littlewood maximal theorem states

$$\sigma(\{(Y,s) \mid M_\sigma(\tilde{\chi})(Y,s) \geq \sqrt{\varepsilon}\}) \leq C(n)\frac{\|\tilde{\chi}\|_{L^1}}{\sqrt{\varepsilon}} \leq C(n)\sqrt{\varepsilon}.$$

As such, there exists a compact set $E \subset \partial\tilde{\Omega} \cap \Delta_1(0,0)$, such that $M_\sigma(\tilde{\chi})(Y,s) \leq \sqrt{\varepsilon}$ for all $(Y,s) \in E$ and

$$\sigma(\Delta_1(0,0)\backslash E) \leq \varepsilon + C(n)\sqrt{\varepsilon} < \tilde{C}(n)\sqrt{\varepsilon}. \tag{3.5.23}$$

Let $\{Q_i\}$ be a Whitney decomposition of $\mathbb{R}^{n+1}\backslash E$. That is to say,

1. $Q_i := Q_{r_i}(P_i, \eta_i)$ is a parallelogram whose cross section at any time is a cube of side length $2r_i$ centered at $P_i$ and whose length (in the time direction) is $2r_i^2$, centered around the time $\eta_i$.

2. $Q_i \cap Q_j = \emptyset, i \neq j$

3. $10^{-10n}d(Q_i, E) \leq r_i \leq 10^{-5n}d(Q_i, E)$,

4. For each $i, \{j \mid \overline{Q_j} \cap \overline{Q}_i \neq \emptyset\}$ has cardinality at most $c$

5. $\mathbb{R}^{n+1}\backslash E = \bigcup \overline{Q_i}$.

For $(Q,\tau) \in E$ and $0 < \rho < 1/2$, let $\xi(Q,\tau,\rho) = \{i \mid \overline{Q}_i \cap \Delta_\rho(Q,\tau) \neq \emptyset\}$. We claim

$$\gamma(Q,\tau,\rho) \leq c(n)\left(\tilde{\gamma}(Q,\tau,\frac{3}{2}\rho) + \sum_{i\in\xi(Q,\tau,\rho)} \left(\frac{r_i}{\rho}\right)^{n+3}\right). \tag{3.5.24}$$

To wit, let $P$ be a plane containing a line parallel to the $t$ axis such that $\tilde{\gamma}(Q,\tau,3\rho/2)$ is achieved by $P$. By definition

$$\tilde{\gamma}(Q,\tau,\frac{3}{2}\rho) = \left(\frac{3}{2}\right)^{-n-3}\rho^{-n-3}\int_{\partial\tilde{\Omega}\cap C_{3\rho/2}(Q,\tau)} d((Y,s),P)^2 d\tilde{\sigma}(Y,s).$$

On the other hand

$$\begin{aligned}\gamma(Q,\tau,\rho) \leq & \rho^{-n-3}\left(\int_{E\cap C_\rho(Q,\tau)} d((Y,s),P)^2 d\sigma + \int_{\Delta_\rho(Q,\tau)\setminus E} d((Y,s),P)^2 d\sigma\right) \\ \leq & \tilde{\gamma}(Q,\tau,\rho) + \rho^{-n-3}\sum_{i\in\xi}\int_{\overline{Q}_i\cap\Delta_\rho(Q,\tau)} d((Y,s),P)^2 d\sigma,\end{aligned} \tag{3.5.25}$$

as $d\sigma = d\tilde{\sigma}$ on $E$.

Note that the parabolic diameter of $Q_i$ is $\leq c(n)r_i$. Hence if

$$\xi_1(Q,\tau,\rho) := \{i\in\xi(Q,\tau,\rho) \mid \exists (Y,s)\in\overline{Q}_i \text{ s.t. } d((Y,s),P) < r_i\}$$

then $d((Y,s),P)\leq c'(n)r_i$ for all $(Y,s)\in\overline{Q}_i$ and all $i\in\xi_1(Q,\tau,\rho)$. Therefore,

$$\rho^{-n-3}\sum_{i\in\xi_1(Q,\tau,\rho)}\int_{\overline{Q}_i\cap\Delta_\rho(Q,\tau)} d((Y,s),P)^2 d\sigma \leq C(n,M)\sum_{i\in\xi_1(Q,\tau,\rho)}(r_i/\rho)^{n+3}. \tag{3.5.26}$$

If $i\in\xi(Q,\tau,\rho)\setminus\xi_1(Q,\tau,\rho)$ let $(Y_i^*,s_i^*)\in\overline{Q_i}$ be such that $d(\overline{Q}_i,E) = d((Y_i^*,s_i^*),E) =: \delta_i$. We estimate $\sup_{(Y,s)\in\overline{Q}_i} d((Y,s),P)\leq d((Y_i^*,s_i^*),P) + c(n)r_i \leq \tilde{c}(n)d((Y_i^*,s_i^*),P)$ (because $i\notin\xi_1(Q,\tau,\rho)$). This implies,

$$\int_{\overline{Q}_i\cap\Delta_\rho(Q,\tau)} d((Y,s),P)^2 d\sigma(Y,s) \leq c(n)r_i^{n+1}d((Y_i^*,s_i^*),P)^2.$$

The distance between $(Y_i^*,s_i^*)$ and $E$ is $\delta_i$, as such, $C_{\delta_i/9}(\tilde{Y}_i,\tilde{s}_i)\subset C_{10\delta_i/9}(Y_i^*,s_i^*)$ for some $(\tilde{Y}_i,\tilde{s}_i)\in E\subset\partial\tilde{\Omega}$. Furthermore, recall $\delta_i\simeq r_i$, hence, $\tilde{\sigma}(C_{\delta/9}(\tilde{Y}_i,\tilde{s}_i)\cap\partial\tilde{\Omega})\geq c(n)\delta_i^{n+1}\geq c' r_i^{n+1}$. Arguing as above, $d((Y,s),P)+c(n)\delta_i\geq d((Y_i^*,s_i^*),P)$ for any $(Y,s)\in$

$C_{10\delta_i/9}(Y_i^*, s_i^*)$. Putting all of this together,

$$\sum_{i\in\xi(Q,\tau,\rho)\setminus\xi_1(Q,\tau,\rho)} \int_{\overline{Q}_i\cap\Delta_\rho(Q,\tau)} d((Y,s),P)^2 d\sigma \leq \sum_{i\in\xi(Q,\tau,\rho)\setminus\xi_1(Q,\tau,\rho)} c(n) r_i^{n+1} d((Y_i^*,s_i^*),P)^2$$
$$\leq \sum_{i\in\xi(Q,\tau,\rho)\setminus\xi_1(Q,\tau,\rho)} c(n) \left( \int_{C_{\delta_i/9}(\tilde{Y}_i,\tilde{s}_i)\cap\partial\tilde{\Omega}} d((Y,s),P)^2 d\tilde{\sigma} + \delta_i^2 r_i^{n+1} \right).$$

Observe that, if $i \in \xi(Q,\tau,\rho)$ then, $\delta_i \leq \rho$ and $\bigcup_{i\in\xi(Q,\tau,\rho)\setminus\xi_1(Q,\tau,\rho)} C_{\delta_i/9}(\tilde{Y}_i,\tilde{s}_i) \subset C_{3\rho/2}(Q,\tau)$. Furthermore for each $i \in \xi(Q,\tau,\rho)$, $\#\{j \in \xi(Q,\tau,\rho)\setminus\xi_1(Q,\tau,\rho) \mid \overline{C}_{\delta_i/9}(\tilde{Y}_i,\tilde{s}_i)\cap \overline{C}_{\delta_j/9}(\tilde{Y}_j,\tilde{s}_j)\} < c(n)$. Plugging these estimates into the offset equation above yields

$$\rho^{-n-3} \sum_{i\in\xi\setminus\xi_1} \int_{\overline{Q}_i\cap\Delta_\rho(Q,\tau)} d((Y,s),P)^2 d\sigma \leq c(n) \left( \tilde{\gamma}(Q,\tau,\frac{3}{2}\rho) + \sum_{i\in\xi\setminus\xi_1} \left(\frac{r_i}{\rho}\right)^{n+3} \right). \tag{3.5.27}$$

Our claim, equation (3.5.24), follows from equations (3.5.25), (3.5.26) and (3.5.27).

By definition, if $i \in \xi(Q,\tau,\rho)$ then $\rho \geq d(\overline{Q}_i,(Q,\tau))$. Integrate equation (3.5.24) in $\rho$ from 0 to 1/2 and over $(Q,\tau) \in E \cap C_{1/2}(0,0)$ to obtain

$$\begin{aligned} &\nu((E\cap C_{1/2}(0,0)) \times [0,1/2]) \leq C(\tilde{\nu}((E\cap C_{3/4}(0,0)) \times [0,3/4]) \\ &+ \int_{E\cap C_{1/2}(0,0)} \left( \sum_{i\in\xi(Q,\tau,1/2)} \int_{d(\overline{Q}_i,(Q,\tau))}^{1/2} (r_i/\rho)^{n+3}\rho^{-1} d\rho \right) d\sigma(Q,\tau). \end{aligned} \tag{3.5.28}$$

Here, and for the rest of the proof, $C$ will refer to a constant which may depend on the dimension, $\|\nu\|$ and the Ahlfors regularity constant of $\Omega$ but not on $\Psi$ or $\varepsilon$.

Evaluate the integral in $\rho$ to bound

$$\int_{E\cap C_{1/2}(0,0)} \left( \sum_{i\in\xi(Q,\tau,1/2)} \int_{d(\overline{Q}_i,(Q,\tau))}^{1/2} (r_i/\rho)^{n+3}\rho^{-1} d\rho \right) d\sigma$$

by

$$C \sum_{i \in \xi(0,0,1)} \int_{E \cap C_{1/2}(0,0)} \left( \frac{r_i}{d(\overline{Q}_i, (Q,\tau))} \right)^{n+3} d\sigma.$$

For every $\lambda \geq c(n) r_i$, let

$$E_\lambda := \{(Q,\tau) \in E \mid d(\overline{Q}_i, (Q,\tau)) \leq \lambda\}.$$

Trivially $E_\lambda = \{(Q,\tau) \in E \mid r_i/d(\overline{Q}_i,(Q,\tau)) \geq r_i/\lambda\}$. By construction of the Whitney decomposition, $\mathrm{diam}(Q_i) \leq c(n)\lambda$, so Ahlfors regularity implies

$$\sigma(E_\lambda) \leq C\lambda^{n+1}.$$

Recall that $r_i/d(Q_i,(Q,\tau)) \leq r_i/\delta_i \leq 10^{-5n}$. Let $\gamma = r_i/\lambda$ and evaluate,

$$\begin{aligned} \int_{E\cap C_{1/2}(0,0)} \left( \frac{r_i}{d(\overline{Q}_i,(Q,\tau))}\right)^{n+3} d\sigma &\lesssim_n \int_0^{10^{-5n}} \gamma^{n+2} \sigma(\{(Q,\tau) \in E \mid \frac{r_i}{d(\overline{Q}_i,(Q,\tau))} \geq \gamma\}) d\gamma \\ &\lesssim_n \int_0^1 \gamma^{n+2} r_i^{n+1} \gamma^{-n-1} d\gamma \leq c(n) r_i^{n+1}. \end{aligned}$$

Now recall, that $Q_i \in \xi(0,0,3/4)$ form a cover of $\Delta_{3/4}(0,0)\backslash E$. In light of equation (3.5.23), $\sum_{i\in\xi(0,0,3/4)} r_i^{n+1} \leq c(n)\sigma(\Delta_1(0,0)\backslash E) \leq C\sqrt{\varepsilon}$. Which allows us to bound,

$$\int_{E\cap C_{1/2}(0,0)} \left( \sum_{i \in \xi(Q,\tau,1/2)} \int_{d(\overline{Q}_i,(Q,\tau))}^{1/2} (r_i/\rho)^{n+3} \rho^{-1} d\rho \right) d\sigma \leq C\sqrt{\varepsilon}.$$

Plugging the above inequality into equation (3.5.28) proves

$$\nu((E\cap C_{1/2}(0,0)) \times [0,1/2]) \leq C(\tilde{\nu}((E \cap C_{3/4}(0,0)) \times [0,3/4]) + \sqrt{\varepsilon}). \tag{3.5.29}$$

It remains to estimate $\nu((\Delta_{1/2}(0,0)\backslash E) \times [0,1/2])$. By the Cauchy-Schwartz inequality

$$\begin{aligned}
\nu((\Delta_{1/2}(0,0)\backslash E) \times [0,1/2]) &= \int_{\Delta_{1/2}(0,0)} \chi_{E^c}(Q,\tau) \left( \int_0^{1/2} \gamma(Q,\tau,\rho)\rho^{-1} d\rho \right) d\sigma(Q,\tau) \\
&\leq \sqrt{\sigma(\Delta_{1/2}(0,0)\backslash E)} \left( \int_{\Delta_{1/2}(0,0)} \left( \int_0^{1/2} \gamma(Q,\tau,\rho)\rho^{-1} d\rho \right)^2 d\sigma(Q,\tau) \right)^{1/2}.
\end{aligned} \tag{3.5.30}$$

As above, let $f(Q,\tau) := \int_0^{1/2} \gamma(Q,\tau,\rho)\rho^{-1} d\rho$. We claim that $f(Q,\tau) \in \mathrm{BMO}(C_{1/2}(0,0))$. Let $C_\rho(P,\eta) \subset C_{1/2}(0,0)$ with $(P,\eta) \in \partial\Omega$. Define $k \equiv k(\rho,P,\eta) = \int_\rho^{1/2} \gamma(P,\eta,r) r^{-1} dr$. Additionally, let $f_1(Q,\tau) := \int_0^\rho \gamma(Q,\tau,r)r^{-1}dr$ and $f_2(Q,\tau) = f(Q,\tau) - f_1(Q,\tau)$ for $(Q,\tau) \in C_\rho(P,\eta)$. By the triangle inequality,

$$\begin{aligned}
\int_{\Delta_\rho(P,\eta)} |f(Q,\tau) - k| d\sigma &\leq \int_{\Delta_\rho(P,\eta)} f_1(Q,\tau) d\sigma + \int_{\Delta_\rho(P,\eta)} |f_2(Q,\tau) - k| d\sigma \\
&\leq \|\nu\|\rho^{n+1} + \int_{\Delta_\rho(P,\eta)} |f_2(Q,\tau) - k| d\sigma.
\end{aligned} \tag{3.5.31}$$

If $(Q,\tau) \in \Delta_\rho(P,\eta)$ then $\sigma((\Delta_r(Q,\tau) \cup \Delta_r(P,\eta))\backslash(\Delta_r(Q,\tau) \cap \Delta_r(P,\eta))) \leq C\rho^{n+1}$ (by Ahlfors regularity). Therefore, $|\gamma(P,\eta,r) - \gamma(Q,\tau,r)| \leq C\frac{\rho^{n+1}}{r^{n+1}}$. Hence,

$$\begin{aligned}
\int_{C_\rho(P,\eta)} |f_2(Q,\tau) - k| d\sigma(Q,\tau) &\leq \int_{C_\rho(P,\eta)} \int_\rho^{1/2} |\gamma(P,\eta,r) - \gamma(Q,\tau,r)| \frac{dr}{r} d\sigma(Q,\tau) \\
&\leq C\rho^{n+1} \int_\rho^{1/2} \frac{\rho^{n+1}}{r^{n+1}} r^{-1} dr \leq C\rho^{n+1}.
\end{aligned}$$

Together with equation (3.5.31), this proves $\|f(Q,\tau)\|_{\mathrm{BMO}(C_{1/2}(0,0))} \leq C$.

Let $k_{1/2} := ⨍_{C_{1/2}(0,0)} f(Q,\tau) d\sigma$ (hence $k_{1/2} \leq \|\nu\|$). By the definition of $f(Q,\tau) \in \mathrm{BMO}$,

$$\begin{aligned}
\int_{C_{1/2}(0,0)} |f(Q,\tau) - k_{1/2}|^2 d\sigma &\leq c(n) \|f(Q,\tau)\|^2_{\mathrm{BMO}(C_{1/2}(0,0))} \Rightarrow \\
\int_{C_{1/2}(0,0)} |f(Q,\tau)|^2 d\sigma(Q,\tau) &\leq c(n)(\|f(Q,\tau)\|^2_{\mathrm{BMO}(C_{1/2}(0,0))} + \|\nu\|^2).
\end{aligned} \tag{3.5.32}$$

Combine equations (3.5.30) and (3.5.32) to produce

$$\nu(\{[\partial\Omega \cap C_{1/2}(0,0)\backslash E] \times [0,1/2]\}) \leq C\varepsilon^{1/4} \tag{3.5.33}$$

which, with (3.5.29), is the desired result. □

We are now ready to prove Proposition 3.5.1, and by extension, complete the proof of Theorem 3.1.9.

*Proof of Proposition 3.5.1.* Fix an $\varepsilon > 0$ and let $r_i \downarrow 0$ and $(Q_i, \tau_i) \in \partial\Omega \cap K$ for any compact set $K$. For each $i$, apply Lemma 3.5.2 inside of $C_{r_i}(Q_i, \tau_i)$. This produces a sequence of functions, $\{\psi_i\}$, with bounded Lipschitz norms whose graphs are good approximations to $C_{r_i}(Q_i, \tau_i) \cap \partial\Omega$. We write, for ease of notation, $P_i \equiv P((Q_i, \tau_i), r_i)$. As there is no harm in a rotation (and we will be considering each $i$ seperately) we may assume that $P_i \equiv \{x_n = 0\}$. We can define $\Phi_i(x,t) := \frac{1}{r_i}\psi_i(r_i x + q_i, r_i^2 t + \tau_i) - \frac{(Q_i)_n}{r_i}$. Then, after a rotation which possibly depends on $i$, $\Omega_i$ and $\Phi_i$ satisfy the requirements of Lemma 3.5.3. In particular, there exists an $i_0(\varepsilon) > 0$ such that for $i \geq i_0$,

$$\frac{\nu(\Delta_{r_i/2}(Q_i, \tau_i) \times [0, r_i/2])}{r_i^{n+1}} \leq K_{n,\|\nu\|,\varepsilon} \int_0^{3/4} \int_{C_{3/4}(0,0)} \gamma_{\Phi_i}(x,t,r) dx dt \frac{dr}{r} + K_{n,\|\nu\|}\varepsilon^{1/2}. \tag{3.5.34}$$

It is important to note that while both constants above can depend on the dimension, $\|\nu\|$ and the Ahlfors regularity of $\Omega$, only $K_{n,\|\nu\|,\varepsilon}$ will depend on $\varepsilon$ and both constants are independent of $i$.

Conclusion (4) of Lemma 3.5.2 implies that $f_i(x,t) := \int_0^{3/4} \gamma_{\Phi_i}(x,t,r) r^{-1} dr$ is uniformly integrable on $C_{3/4}(0,0) \cap \{x_n = 0\}$. Furthermore the $\Phi_i$ are uniformly bounded in the $\mathrm{Lip}(1,1/2)$ norm so by the Arzelà-Ascoli theorem there is some $\Phi_\infty$ such that $\Phi_i \rightrightarrows \Phi_\infty$. It follows that $f_i(x,t) \to f_\infty(x,t) := \int_0^{3/4} \gamma_{\Phi_\infty}(x,t,r) r^{-1} dr$. Thus, the dominated convergence theorem implies

$$\limsup_{i\to\infty} \frac{\nu(C_{r_i/2}(Q_i,\tau_i)\times[0,r_i/2])}{r_i^{n+1}} \leq K_{n,\|\nu\|,\varepsilon}\int_{C_{3/4}(0,0)} f_\infty(x,t)dxdt + K_{n,\|\nu\|}\varepsilon^{1/2}. \quad (3.5.35)$$

On the other hand, condition (3) in Lemma 3.5.2 and vanishing Reifenberg flatness tells us that the graph of $\Phi_i$ in the cylinder $C_1(0,0)$ is contained in increasingly smaller neighborhoods of $P_i$. Hence, $\Phi_\infty(x,t)\equiv 0$ inside of $C_1(0,0)$, and $f_\infty(x,t)\equiv 0$ in $C_{1/2}(0,0)$. Plugging this into equation (3.5.35) yields the bound,

$$\limsup_{i\to\infty} \frac{1}{r_i^{n+1}}\nu(\Delta_{r_i/2}(Q_i,\tau_i)\times[0,r_i/2]) \leq K_{n,\|\nu\|}\varepsilon^{1/2}. \quad (3.5.36)$$

Since $\varepsilon$ is arbitrarily small the result follows. □

## 3.6 Initial Hölder Regularity

We turn our attention to proving Proposition 3.1.11 and assume that $\log(h)$ (or $\log(k^{(X_0,t_0)})$) is Hölder continuous. As before, $\Omega$ will be a $\delta$-Reifenberg flat parabolic regular domain. We will state and prove all the results in the infinite pole setting, however, almost no modifications are needed for kernels with a finite pole.

This section is devoted to proving an initial Hölder regularity result:

**Proposition 3.6.1.** *Let $\Omega\subset\mathbb{R}^{n+1}$ be a parabolic regular domain and $\alpha\in(0,1)$ such that $\log(h)\in\mathbb{C}^{\alpha,\alpha/2}(\mathbb{R}^{n+1})$. There is a $\delta_n>0$ such that if $\delta_n\geq\delta>0$ and $\Omega$ is $\delta$-Reifenberg flat then $\Omega$ is a $\mathbb{C}^{1+\alpha,(1+\alpha)/2}(\mathbb{R}^{n+1})$ domain.*

We follow closely the structure and exposition of Appendix B.1, occasionally dealing with additional complications introduced by the Hölder condition. We should also mention that this section is strongly influenced by the work of Andersson and Weiss in (AW09) and Alt and Caffarelli in (AC81). To begin, we introduce flatness conditions (these are in the vein of Definitions B.1.1 and B.1.3 but adapted to the Hölder regularity setting).

First, "current flatness" (compare to Definition 7.1 in (AC81)).

**Definition 3.6.2.** *For $0 < \sigma_i \leq 1, \kappa > 0$ we say that $u \in HCF(\sigma_1, \sigma_2, \kappa)$ in $C_\rho(X,t)$ in the direction $\nu \in \mathbb{S}^{n-1}$ if for $(Y,s) \in C_\rho(X,t)$*

- $(X,t) \in \partial\{u > 0\}$
- $u((Y,s)) = 0$ *whenever* $(Y - X) \cdot \nu \leq -\sigma_1 \rho$
- $u((Y,s)) \geq h(X,t)\left((Y-X)\cdot\nu - \sigma_2\rho\right)$ *whenever* $(Y-X)\cdot\nu - \sigma_2\rho \geq 0$.
- $|\nabla u(Y,s)| \leq h(X,t)(1+\kappa)$
- $\text{osc}_{(Q,\tau)\in\Delta_\rho(X,t)} h(Q,\tau) \leq \kappa h(X,t)$.

In some situations we will not have a estimate on the growth of $u$ in the positive side. Thus "weak current flatness":

**Definition 3.6.3.** *For $0 < \sigma_i \leq 1, \kappa > 0$ we say that $u \in \widetilde{HCF}(\sigma_1, \sigma_2, \kappa)$ in $C_\rho(X,t)$ in the direction $\nu \in \mathbb{S}^{n-1}$ if for $(Y,s) \in C_\rho(X,t)$*

- $(X,t) \in \partial\{u > 0\}$
- $u((Y,s)) = 0$ *whenever* $(Y - X) \cdot \nu \leq -\sigma_1 \rho$
- $u((Y,s)) \geq 0$ *whenever* $(Y-X)\cdot\nu - \sigma_2\rho \geq 0$.
- $|\nabla u(Y,s)| \leq h(X,t)(1+\kappa)$
- $\text{osc}_{(Q,\tau)\in\Delta_\rho(X,t)} h(Q,\tau) \leq \kappa h(X,t)$.

Finally, in our proofs we will need to consider functions satisfying a "past flatness" condition (first introduced for constant $h$ in (AW09), Definition 4.1).

**Definition 3.6.4.** *For $0 < \sigma_i \leq 1, \kappa > 0$ we say that $u \in HPF(\sigma_1, \sigma_2, \kappa)$ in $C_\rho(X,t)$ in the direction $\nu \in \mathbb{S}^{n-1}$ if for $(Y,s) \in C_\rho(X,t)$*

- $(X, t-\rho^2) \in \partial\Omega$
- $u((Y,s)) = 0$ *whenever* $(Y - X)\cdot \nu \leq -\sigma_1 \rho$
- $u((Y,s)) \geq h(X, t-\rho^2)\left((Y-X)\cdot\nu - \sigma_2\rho\right)$ *whenever* $(Y-X)\cdot\nu - \sigma_2\rho \geq 0$.
- $|\nabla u(Y,s)| \leq h(X, t-\rho^2)(1+\kappa)$
- $\mathrm{osc}_{(Q,\tau)\in\Delta_\rho(X,t)} h(Q,\tau) \leq \kappa h(X, t-\rho^2)$.

Proposition 3.6.1 will be straightfoward once we prove three lemmas. The first two allow us to conclude greater flatness on a particular side given flatness on the other (they are analogues of Lemmas B.1.4 and B.1.5 in the Hölder setting). We will postpone their proofs until later subsections.

**Lemma 3.6.5.** *Let* $0 < \kappa \leq \sigma \leq \sigma_0$ *where* $\sigma_0$ *depends only on dimension. If* $u \in \widetilde{HCF}(\sigma, 1/2, \kappa)$ *in* $C_\rho(Q,\tau)$ *in the direction* $\nu$*, then there is a constant* $C_1 > 0$ *(depending only on dimension) such that* $u \in HCF(C_1\sigma, C_1\sigma, \kappa)$ *in* $C_{\rho/2}(Q,\tau)$ *in the direction* $\nu$*.*

**Lemma 3.6.6.** *Let* $\theta \in (0,1)$ *and assume that* $u \in HCF(\sigma,\sigma,\kappa)$ *in* $C_\rho(Q,\tau)$ *in the direction* $\nu$*. There exists a constant* $0 < \sigma_\theta < 1/2$ *such that if* $\sigma < \sigma_\theta$ *and* $\kappa \leq \sigma_\theta\sigma^2$ *then* $u \in \widetilde{HCF}(\theta\sigma, \theta\sigma, \kappa)$ *in* $C_{c(n)\rho\theta}(Q,\tau)$ *in the direction* $\overline{\nu}$ *where* $|\overline{\nu} - \nu| \leq C(n)\sigma$*. Here* $\infty > C(n), c(n) > 0$ *are constants depending only on dimension.*

The third lemma is an adaptation of Proposition 3.4.4 and tells us that $|\nabla u(X,t)|$ is bounded above by $h(Q,\tau)$ as $(X,t)$ gets close to $(Q,\tau)$.

**Lemma 3.6.7.** *Let* $u, \Omega, h$ *be as in Proposition 3.6.1. Then there exists a constant* $C > 0$*, which is uniform in* $(Q,\tau) \in \partial\Omega$ *on compacta, such that for all* $r < 1/4$*,*

$$\sup_{(X,t)\in C_r(Q,\tau)} |\nabla u(X,t)| \leq h(Q,\tau) + Cr^{\min\{3/4,\alpha\}}.$$

*Proof.* Fix an $R >> 1$ and $(Q,\tau) \in \partial\Omega$. Lemma 3.3.7 says that there is a uniform constant $C > 0$ such that, for any $(X,t) \in \Omega$ with $\|(X,t) - (Q,\tau)\| \leq R/2$,

$$|\nabla u(X,t)| \leq \int_{\Delta_{2R}(Q,\tau)} h(P,\eta) d\hat{\omega}^{(X,t)}(P,\eta) + C\frac{\|(X,t) - (Q,\tau)\|^{3/4}}{R^{1/2}}. \tag{3.6.1}$$

Let $\|(X,t) - (Q,\tau)\| \leq r$ and let $k_0 \in \mathbb{N}$ be such that $2^{-k_0-1} \leq r < 2^{-k_0}$. The inequality

$$1 - \widehat{\omega}^{(X,t)}(C_{2^{-j}}(Q,\tau)) \leq C2^{3j/4}r^{3/4}, \forall j < k_0 - 2, \tag{3.6.2}$$

follows from applying Lemma 3.2.1 to $1 - \widehat{\omega}^{(Y,s)}(C_{2^{-j}}(Q,\tau))$. We can write

$$\begin{aligned}\int_{\Delta_{2R}(Q,\tau)} h(P,\eta) d\hat{\omega}^{(X,t)}(P,\eta) &\leq h(Q,\tau) + C\|h\|_{\mathbb{C}^{\alpha,\alpha/2}} \int_{\Delta_{4r}(Q,\tau)} (4r)^{\alpha} d\hat{\omega}^{(X,t)} \\ + C\|h\|_{\mathbb{C}^{\alpha,\alpha/2}} &\left( \int_{\Delta_{2R}(Q,\tau)\setminus\Delta_1(Q,\tau)} d\hat{\omega}^{(X,t)} + \sum_{j=0}^{k_0-2} \int_{\Delta_{2^{-j}}(Q,\tau)\setminus\Delta_{2^{-(j+1)}}(Q,\tau)} 2^{-j\alpha} d\hat{\omega}^{(X,t)} \right).\end{aligned} \tag{3.6.3}$$

We may bound

$$\hat{\omega}^{(X,t)}(\Delta_{2R}(Q,\tau)\setminus\Delta_1(Q,\tau)) \leq 1 - \widehat{\omega}^{(X,t)}(\Delta_1(Q,\tau)) \overset{eq.(3.6.2)}{\leq} C_R r^{3/4}$$

$$\hat{\omega}^{(X,t)}(\Delta_{2^{-j}}(Q,\tau)\setminus\Delta_{2^{-(j+1)}}(Q,\tau)) \leq 1 - \widehat{\omega}^{(X,t)}(\Delta_{2^{-j}}(Q,\tau)) \overset{eq.(3.6.2)}{\leq} C2^{3j/4}r^{3/4}.$$

Plug these estimates into equation (3.7.4) to obtain

$$\int_{\Delta_{2R}(Q,\tau)} h(P,\eta) d\hat{\omega}^{(X,t)}(P,\eta) \leq h(Q,\tau) + Cr^{\alpha} + C_R r^{3/4} + Cr^{3/4} \sum_{j=0}^{[\log_2(r^{-1})]} 2^{j(3/4-\alpha)}. \tag{3.6.4}$$

If $\alpha > 3/4$ then we can let the sum above run to infinity and evaluate to get the desired result. If $\alpha < 3/4$ then the geometric sum above evaluates to $\approx \frac{r^{\alpha-3/4}-1}{2^{3/4-\alpha}-1} \leq 4r^{\alpha-3/4}$. Plug this into estimate (3.7.5) and we are done. Finally, if $\alpha = 3/4$ then we may apply Lemma 3.2.1 with $\varepsilon$ just slightly less than 1/4 to get a version of equation (3.6.2) with a different exponent, which will allow us to repeat the above argument without issue. □

These three results allow us to iteratively improve the flatness of the free boundary.

**Corollary 3.6.8.** *For every $\theta \in (0,1)$ there is a $\sigma_{n,\alpha} > 0$ and a constant $c_\theta \in (0,1)$, which depends only on $\theta, \alpha$ and $n$, such that if $u \in \widetilde{HCF}(\sigma, 1/2, \kappa)$ in $C_\rho(Q,\tau)$ in direction $\nu$ then $u \in \widetilde{HCF}(\theta\sigma, \theta\sigma, \theta^2\kappa)$ in $C_{c_\theta \rho}(Q,\tau)$ in direction $\overline{\nu}$ as long as $\sigma \leq \sigma_{n,\alpha}$, $\kappa \leq \sigma_{n,\alpha}\sigma^2$ and $\mathrm{osc}_{\Delta_{s\rho}(0,0)} h \leq \kappa s^\alpha h(0,0)$. Furthermore $\overline{\nu}$ satisfies $|\overline{\nu} - \nu| \leq C\sigma$ where $C$ depends only on dimension. Finally, there is constant $\tilde{C} > 1$, which depends only on $n$, and a number $\gamma \in (0,1)$, which depends only on $n, \alpha$, such that $\tilde{C} c_\theta^\gamma \geq \theta \geq c_\theta^{\alpha/2}$.*

*Proof.* We may assume that $\rho = 1, (Q,\tau) = (0,0)$ and $\nu = e_n$. By Lemma 3.6.5 we know that $u \in HCF(C_1\sigma, C_1\sigma, \kappa)$ in $C_{1/2}(0,0)$ in direction $e_n$. Let $\theta_1 \in (0,1)$ be chosen later (to depend on the dimension and $\alpha$), and set $\sigma_{n,\alpha} := \sigma_{\theta_1}/C_1$ where $\sigma_{\theta_1}$ is the constant given by Lemma 3.6.6. Then if $\sigma < \sigma_{n,\alpha}$ and $\kappa \leq \sigma_{n,\alpha}\sigma^2$, Lemma 3.6.6 implies $u \in \widetilde{HCF}(C_1\theta_1\sigma, C_1\theta_1\sigma, \kappa)$ in $C_{\tilde{c}\theta_1}(0,0)$ in the direction $\nu_1$ where $|\nu_1 - e_n| \leq C(n)\sigma$.

We turn to improving the bound on $\nabla u$. Observe that $U = \max\{|\nabla u(X,t)| - h(0,0), 0\}$ is an adjoint-subcaloric function in $C_1(0,0)$. Let $V$ be the solution to the adjoint heat equation such that $V = \kappa h(0,0)\chi_{x_n \geq -\sigma}$ on the adjoint parabolic boundary of $C_1(0,0)$. That $u \in \widetilde{HCF}(\sigma, 1/2, \kappa)$ implies $U \leq V$ on $\partial_p C_1(0,0)$. The maximum theorem and Harnack inequality then imply $U \leq V \leq (1-c)\kappa h(0,0)$ on all of $C_{1/2}(0,0)$, where $c$ depends only on dimension. Furthermore, by assumption,

$$\mathrm{osc}_{\Delta_{\tilde{c}\theta_1}(0,0)} h \leq \kappa(\tilde{c}\theta_1)^\alpha h(0,0).$$

Hence, if $\theta_0 = \sqrt{1-c}$ and $\theta_1 = \min\{\theta_0/C_1, \theta_0^{2/\alpha}/\tilde{c}\}$ we have that $u \in \widetilde{HCF}(\theta_0\sigma, \theta_0\sigma, \theta_0^2\kappa)$ in $C_{\tilde{c}\theta_1}(0,0)$ in direction $\nu_1$.

Iterate this scheme $m$ times to get that $u \in \widetilde{HCF}(\theta_0^m\sigma, \theta_0^m\sigma, \theta_0^{2m}\kappa)$ in $C_{\tilde{c}^m\theta_1^m}(0,0)$ in the direction $\nu_m$ where $|e_n - \nu_m| \leq C\sigma\sum_{j=0}^\infty \theta_0^j \leq C(n)\sigma$. Let $m$ be large so that $\theta_0^m \leq \theta \leq \theta_0^{m-1}$. Then, since $\theta_0, \tilde{c}, \theta_1$ are constants which depend only on the dimension, $n$, and $\alpha$, we see that $(\tilde{c}\theta_1)^m = c_\theta$ where $c_\theta$ depends on $\theta, n, \alpha$. By the definition of $\theta_1$, there is some $\chi \geq 2/\alpha > 1$ (but which depends only on $n, \alpha$) such that $\tilde{c}\theta_1 = \theta_0^\chi$. Then

$$(c_\theta)^{\alpha/2} \leq (c_\theta)^{1/\chi} = (\tilde{c}\theta_1)^{\frac{m}{\chi}} = \theta_0^m \leq \theta \leq \theta_0^{m-1} = \frac{1}{\theta_0}\theta_0^m = \frac{1}{\theta_0}c_\theta^{1/\chi}.$$

Letting $\tilde{C} = \frac{1}{\theta_0}$ and $\gamma = \frac{1}{\chi}$ implies that

$$c_\theta^{\alpha/2} \leq \theta \leq \tilde{C}c_\theta^\gamma. \tag{3.6.5}$$

which are the desired bounds on $c_\theta$. □

Proposition 3.6.1 then follows from a standard argument:

*Proof of Proposition 3.6.1.* We want to apply Corollary 3.6.8 iteratively. But before we can start the iteration, we must show that the hypothesis of that result are satisfied.

By Lemma 3.6.7 and that fact that $\log(h)$ is Hölder continuous, there exists a constant $C > 0$ for any compact set $K$ such that $\forall (Q,\tau) \in \partial\Omega \cap K$ and $1/4 > \rho > 0$,

$$\begin{aligned} |\nabla u(X,t)| \leq & h(Q,\tau) + C\rho^{\alpha/2} \\ \mathrm{osc}_{\Delta_{s\rho}(Q,\tau)} h \leq & Ch(Q,\tau)s^\alpha \rho^\alpha, \ \forall s \in (0,1]. \end{aligned} \tag{3.6.6}$$

Fix a compact set $K$ and a $\sigma_0 \leq \sigma_{n,\alpha}$ (where $\sigma_{n,\alpha}$ is as in Corollary 3.6.8). As $\Omega$ is vanishing Reifenberg flat, there exists an $R := R_{\sigma,K} > 0$ with the property that for all $\rho < R$ and $(Q,\tau) \in \partial\Omega \cap K$, there is a plane $P$ (containing a line parallel to the time axis and going through $(Q,\tau)$) such that

$$D[C_\rho(Q,\tau) \cap P; C_\rho(Q,\tau) \cap \partial\Omega] \leq \rho\sigma.$$

Then fix a $\kappa_0 \leq \sigma_{n,\alpha}\sigma_0^2$. Obviously, we can choose $\rho_0$ small enough (and smaller than $R_{\sigma,K}$ above) such that $h(Q,\tau) + C\rho_0^{\alpha/2} \leq (1+\kappa_0)h(Q,\tau)$ and $Ch(Q,\tau)\rho_0^\alpha \leq \kappa_0$. Furthermore, this $\rho_0$ can be choosen uniformly over all $(Q, \tau) \in K$. These observations, combined with equation (3.6.6), means for $0 < \rho \leq \rho_0$, $u \in \widetilde{HCF}(\sigma, \sigma, \kappa_0)$ in $C_\rho(Q, \tau)$ for some direction $\nu$.

Then for any $(P, \eta) \in C_{\rho_0}(Q, \tau)$ there is a $\nu_0(P, \eta)$ such that $u \in \widetilde{HCF}(\sigma, \sigma, \kappa_0)$ in $C_{\rho_0}(P, \eta)$ in the direction $\nu_0(P, \eta)$. Let $\theta \in (0, 1)$ and apply Corollary 3.6.8 $m$ times to get that $u \in \widetilde{HCF}(\theta^m \sigma, \theta^m \sigma, \theta^{2m} \kappa_0)$ in $C_{c_\theta^m \rho_0}(P, \eta)$ in the direction $\nu_m(P, \eta)$. We should check that the conditions of Corollary 3.6.8 are fulfilled at every step. In particular, that for any $m$, we have $\operatorname{osc}_{\Delta_{sc_\theta^m \rho_0}(P,\eta)} h \leq \theta^{2m} \kappa_0 h(P, \eta)$. Indeed,

$$\operatorname{osc}_{\Delta_{sc_\theta^m \rho_0}(P,\eta)} h \leq Ch(P, \eta)(s\rho_0 c_\theta^m)^\alpha \leq \kappa_0 h(P, \eta) c_\theta^{2m\alpha/2} \leq \kappa_0 h(P, \eta) \theta^{2m}.$$

The last inequality above follows from equation (3.6.5), and the penultimate one follows from the definition of $\rho_0$.

Letting $m \to \infty$ it is clear that $\partial\Omega$ has a normal vector $\nu(P, \eta)$ at every $(P, \eta) \in C_{\rho_0}(Q, \tau)$ and $|\nu_m(P, \eta) - \nu(P, \eta)| \leq C_\theta \theta^m \sigma$ . Furthermore, if $(P', \eta') \in \Delta_{\rho_0 c_\theta^m}(P, \eta) \backslash \Delta_{\rho_0 c_\theta^{m+1}}(P, \eta)$ then $|\nu_m(P, \eta) - \nu_m(P', \eta')| \leq C\theta^m \sigma$. Hence, $|\nu(P, \eta) - \nu(P', \eta')| \leq C\theta^m \sigma$. By equation (3.6.5) we know that $C\sigma\theta^m \leq C\theta^m \leq \tilde{C} c_\theta^{\gamma m}$. Let $\beta \in (0, 1)$ be such that $\beta(m + 1) = \gamma m$. Hence, $|\nu(P, \eta) - \nu(P', \eta')| \leq C\|(P, \eta) - (P', \eta')\|^\beta$ which is the desired result. □

### *3.6.1 Flatness of the zero side implies flatness of the positive side: Lemma 3.6.5*

Before we begin we need two technical lemmas. The first allows us to conclude regularity in the time dimension given regularity in the spatial dimensions.

**Lemma 3.6.9.** *If $f$ satisfies the (adjoint)-heat equation in $\mathcal{O}$ and is zero outside $\mathcal{O}$ then*

$$\|f\|_{\mathbb{C}^{1,1/2}(\mathbb{R}^{n+1})} \leq c\|\nabla f\|_{L^\infty(\mathcal{O})},$$

*where* $0 < c < \infty$ *depends only on the dimension.*

*Proof.* It suffices to show that for any $(X,t),(X,s) \in \mathcal{O}$ we have $|f(X,t) - f(X,s)| \leq C|s-t|^{1/2}$ where $C$ does not depend on $X, t$ or $s$. Assume $s > t$ and let $r \equiv \sqrt{s-t}$. Before our analysis we need a basic estimate:

$$\left| \fint_{B'((X,t),r)} f_t dX \right| = \left| \fint_{B'((X,t),r)} \Delta f dX \right| = \left| \frac{c_n}{r} \fint_{\partial B'((X,t),r)} \nabla f \cdot \nu \right| \leq \frac{C_n \|\nabla f\|_{L^\infty(\mathcal{O})}}{r} \tag{3.6.7}$$

as long as $B'((X,t),r) := \{(Y,t) \mid |Y - X| \leq r\} \subset \mathcal{O}$.

There are two cases:

**Case 1:** $\{(Y,\tau) \mid |Y-X| \leq r, t \leq \tau \leq s\} \subset \mathcal{O}$. By Lipschitz continuity,

$$\left| f(X,t) - \fint_{B'((X,t),r)} f(Y,t) dY \right| \leq C \|\nabla f\|_{L^\infty} r.$$

Note that by Fubini's theorem and the mean value theorem there is a $\tilde{t} \in [t,s]$ such that

$$\begin{aligned}\left| \fint_{B'((X,t),r)} f(Y,t) dY - \fint_{B'((X,s),r)} f(Y,s) dY \right| =& \left| \fint_{\{|Y-X| \leq r\}} \int_t^s \partial_\tau f(Y,\tau) d\tau dY \right| \\ =& (s-t) \left| \fint_{B'((X,\tilde{t}),r)} f_\tau(Y,\tilde{t}) dY \right|.\end{aligned}$$

We may combine the two equations above to conclude,

$$\begin{aligned}|f(X,t) - f(X,s)| \leq& C\|\nabla f\|_{L^\infty} r + (s-t) \left| \fint_{B'((X,\tilde{t}),r)} f_\tau(Y,\tilde{t}) dY \right| \\ \overset{\text{eqn (3.6.7)}}{\leq}& C\|\nabla f\|_{L^\infty}(r + \frac{(s-t)}{r}) = C\|\nabla f\|_{L^\infty(\mathcal{O})} \sqrt{s-t}.\end{aligned}$$

**Case 2:** $\{(Y,\tau) \mid |Y-X| \leq r, t \leq \tau \leq s\} \not\subset \mathcal{O}$. If neither $B'((X,t),r)$ or $B'((X,s),r)$ are contained in $\mathcal{O}$ then $|f(X,t) - f(X,s)| \leq |f(X,t)| + |f(X,s)| \leq \|\nabla f\|_{L^\infty} r$ by the Lipschitz continuity of $u$. Therefore, without loss of generality we may assume $B'((X,t),r) \subset \mathcal{O}$ . Let $t \leq t^* \leq s$ be such that $t^* = \inf_{t \leq a} B'((X,a),r) \not\subset \mathcal{O}$. Since $B'((X,a),r) \subset \mathcal{O}$ is an open condition, we can argue as in Case 1 so that, $|f(X,t)-f(X,t^*)| \leq C\|\nabla f\|_{L^\infty(\mathcal{O})}(t^* - t)/r \leq C\|\nabla f\|_{L^\infty(\mathcal{O})}\sqrt{s-t}$. On the other hand, as $B'((X,t^*),r) \not\subset \mathcal{O}$ we know $|f(X,t^*)| \leq \|\nabla f\|_{L^\infty}|s-t|^{1/2}$. Therefore, $|f(X,t)| \leq C\|\nabla f\|_{L^\infty}|s-t|^{1/2}$. Arguing similarly at $s$ we are done. □

This second lemma allows us to bound from below the normal derivative of a solution at a smooth point of $\partial\Omega$.

**Lemma 3.6.10.** *Let $(Q,\tau) \in \partial\Omega$ be such that there exists a tangent ball (in the Euclidean sense) $B$ at $(Q,\tau)$ contained in $\overline{\Omega}^c$. Then*

$$\limsup_{\Omega \ni (X,t)\to(Q,\tau)} \frac{u(X,t)}{d((X,t),B)} \geq h(Q,\tau).$$

*Proof.* Without loss of generality set $(Q,\tau) = (0,0)$ and let $(X_k,t_k) \in \Omega$ be a sequence that achieves the supremum, $\ell$. Let $(Y_k,s_k) \in B$ be such that $d((X_k,t_k),B) = \|(X_k,t_k) - (Y_k,s_k)\| =: r_k$. Define $u_k(X,t) := \frac{u(r_kX+Y_k, r_k^2t+s_k)}{r_k}$, $\Omega_k := \{(Y,s) \mid Y = (X-Y_k)/r_k, s = (t-s_k)/r_k^2, \text{ s.t. } (X,t) \in \Omega\}$ and $h_k(X,t) := h(r_kX+Y_k, r_k^2t+s_k)$. Then

$$\int_{\mathbb{R}^{n+1}} u_k(\Delta\phi - \partial_t\phi)dXdt = \int_{\partial\Omega_k} h_k\phi d\sigma. \tag{3.6.8}$$

As $k\to\infty$ we can guarentee that $(r_kX+Y_k, r_k^2t+s_k) \in C_{1/100}(0,0)$. Apply Lemma 3.6.7 to conclude that, for $(X,t) \in C_1(0,0)$,

$$|\nabla u_k(X,t)| = |\nabla u(r_kX+Y_k, r_k^2t+s_k)| \leq Ch(0,0) + \left(\frac{1}{100}\right)^\beta.$$

In particular, the $u_k$ are uniformly Lipschitz continuous. By Lemma 3.6.9 the $u_k$ are bounded uniformly in $\mathbb{C}^{1,1/2}$. Therefore, perhaps passing to a subsequence, $u_k \to u_0$ uniformly on compacta. In addition, as there exists a tangent ball at $(0,0)$, $\Omega_k \to \{x_n > 0\}$ in the Hausdorff distance norm (up to a rotation). We may assume, passing to a subsequence, that $\frac{X_k - Y_k}{r_k} \to Z_0, \frac{t_k - s_k}{r_k^2} \to t_0$ with $(Z_0, t_0) \in C_1(0,0) \cap \{x_n > 0\}$ and $u_0(Z_0, t_0) = \ell$. Furthermore, by the definition of supremum, for any $(Y, s) \in \{x_n > 0\}$ we have

$$\begin{aligned} u_0(Y,s) &= \lim_{k\to\infty} u(r_k Y + y_k, r_k^2 s + s_k)/r_k \\ &\leq \lim_{k\to\infty} \ell \frac{\text{pardist}((r_k Y + y_k, r_k^2 s + s_k), B)}{r_k} \\ &= \lim_{k\to\infty} \ell \text{pardist}((Y,s), B_k) \\ &= \ell y_n, \end{aligned} \tag{3.6.9}$$

where $B_k$ is defined like $\Omega_k$ above.

Let $\phi \in C_0^\infty(\mathbb{R}^{n+1})$ be positive, then

$$\begin{aligned} \int_{\{x_n>0\}} \ell x_n (\Delta\phi - \partial_t \phi) dX dt &\geq \int_{\{x_n>0\}} u_0(X,t)(\Delta\phi - \partial_t\phi) dX dt \\ &= \lim_{k\to\infty} \int_{\Omega_k} u_k(X,t)(\Delta\phi - \partial_t\phi) dX dt \\ &= \lim_{k\to\infty} \int_{\partial\Omega_k} h_k \phi d\sigma. \end{aligned} \tag{3.6.10}$$

Integrating by parts yields

$$\begin{aligned} \ell \int_{\{x_n=0\}} \phi dx dt &= \int_{\{x_n>0\}} \ell x_n (\Delta\phi - \partial_t\phi) dX dt \\ &\overset{eqn.\ (3.6.10)}{\geq} \lim_{k\to\infty} \int_{\partial\Omega_k} h_k \phi d\sigma \\ &\geq \lim_{k\to\infty} \left( \inf_{(P,\eta)\in \text{supp}\, \phi} h(r_k P + Y_k, r_k^2 \eta + s_k) \right) \int_{\{x_n=0\}} \phi dx dt \end{aligned}$$

Hence, $\ell \geq \lim_{k\to\infty} h(Y_k, s_k) - C r_k^\alpha$, by the Hölder continuity of $h$. As $(Y_k, s_k) \to (Q, \tau)$ and $r_k \downarrow 0$ the desired result follows. □

We will first show that for "past flatness", flatness on the positive side gives flatness on the zero side.

**Lemma 3.6.11.** *[Compare with Lemma 5.2 in (AW09)] Let $0 < \kappa \leq \sigma/4 \leq \sigma_0$ where $\sigma_0$ depends only on dimension. Then if $u \in HPF(\sigma, 1, \kappa)$ in $C_\rho(\tilde{X}, \tilde{t})$ in the direction $\nu$, there is a constant $C$ such that $u \in HPF(C\sigma, C\sigma, 3\kappa)$ in $C_{\rho/2}(\tilde{X} + \alpha\nu, \tilde{t})$ in the direction $\nu$ for some $|\alpha| \leq C\sigma\rho$.*

*Proof.* Let $(\tilde{X}, \tilde{t}) = (0, 0), \rho = 1$ and $\nu = e_n$. First we will construct a regular function which touches $\partial\Omega$ at one point.

Define

$$\eta(x,t) = e^{\frac{16(|x|^2+|t+1|)}{16(|x|^2+|t+1|)-1}}$$

for $16(|x|^2 + |t+1|) < 1$ and $\eta(x,t) \equiv 0$ otherwise. Let $D := \{(x, x_n, t) \in C_1(0,0) \mid x_n > -\sigma + s\eta(x,t)\}$. Now pick $s$ to be the largest such constant that $C_1(0,0) \cap \Omega \subset D$. As $(0,-1) \in \partial\{u > 0\}$, there must be a touching point $(X_0, t_0) \in \partial D \cap \partial\Omega \cap \{-1 \leq t \leq -15/16\}$ and $s \leq \sigma$.

Define the barrier function $v$ as follows:

$$\begin{aligned} \Delta v + \partial_t v &= 0 \text{ in } D, \\ v &= 0 \text{ in } \partial_p D \cap C_1(0,0) \\ v &= h(0,-1)(1+\sigma)(\sigma + x_n) \text{ in } \partial_p D \cap \partial C_1(0,0). \end{aligned} \tag{3.6.11}$$

Note that on $\partial_p D \cap C_1(0)$ we have $u = 0$ because $D$ contains the positivity set. Also, as $|\nabla u| \leq h(0,-1)(1+\kappa) \leq h(0,-1)(1+\sigma)$, it must be the case that $u(X,t) \leq h(0,-1)(1+\sigma)\max\{0, \sigma + x_n\}$ for all $(X,t) \in C_1(0,0)$. As $v \geq u$ on $\partial_p D$ it follows that $v \geq u$ on all of $D$ (by the maximum principle for subadjoint-caloric functions). We now want to estimate the normal derivative of $v$ at $(X_0, t_0)$. To estimate from below, apply Lemma 3.6.10,

$$h(X_0, t_0) \leq \limsup_{(X,t)\to(X_0,t_0)} \frac{u(X,t)}{\text{pardist}((X,t),B)} \leq -\partial_\nu v(X_0, t_0) \tag{3.6.12}$$

where $\nu$ is the normal pointing out of $D$ at $(X_0, t_0)$ and $B$ is the tangent ball at $(X_0, t_0)$ to

$D$ contained in $D^c$.

To estimate from above, first consider $F(X,t) := (1+\sigma)h(0,-1)(\sigma + x_n) - v$. On $\partial_p D$,

$$-(1+\sigma)h(0,-1)\sigma \leq v - (1+\sigma)h(0,-1)x_n \leq (1+\sigma)h(0,-1)\sigma \tag{3.6.13}$$

thus (by the maximum principle) $0 \leq F(X,t) \leq 2(1+\sigma)h(0,-1)\sigma$. As $\partial D$ is piecewise smooth domain, standard parabolic regularity gives $\sup_D |\nabla F(X,t)| \leq K(1+\sigma)h(0,-1)\sigma$. Note, since $s \leq \sigma$, that $-\sigma + s\eta(x,t)$ is a function whose $\text{Lip}(1,1)$ norm is bounded by a constant. Therefore, $K$ does not depend on $\sigma$.

Hence,

$$\begin{aligned} |\nabla v| - (1+\sigma)h(0,-1) \leq & |\nabla v - (1+\sigma)h(0,-1)e_n| \leq K(1+\sigma)h(0,-1)\sigma \\ \overset{\text{eqn (3.6.12)}}{\Rightarrow} \quad h(X_0,t_0) \leq & -\partial_\nu v(Z) \leq (1+K\sigma)(1+\sigma)h(0,-1). \end{aligned} \tag{3.6.14}$$

We want to show that $u \geq v - \tilde{K}(1+\sigma)h(0,-1)\sigma x_n$ for some large constant $\tilde{K}$ to be choosen later, depending only on the dimension. Let $\tilde{Z} := (Y_0, s_0)$ with $s_0 \in (-3/4, 1)$, $|y_0| \leq 1/2$ and $(Y_0)_n = 3/4$ and assume, in order to obtain a contradiction, that $u \leq v - \tilde{K}(1+\sigma)h(0,-1)\sigma x_n$ at every point in $\{(Y,s_0) \mid |Y - Y_0| \leq 1/8\}$. We construct a barrier function, $w \equiv w_{\tilde{Z}}$, defined by

$$\begin{aligned} \Delta w + \partial_t w = 0 \text{ in } D \cap \{t < s_0\}, \\ w = x_n \text{ on } \partial_p(D \cap \{t < s_0\}) \cap \{(Y,s_0) \mid |Y - Y_0| < 1/8\}, \\ w = 0 \text{ on } \partial_p(D \cap \{t < s_0\}) \backslash \{(Y,s_0) \mid |Y - Y_0| < 1/8\}. \end{aligned}$$

By our initial assumption (and the definition of $w$), $v - \tilde{K}\sigma(1+\sigma)h(0,-1)w \geq u$ on $\partial_p(D \cap \{t < s_0\})$ and, therefore, $v - \tilde{K}\sigma(1+\sigma)h(0,-1)w \geq u$ on all of $D \cap \{t < s_0\}$. Since $t_0 \leq -15/16$ we know $(X_0, t_0) \in \partial_p(D \cap \{t < s_0\})$. Furthermore, the Hopf lemma gives an $\alpha > 0$ (independent of $\tilde{Z}$) such that $\partial_\nu w(X_0, t_0) \leq -\alpha$. With these facts in mind, apply

Lemma 3.6.10 at $(X_0, t_0)$ and recall estimate (3.6.14) to estimate,

$$\begin{aligned} h(X_0,t_0) = & \limsup_{(X,t)\to(X_0,t_0)} \frac{u(X,t)}{\mathrm{pardist}((X,t),B)} \\ \leq & -\partial_\nu v(X_0,t_0) + K(1+\sigma)h(0,-1)\sigma\partial_\nu w(X_0,t_0) \\ \leq & (1+K\sigma)(1+\sigma)h(0,-1) - \tilde{K}\alpha(1+\sigma)h(0,-1)\sigma \leq (1-2\sigma)h(0,-1) \end{aligned} \tag{3.6.15}$$

if $\tilde{K} \geq (K+3)/\alpha$. On the other hand, our assumed flatness tells us that $h(X_0,t_0)-h(0,-1) \geq -\kappa h(0,-1) \geq -\sigma h(0,-1)$. Together with equation (3.6.15) this implies $-\sigma h(0,-1) \leq -2\sigma h(0,-1)$, which is absurd.

Hence, there exists a point, call it $(\overline{Y}, s_0)$, such that $|\overline{Y} - Y_0| \leq 1/8$ and

$$(u-v)(\overline{Y}, s_0) \geq -\tilde{K}\sigma(1+\sigma)h(0,-1)(\overline{Y})_n \overset{(\overline{Y})_n \leq 1}{\geq} -\tilde{K}(1+\sigma)h(0,-1)\sigma.$$

Apply the parabolic Harnack inequality to obtain,

$$\begin{aligned} &\inf_{|X-Y_0|<1/8}(u-v)(X,s_0-1/32) \geq c_n \sup_{|\tilde{X}-Y_0|<1/8}(u-v)(\tilde{X},s_0) \geq -K'(1+\sigma)h(0,-1)\sigma \\ &\overset{(3.6.13)}{\Rightarrow} u(X,s_0-1/32) \geq (1+\sigma)h(0,-1)(x_n-\sigma) - \overline{C}(1+\sigma)h(0,-1)\sigma, \end{aligned}$$

for all $X$ such that $|X - Y_0| < 1/8$ and $\overline{C}$ which depends only on the dimension. Ranging over all $s_0 \in (-3/4, 1)$ and $|y_0| \leq 1/2$ the above implies

$$u(X,t) \geq (1+\sigma)h(0,-1)x_n - C(1+\sigma)h(0,-1)\sigma,$$

whenever $(X,t)$ satisfies $|x| < 1/2, |x_n - 3/4| < 1/8, t \in (-1/2, 1/2)$. As $|\nabla u| \leq (1+\sigma)h(0,-1)$ we can conclude, for any $(X,t)$ such that $|x| < 1/2, t \in (-1/2,1/2)$ and $3/4 \geq x_n \geq C\sigma$, that

$$u(X,t) \geq u(x,3/4,t) - (1+\sigma)h(0,-1)(3/4-x_n) \geq (1+\sigma)h(0,-1)(x_n - C\sigma). \tag{3.6.16}$$

We now need to find an $\alpha$ such that $(0, \alpha, -1/4) \in \partial\Omega$. By the initial assumed flatness, and equation (3.6.16), $\alpha \in \mathbb{R}$ exists and $-\sigma \leq \alpha \leq C\sigma$ (here we pick $\sigma_0$ is small enough such that $C\sigma_0 < 1/4$).

Furthermore, by the assumed flatness in $C_1(0,0)$,

$$\begin{aligned} &h(0,\alpha,-1/4) - h(0,0,-1) \geq -\kappa h(0,0,-1) \\ &\Rightarrow 3h(0,\alpha,-1/4) \geq (1-\kappa)^{-1} h(0,\alpha,-1/4) \geq h(0,0,-1). \end{aligned} \tag{3.6.17}$$

Hence,

$$\text{osc}_{C_{1/2}(0,\alpha,0)} h \leq \text{osc}_{C_1(0,0)} h \leq \kappa h(0,0,-1) \overset{\text{eqn (3.6.17)}}{\leq} 2\kappa h(0,\alpha,-1/4).$$

In summary we know,

- $(0,\alpha,-1/4) \in \partial\Omega,\ |\alpha| < C\sigma$
- $x_n - \alpha \leq -3C\sigma/2 \Rightarrow x_n \leq -\sigma \Rightarrow u(X,t) = 0.$
- When $x_n - \alpha \geq 2C\sigma \Rightarrow x_n \geq C\sigma$ hence equation (3.6.16) implies $u(X,t) \geq ((1+\sigma)h(0,-1))(x_n - C\sigma) \geq (1+2\kappa)h(0,\alpha,-1/4)(x_n - \alpha - 2C\sigma)$.
- As written above $\text{osc}_{C_{1/2}(0,\alpha,0)} h \leq 3\kappa h(0,\alpha,-1/4)$.
- Finally $\sup_{C_{1/2}(0,\alpha,0)} |\nabla u| \leq \sup_{C_1(0,0)} |\nabla u| \leq (1+\kappa)h(0,-1) \leq \frac{1+\kappa}{1-\kappa} h(0,\alpha,-1/4) \leq (1+3\kappa)h(0,\alpha,-1/4)$, where the penultimate inequality follows by (3.6.17).

Therefore $u \in HPF(2C\sigma, 2C\sigma, 3\kappa)$ in $C_{1/2}(0,\alpha,0)$ which is the desired result. □

Lemma 3.6.5 is the current version of the above and follows almost identically. Thus we will omit the full proof in favor of briefly pointing out the ways in which the argument differs.

**Lemma** (Lemma 3.6.5). *Let $0 < \kappa \leq \sigma \leq \sigma_0$ where $\sigma_0$ depends only on dimension. If $u \in \widetilde{HCF}(\sigma, 1/2, \kappa)$ in $C_\rho(Q, \tau)$ in the direction $\nu$, then there is a constant $C_1 > 0$ (depending only on dimension) such that $u \in HCF(C_1\sigma, C_1\sigma, \kappa)$ in $C_{\rho/2}(Q, \tau)$ in the direction $\nu$.*

*Proof of Lemma 3.6.5.* We begin in the same way; let $(Q, \tau) = (0, 0), \rho = 1$ and $\nu = e_n$. Then we recall the smooth function

$$\eta(x, t) = e^{\frac{16(|x|^2+|t+1|)}{16(|x|^2+|t+1|)-1}}$$

for $16(|x|^2 + |t + 1|) < 1$ and $\eta(x, t) \equiv 0$ otherwise. Let $D := \{(x, x_n, t) \in C_1(0, 0) \mid x_n > -\sigma + s\eta(x, t)\}$. Now pick $s$ to be the largest such constant that $C_1(0, 0) \cap \Omega \subset D$. Since $|x_n| > 1/2$ implies that $u(X, t) > 0$ there must be some touching point $(X_0, t_0) \in \partial D \cap \partial\Omega \cap \{-1 \leq t \leq -15/16\}$. Furthermore, we can assume that $s < \sigma + 1/2 < 2$.

The proof then proceeds as above until equation (3.6.16). In the setting of “past flatness” we need to argue further; the boundary point is at the bottom of the cylinder, so after the cylinder shrinks we need to search for a new boundary point. However, in current flatness the boundary point is at the center of the cylinder so after equation (3.6.16) we have completed the proof of Lemma 3.6.5. In particular, this explains why there is no increase from $\kappa$ to $3\kappa$ in the current setting. □

### *3.6.2 Flatness on Both Sides Implies Greater Flatness on the Zero Side: Lemma 3.6.6*

In this section we prove Lemma 3.6.6. The outline of the argument is as follows: arguing by contradiction, we obtain a sequence $u_k$ whose free boundaries, $\partial\{u_k > 0\}$, approach the graph of a function $f$. Then we prove that this function $f$ is $C^\infty$ which will lead to a contradiction.

Throughout this subsection, $\{u_k\}$ is a sequence of adjoint caloric functions such that

$\partial\{u_k > 0\}$ is a parabolic regular domain and such that, for all $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$,

$$\int_{\{u_k>0\}} u_k(\Delta\varphi - \partial_t\varphi)dXdt = \int_{\partial\{u_k>0\}} h_k\varphi d\sigma.$$

We will also assume the $h_k$ satisfy $\|\log(h_k)\|_{\mathbb{C}^{\alpha,\alpha/2}} \leq C\|\log(h)\|_{\mathbb{C}^{\alpha,\alpha/2}}$ and $h_k(0,0) = h(0,0)$. While we present these arguments for general $\{u_k\}$ it suffices to think of $u_k(X,t) := \frac{u(r_kX, r_k^2t)}{r_k}$ for some $r_k \downarrow 0$.

**Lemma 3.6.12.** *[Compare with Lemma 6.1 in (AW09)] Suppose that* $u_k \in HCF(\sigma_k, \sigma_k, \tau_k)$ *in* $C_{\rho_k}(0,0)$ *in direction* $e_n$*, with* $\sigma_k \downarrow 0$ *and* $\tau_k/\sigma_k^2 \to 0$*. Define* $f_k^+(x,t) = \sup\{d \mid (\rho_k x, \sigma_k\rho_k d, \rho_k^2 t) \in \{u_k = 0\}\}$ *and* $f_k^-(x,t) = \inf\{d \mid (\rho_k x, \sigma_k\rho_k d, \rho_k^2 t) \in \{u_k > 0\}\}$*. Then, passing to subsequences,* $f_k^+, f_k^- \to f$ *in* $L^\infty_{\text{loc}}(C_1(0,0))$ *and* $f$ *is continuous.*

*Proof.* By scaling each $u_k$ we may assume $\rho_k \equiv 1$. Then define

$$D_k := \{(y,d,t) \in C_1(0,0) \mid (y, \sigma_k d, t) \in \{u_k > 0\}\}.$$

Let

$$f(x,t) := \liminf_{\substack{(y,s)\to(x,t)\\ k\to\infty}} f_k^-(y,s),$$

so that, for every $(y_0,t_0)$, there exists a $(y_k,t_k) \to (y_0,t_0)$ such that $f_k^-(y_k,t_k) \overset{k\to\infty}{\to} f(y_0,t_0)$.

Fix a $(y_0,t_0)$ and note, as $f$ is lower semicontinuous, for every $\varepsilon > 0$, there exists a $\delta > 0, k_0 \in \mathbb{N}$ such that

$$\{(y,d,t) \mid |y-y_0| < 2\delta, |t-t_0| < 4\delta^2, d \leq f(y_0,t_0) - \varepsilon\} \cap \overline{D}_k = \emptyset,\ \forall k \geq k_0.$$

Consequently

$$x_n - f(y_0,t_0) \leq -\varepsilon \Rightarrow u_k(x, \sigma_k x_n, s) = 0,\ \forall (X,s) \in C_{2\delta}(Y_0,t_0). \tag{3.6.18}$$

Together with the definition of $f$, equation (3.6.18) implies that there exist $\alpha_k \in \mathbb{R}$ with $|\alpha_k| < 2\varepsilon$ such that $(y_0, \sigma_k(f(y_0,t_0)+\alpha_k), t_0-\delta^2) \in \partial\{u_k > 0\}$. Furthermore, for any $(Y,s) \in C_1(0,0)$ by assumption $h(0,0) - h_k(Y,s) \le \tau_k h(0,0) \Rightarrow h(0,0) \le (1+\frac{4}{3}\tau_k)h_k(Y,s)$ for $k$ large enough. This observation, combined with equation (3.6.18) allows us to conclude, $u_k(\cdot, \sigma_k \cdot, \cdot) \in HPF(3\sigma_k \frac{\varepsilon}{\delta}, 1, 4\tau_k)$ in $C_\delta(y_0, \sigma_k(f(y_0,t_0)+\alpha_k), t_0)$, for $k$ large enough.

As $\tau_k/\sigma_k^2 \to 0$ the conditions of Lemma 3.6.11 hold for $k$ large enough. Therefore, $u_k(\cdot,\sigma_k\cdot,\cdot) \in HPF(C\sigma_k\frac{\varepsilon}{\delta}, C\sigma_k\frac{\varepsilon}{\delta}, 8\tau_k)$ in $C_{\delta/2}(y_0, \sigma_k f(y_0,t_0)+\tilde{\alpha}_k, t_0)$ where $|\tilde{\alpha}_k| \le C\sigma_k\varepsilon$. Thus whenever $z_n - (\sigma_k f(y_0,t_0)+\tilde{\alpha}_k) \ge C\varepsilon\sigma_k/2$, we have $u_k(z, \sigma_k z_n, t) > 0$ for $(Z,t) \in C_{\delta/2}(y_0, \sigma_k f_k^-(y_0,t_0)+\tilde{\alpha}_k, t_0)$. In other words

$$\sup_{(Z,s)\in C_{\delta/2}(y_0,\sigma_k f(y_0,t_0)+\tilde{\alpha}_k,t_0)} f_k^+(z,s) \le f(y_0,t_0) + 3C\varepsilon. \tag{3.6.19}$$

As $f_k^+ \ge f_k^-$, if

$$\tilde{f}(y_0,t_0) := \limsup_{\substack{(y,s)\to(y_0,t_0)\\ k\to\infty}} f_k^+(y,s),$$

it follows (in light of equation (3.6.19)) that $\tilde{f} = f$. Consequently, $f$ is continuous and $f_k^+, f_k^- \to f$ locally uniformly on compacta. □

We now show that $f$ is given by the boundary values of $w$, a solution to the adjoint heat equation in $\{x_n > 0\}$.

**Lemma 3.6.13.** *[Compare with Proposition 6.2 in (AW09)] Suppose $u_k \in HCF(\sigma_k, \sigma_k, \tau_k)$ in $C_{\rho_k}(0,0)$, with $\rho_k \ge 0, \sigma_k \downarrow 0$ and $\tau_k/\sigma_k^2 \to 0$. Further assume that, after relabeling, $k$ is the subsequence given by Lemma 3.6.12. Define*

$$w_k(x,d,t) := \frac{u_k(\rho_k x, \rho_k d, \rho_k^2 t) - (1+\tau_k)h(0,0)\rho_k d}{h(0,0)\sigma_k}.$$

*Then, $w_k$ is bounded on $C_1(0,0)\cap\{x_n > 0\}$ (uniformly in $k$) and converges, in the $\mathbb{C}^{2,1}$-norm, on compact subsets of $C_1(0,0)\cap\{x_n > 0\}$ to $w$. Furthermore, $w$ is a solution to the adjoint-*

*heat equation and $w(x,d,t)$ is non-increasing in $d$ when $d>0$. Finally $w(x,0,t) = -f(x,t)$ and $w$ is continuous in $\overline{C_{1-\delta}(0,0)\cap\{x_n>0\}}$ for any $\delta>0$.*

*Proof.* As before we rescale and set $\rho_k \equiv 1$. Since $|\nabla u_k| \leq h(0,0)(1+\tau_k)$ and $x_n \leq -\sigma_k \Rightarrow u_k = 0$ it follows that $u_k(X,t) \leq h(0,0)(1+\tau_k)(x_n+\sigma_k)$. Which implies $w_k(X,t) \leq 1+\tau_k$. On the other hand, when $0 < x_n \leq \sigma_k$ we have $u_k(X,t) - (1+\tau_k)h(0,0)x_n \geq -(1+\tau_k)h(0,0)x_n \geq -(1+\tau_k)h(0,0)\sigma_k$, hence $w_k \geq -1-\tau_k$. Finally, if $x_n \geq \sigma_k$ we have $u_k(X,t)-(1+\tau_k)h(0,0)x_n \geq (1+\tau_k)h(0,0)(x_n-\sigma_k)-(1+\tau_k)h(0,0)x_n \Rightarrow w_k \geq -(1+\tau_k)$. Thus, for $k$ large enough, $|w_k| \leq 2$ in $C_1(0,0)\cap\{x_n>0\}$.

By definition, $w_k$ is a solution to the adjoint-heat equation in $C_1(0,0)\cap\{x_n>\sigma_k\}$. So for any $K \subset\subset \{x_n>0\}$ the $\{w_k\}$ are, for large enough $k$, a uniformly bounded sequence of solutions to the adjoint-heat equation on $K$. As $|w_k| \leq 2$, standard estimates for parabolic equations tell us that $\{w_k\}$ is uniformly bounded in $\mathbb{C}^{2+\alpha,1+\alpha/2}(K)$. Therefore, perhaps passing to a subsequence, $w_k \to w$ in $\mathbb{C}^{2,1}(K)$. Furthermore, $w$ must also be a solution to the adjoint heat equation in $K$ and $|w| \leq 1$. A diagonalization argument allows us to conclude that $w$ is adjoint caloric on all of $\{x_n>0\}$.

Compute that $\partial_n w_k = (\partial_n u_k - (1+\tau_k)h(0,0))/(h(0,0)\sigma_k) \leq 0$, which implies $\partial_n w \leq 0$ on $\{x_n>0\}$. As such, $w(x,0,t) := \lim_{d\to 0^+} w(x,d,t)$ exists. We will now show that this limit is equal to $-f(x,t)$ (which, recall, is a continuous function). If true, then regularity theory for adjoint-caloric functions immediately implies that $w$ is continuous in $\overline{C_{1-\delta}(0,0)\cap\{x_n>0\}}$.

First we show that the limit is less that $-f(x,t)$. Let $\varepsilon>0$ and pick $0<\alpha\leq 1/2$ small enough so that $|w(x,\alpha,t)-w(x,0,t)| < \varepsilon$. For $k$ large enough we have $\alpha/\sigma_k > f(x,t)+1 > f_k^-(x,t)$ therefore,

$$\begin{aligned} w(x,0,t) &\leq w(x,\alpha,t)+\varepsilon = w_k(x,\sigma_k\frac{\alpha}{\sigma_k},t)+\varepsilon+o_k(1) \\ &= (w_k(x,\sigma_k\frac{\alpha}{\sigma_k},t) - w_k(x,\sigma_k f_k^-(x,t),t)) + w_k(x,\sigma_k f_k^-(x,t),t)+\varepsilon+o_k(1) \\ &\overset{\partial_n w_k \leq 0}{\leq} w_k(x,\sigma_k f_k^-(x,t),t)+o_k(1)+\varepsilon. \end{aligned} \tag{3.6.20}$$

Note, $w_k(x, \sigma_k f_k^-(x,t), t) = -(1+\tau_k) f_k^-(x,t) \to -f(x,t)$ uniformly in $C_{1-\delta}(0,0)$. In light of (3.6.20), this observation implies $w(x,0,t) \leq -f(x,t) + \varepsilon$. Since $\varepsilon > 0$ was arbitrary we have $w(x,0,t) \leq -f(x,t)$.

To show $w(x,0,t) \geq -f(x,t)$ we first define, for $S > 0, k \in \mathbb{N}$,

$$\tilde{\sigma}_k = \frac{1}{S} \sup_{(Y,s) \in C_{2S\sigma_k}(x, \sigma_k f_k^-(x, t - S^2\sigma_k^2), t - S^2\sigma_k^2)} (f_k^-(x, t - S^2\sigma_k^2) - f_k^-(y,s)).$$

Observe that if $k$ is large enough (depending on $S, \delta$) then $(x, t - S^2\sigma_k^2) \in C_{1-\delta}(0,0)$.

Then, by construction, $\forall (Y,s) \in C_{2S\sigma_k}(x, \sigma_k f_k^-(x, t - S^2\sigma_k^2), t - S^2\sigma_k^2)$,

$$y_n - \sigma_k f_k^-(x, t - S^2\sigma_k^2) \leq -S\sigma_k \tilde{\sigma}_k \Rightarrow y_n \leq \sigma_k f_k^-(y,s) \Rightarrow u_k(Y,s) = 0.$$

Bounding the oscillation of $h_k$ as in the proof of Lemma 3.6.12, $u_k \in HPF(\tilde{\sigma}_k, 1, 4\tau_k) \subset HPF(\overline{\sigma}_k, 1, 4\tau_k)$ in $C_{S\sigma_k}(x, \sigma_k f_k^-(x, t - S^2\sigma_k^2), t)$, where $\overline{\sigma}_k = \max\{16\tau_k, \tilde{\sigma}_k\}$. Note, by Lemma 3.6.12, $\tilde{\sigma}_k \to 0$ and, therefore, $\overline{\sigma}_k \to 0$.

Apply Lemma 3.6.11 to conclude that

$$u_k \in HPF(C\overline{\sigma}_k, C\overline{\sigma}_k, 8\tau_k) \text{ in } C_{S\sigma_k/2}(x, \sigma_k f_k^-(x, t - S^2\sigma_k^2) + \alpha_k, t) \text{ where } |\alpha_k| \leq CS\sigma_k\overline{\sigma}_k. \tag{3.6.21}$$

Define $D_k \equiv f_k^-(x, t - S^2\sigma_k^2) + \alpha_k/\sigma_k + S/2$. Pick $S > 0$ large such that $D_k \geq 1$ and then, for large enough $k$, we have $D_k - \alpha_k/\sigma_k - f_k^-(x, t - S^2\sigma_k^2) - CS\overline{\sigma}_k > 0$ and $(x, \sigma D_k, t) \in C_{S\sigma_k/2}(x, \sigma_k f_k^-(x, t - S^2\sigma_k^2) + \alpha_k, t)$. The flatness condition, (3.6.21), gives

$$\begin{aligned} u_k(x, \sigma_k D_k, t) &\geq h_k(x, \sigma_k f_k^-(x, t - S^2\sigma_k^2) + \alpha_k, t - S^2\sigma_k^2/4)(S\sigma_k - CS\overline{\sigma}_k\sigma_k)/2 \\ &\geq (1-\tau_k) h(0,0) S\sigma_k (1 - C\overline{\sigma}_k)/2. \end{aligned} \tag{3.6.22}$$

Plugging this into the definition of $w_k$,

$$\begin{aligned} w_k(x, \sigma_k D_k, t) &\geq (1-\tau_k)S(1-C\overline{\sigma}_k)/2 - (1+\tau_k)D_k \\ &= (1-\tau_k)S(1-C\overline{\sigma}_k)/2 - (1+\tau_k)(f_k^-(x, t-S^2\sigma_k^2) + \alpha_k/\sigma_k + S/2) \\ &= -f_k^-(x, t-S^2\sigma_k^2) + o_k(1) = -f(x,t) + o_k(1). \end{aligned} \tag{3.6.23}$$

We would like to replace the left hand side of equation (3.6.23) with $w_k(x, \alpha, t)$, where $d$ does not depend on $k$. We accomplish this by means of barriers; for $\varepsilon > 0$ define $z_\varepsilon$ to be the unique solution to

$$\begin{aligned} \partial_t z_\varepsilon + \Delta z_\varepsilon = 0, \text{ in } C_{1-\delta}(0,0) \cap \{x_n > 0\} \\ z_\varepsilon = g_\varepsilon, \text{ on } \partial_p(C_{1-\delta}(0,0) \cap \{x_n > 0\}) \cap \{x_n = 0\} \\ z_\varepsilon = -2, \text{ on } \partial_p(C_{1-\delta}(0,0) \cap \{x_n > 0\}) \cap \{x_n > 0\}, \end{aligned} \tag{3.6.24}$$

where $g_\varepsilon \in C^\infty(C_{1-\delta}(0,0))$ and $-f(x,t) - 2\varepsilon < g_\varepsilon(x,t) < -f(x,t) - \varepsilon$. By standard parabolic theory, for any $\varepsilon > 0$ there exists an $\alpha > 0$ (which depends on $\varepsilon > 0$) such that $|x_n| < \alpha$ implies $|z_\varepsilon(x, x_n, t) - z_\varepsilon(x, 0, t)| < \varepsilon/2$. Pick $k$ large enough so that $\sigma_k < \alpha$. We know $w_k$ solves the adjoint heat equation on $\{x_n \geq \sigma_k\}$ and, by equations (3.6.24) and (3.6.23), $w_k \geq z_\varepsilon$ on $\partial_p(C_{1-\delta}(0,0) \cap \{x_n > \sigma_k\})$. Therefore, $w_k \geq z_\varepsilon$ on all of $C_{1-\delta}(0,0) \cap \{x_n > \sigma_k\}$.

Consequently,

$$w_k(x, \alpha, t) \geq z_\varepsilon(x, \alpha, t) \geq z_\varepsilon(x, 0, t) - \varepsilon/2 \geq -f(x,t) - 3\varepsilon.$$

As $k \to \infty$ we know $w_k(x, \alpha, t) \to w(x, \alpha, t) \leq w(x, 0, t)$. This gives the desired result. $\square$

The next step is to prove that the normal derivative of $w$ on $\{x_n = 0\}$ is zero. This will allow us to extend $w$ smoothly over $\{x_n = 0\}$ and obtain regularity for $f$.

**Lemma 3.6.14.** *Suppose the assumptions of Lemma 3.6.12 are satisfied and that $k$ is the* *subsequence identified in that lemma. Further suppose that $w$ is the limit function identified in Lemma 3.6.13. Then $\partial_n w = 0$, in the sense of distributions, on $C_{1/2}(0,0) \cap \{x_n = 0\}$.*

*Proof.* Rescale so $\rho_k \equiv 1$ and define $g(x,t) = 5 - 8(|x|^2 + |t|)$. For $(x,0,t) \in C_{1/2}(0,0)$ we observe $f(x,0,t) \leq 1 \leq g(x,0,t)$. We shall work in the following set

$$Z := \{(x, x_n, t) \mid |x|, |t| \leq 1, x_n \in \mathbb{R}\}.$$

For any, $\phi(x,t)$ define $Z^+(\phi)$ to be the set of points in $Z$ above the graph $\{(X,t) \mid x_n = \phi(x,t)\}$, $Z^-(\phi)$ as set of points below the graph and $Z^0(\phi)$ as the graph itself. Finally, let $\Sigma_k := \{u_k > 0\} \cap Z^0(\sigma_k g)$.

Recall, for any Borel set $A$, we define the "surface measure", $\mu(A) = \int_{-\infty}^{\infty} \mathcal{H}^{n-1}(A \cap \{s = t\})dt$. If $k$ is sufficiently large, and potentially adding a small constant to $g$, $\mu(Z^0(\sigma_k g) \cap \partial\{u_k > 0\} \cap C_{1/2}(0,0)) = 0$.

There are three claims, which together prove the desired result.

**Claim 1:**

$$\mu(\partial\{u_k > 0\} \cap Z^-(\sigma_k g)) \leq \frac{1}{(1-\tau_k)h_k(0,0)} \left( \int_{\Sigma_k} \partial_n u_k - 1 dxdt + \mu(\Sigma_k) \right) + C\sigma_k^2$$

*Proof of Claim 1*: For any positive $\phi \in C_0^\infty(C_1(0,0))$ we have

$$\begin{aligned}
\int_{\partial\{u_k>0\}} \phi d\mu &\leq \int_{\partial\{u_k>0\}} \phi \frac{h_k(Q,\tau)}{(1-\tau_k)h_k(0,0)} d\mu(Q,\tau) \\
&= \frac{1}{(1-\tau_k)h_k(0,0)} \int_{\{u_k>0\}} u_k(\Delta\phi - \partial_t\phi) dXdt \\
&= -\frac{1}{(1-\tau_k)h_k(0,0)} \int_{\{u_k>0\}} \nabla u_k \cdot \nabla\phi + u_k \partial_t \phi dXdt
\end{aligned} \tag{3.6.25}$$

(we can use integration by parts because, for almost every $t$, $\{u_k > 0\} \cap \{s = t\}$ is a set of finite perimeter). Let $\phi \to \chi_{Z^-(\sigma_k g)} \chi_{C_1}$ (as functions of bounded variation) and, since

$|t| > 3/4$ or $|x|^2 > 3/4$ implies $u(x, \sigma_k g(x,t), t) = 0$, equation (3.6.25) becomes

$$\mu(\partial\{u_k > 0\} \cap Z^-(\sigma_k g)) \leq -\frac{1}{h_k(0,0)(1-\tau_k)}\left(\int_{\Sigma_k} \frac{\nabla u_k \cdot \nu + \sigma_k u_k \operatorname{sgn}(t)}{\sqrt{1+\sigma_k^2(|\nabla_x g(x,t)|^2+1)}} d\mu\right), \tag{3.6.26}$$

where $\nu(x,t) = (\sigma_k \nabla g(x,t), -1)$ points outward spatially in the normal direction.

We address the term with $\operatorname{sgn}(t)$ first; the gradient bound on $u_k$ tells us that $|u_k| \leq C\sigma_k(1+\tau_k)h_k(0,0)$ on $\Sigma_k$, so

$$\left|\frac{\sigma_k}{(1-\tau_k)h_k(0,0)} \int_{\Sigma_k} \frac{\sigma_k u_k \operatorname{sgn}(t)}{\sqrt{1+\sigma_k^2(|\nabla_x g(x,t)|^2+1)}} d\mu\right| \leq C\sigma_k^2. \tag{3.6.27}$$

To bound the other term note that $\frac{d\mu}{\sqrt{1+\sigma_k^2(|\nabla_x g(x,t)|^2+1)}} = dxdt$ where the latter integration takes place over $E_k = \{(x,t) \mid (x, \sigma_k g(x,t), t) \in \Sigma_k\} \subset \{x_n = 0\}$. Then integrate by parts in $x$ to obtain

$$\begin{aligned}\int_{E_k} (\sigma_k \nabla g(x,t), -1) \cdot \nabla u_k(x, \sigma_k g(x,t), t) dxdt &= \int_{\partial E_k} \sigma_k u_k(x, \sigma_k g(x,t), t) \partial_\eta g d\mathcal{H}^{n-2} dt \\ - \int_{E_k} \sigma_k u_k(x, \sigma_k g(x,t), t)\Delta_x g(x,t) &+ \sigma_k^2 \partial_n u_k(x, \sigma_k g(x,t), t)|\nabla g|^2 dxdt \\ - \int_{E_k} \partial_n u_k(x, \sigma_k g(x,t), t) - 1 dxdt &+ \mathcal{L}^n(E_k),\end{aligned} \tag{3.6.28}$$

where $\eta$ is the outward space normal on $\partial E_k$. Since $u_k = 0$ on $\partial\Sigma_k$ the first term zeroes out.

The careful reader may object that $E_k$ may not be a set of finite perimeter and thus our use of integration by parts is not justified. However, for any $t_0$, we may use the coarea formula with $\chi_{\{u(x,\sigma_k g(x,t_0),t_0)>0\}} \in L^1$ and $\sigma_k g(-, t_0)$ smooth to get

$$\begin{aligned}\infty &> \int \sigma_k |\nabla g(x,t_0)| \chi_{\{u(x,\sigma_k g(x,t_0),t_0)>0\}} dx \\ &= \int_{-\infty}^{\infty} \int_{\{(x,t_0) \mid \sigma_k g(x,t_0) = r\}} \chi_{\{u(x,r,t_0)>0\}} d\mathcal{H}^{n-2}(x) dr.\end{aligned}$$

Thus $\{(x,t_0) \mid \sigma_k g(x,t_0) > r\} \cap \{(x,t) \mid u(x, \sigma_k g(x,t_0), t_0) > 0\}$ is a set of finite perimeter for almost every $r$. Equivalently, $\{(x,t_0) \mid \sigma_k(g(x,t_0)+\varepsilon) > 0\} \cap \{(x,t) \mid u(x, \sigma_k(g(x,t_0) + \varepsilon), t_0) > 0\}$ is a set of finite perimeter for almost every $\varepsilon \in \mathbb{R}$. Hence, there exists a $\varepsilon > 0$ aribtrarily small such that if we replace $g$ by $g+\varepsilon$ then $E_k \cap \{t = t_0\}$ will be a set of finite perimeter for almost every $t_0$. Since we can perturb $g$ slightly without changing the above arguments, we may safely assume that $E_k$ is a set of finite perimeter for almost every time slice.

Observe that $\Delta g$ is bounded above by a constant, $|u_k| \leq Ch(0,0)(1+\tau_k)\sigma_k$ on $\Sigma_k$, $|\partial_n u_k| \leq h(0,0)(1+\tau_k)$ and finally $\mu(\Sigma_k) \geq \mathcal{L}^n(E_k)$. Thus,

$$\mu(\partial\{u_k > 0\} \cap Z^-(\sigma_k g)) \leq \frac{1}{(1-\tau_k)h_k(0,0)} \left( \int_{E_k} \partial_n u_k - 1 dxdt + \mu(\Sigma_k) \right) + C\sigma_k^2.$$

As the difference between integrating over $E_k$ and integrating over $\Sigma_k$ is a factor of $\sqrt{1+\sigma_k^2}$ (which is comparable to $1+\sigma_k^2$, for $\sigma_k$ small) we can conclude

$$\mu(\partial\{u_k > 0\} \cap Z^-(\sigma_k g)) \leq \frac{1}{(1-\tau_k)h_k(0,0)} \left( \int_{\Sigma_k} \partial_n u_k - 1 dxdt + \mu(\Sigma_k) \right) + C\sigma_k^2,$$

which is of course the claim.

Note, arguing as in equations (3.6.27) and (3.6.28),

$$\int_{\Sigma_k} \frac{(\sigma_k \nabla_x g(x,t), 0, \sigma_k \mathrm{sgn}(t))}{\sqrt{1 + \|(\sigma_k \nabla_x g(x,t), 0, \sigma_k \mathrm{sgn}(t))\|^2}} \cdot (\nabla_x w_k, 0, w_k) d\mu \overset{k\to\infty}{\to} 0, \tag{3.6.29}$$

which will be useful to us later.

**Claim 2:**

$$\mu(\Sigma_k) - C_2\sigma_k^2 \leq \mu(\partial\{u_k > 0\} \cap Z^-(\sigma_k g)).$$

*Proof of Claim 2:* Let $\nu_k(x,t)$ the inward pointing measure theoretic space normal to $\partial\{u_k > 0\} \cap \{s = t\}$ at the point $x$. Note that for almost every $t$ it is true that $\nu_k$ exists

$\mathcal{H}^{n-1}$ almost everywhere. Defining $\nu_{\sigma_k g}(X,t) = \frac{1}{\sqrt{1+256\sigma_k^2|x|^2}}(-\sigma_k 16x, 1, 0)$, we have

$$\mu(\partial\{u_k > 0\} \cap Z^-(\sigma_k g)) = \int_{\partial\{u_k>0\}\cap Z^-(\sigma_k g)} \nu_k \cdot \nu_k d\mu \geq \int_{\partial^*\{u_k>0\}\cap Z^-(\sigma_k g)} \nu_k \cdot \nu_k d\mu \geq$$

$$\int_{\partial^*\{u_k>0\}\cap Z^-(\sigma_k g)} \nu_k \cdot \nu_{\sigma_k g} d\mu \overset{\text{div thm}}{=} -\int_{Z^-(\sigma_k g)\cap\{u_k>0\}} \text{div } \nu_{\sigma_k g} dX dt + \int_{\Sigma_k} 1 d\mu.$$

In the last equality above we use the fact that on $Z^0(\sigma_k g)$, $\nu_{\sigma_k g}$ agrees with upwards pointing space normal.

We compute $|\text{div } \nu_{\sigma_k g}| = \left|\frac{-16\sigma_k(n-1)}{\sqrt{1+256\sigma_k^2|x|^2}} + \frac{3\sigma_k^3(16*256)|x|^2}{\sqrt{1+256\sigma_k^2|x|^2}^3}\right| \leq C\sigma_k$. As the "width" of $Z^-(\sigma_k g) \cap \{u_k > 0\}$ is of order $\sigma_k$ we get the desired result.

**Claim 3:**

$$\int_{\Sigma_k} |\partial_n w_k| \overset{k\to\infty}{\to} 0.$$

*Proof of Claim 3:* Recall that $\partial_n u_k \leq (1+\tau_k)h(0,0)$, which implies, $\partial_n w_k \leq 0$. To show the limit above is at least zero we compute

$$\begin{aligned}
\int_{\Sigma_k} \partial_n w_k d\mu =& \int_{\Sigma_k} \frac{\partial_n u_k - 1}{\sigma_k h_k(0,0)} d\mu + \frac{\mu(\Sigma_k)}{\sigma_k h_k(0,0)} - \frac{(1+\tau_k)\mu(\Sigma_k)}{\sigma_k} \\
&\overset{\text{Claim1}}{\geq} \frac{(1-\tau_k)\mu(\partial\{u_k > 0\} \cap Z^-(\sigma_k g))}{\sigma_k} - \frac{(1+\tau_k)\mu(\Sigma_k)}{\sigma_k} - C\sigma_k \\
&\overset{\text{Claim2}}{\geq} \frac{(1-\tau_k)\mu(\Sigma_k)}{\sigma_k} - \frac{(1+\tau_k)\mu(\Sigma_k)}{\sigma_k} - \tilde{C}\sigma_k \\
&\geq -C'(\sigma_k + \frac{4\tau_k}{\sigma_k}) \to 0.
\end{aligned} \tag{3.6.30}$$

We can now combine these claims to reach the desired conclusion. We say that $\partial_n w = 0$, in the sense of distributions on $\{x_n = 0\}$, if, for any $\zeta \in C_0^\infty(C_{1/2}(0,0))$,

$$\int_{\{x_n=0\}} \partial_n w \zeta = 0.$$

Claim 3 implies

$$0 = \lim_{k\to\infty} \int_{\Sigma_k} \zeta \partial_n w_k d\mu. \tag{3.6.31}$$

On the other hand equation (3.6.29) (and $\zeta$ bounded) implies

$$\lim_{k\to\infty} \int_{\Sigma_k} \zeta \partial_n w_k d\mu = \lim_{k\to\infty} \int_{\Sigma_k} \zeta \nu_{\Sigma_k} \cdot (\nabla_X w_k, w_k) d\mu, \tag{3.6.32}$$

where $\nu_{\Sigma_K}$ is the unit normal to $\Sigma_k$ (thought of as a Lispchitz graph in $(x,t)$) pointing upwards. Together, equations (3.6.31), (3.6.32) and the divergence theorem in the domain $Z^+(\sigma_k g) \cap C_{1/2}(0,0)$ have as a consequence

$$\begin{aligned}
0 &= \lim_{k\to\infty} \int_{Z^+(\sigma_k g)} \mathrm{div}_{X,t}(\zeta(\nabla_X w_k, w_k)) dX dt \\
&= \lim_{k\to\infty} \int_{Z^+(\sigma_k g)} \nabla_X \zeta \cdot \nabla_X w_k + (\partial_t \zeta) w_k + \zeta(\Delta_X w_k + \partial_t w_k) dX dt \\
&\overset{\Delta w_k + \partial_t w_k = 0}{=} \int_{\{x_n > 0\}} \nabla_X w \cdot \nabla_X \zeta + (\partial_t \zeta) w dX dt \\
&\overset{\text{integration by parts}}{=} \int_{\{x_n = 0\}} w_n \zeta dx dt - \int_{\{x_n > 0\}} \zeta(\Delta_X w + \partial_t w) dX dt.
\end{aligned}$$

As $w$ is adjoint caloric this implies that $\int_{\{x_n=0\}} \partial_n w \zeta = 0$ which is the desired result. □

From here it is easy to conclude regularity of $f$.

**Corollary 3.6.15.** *Suppose the assumptions of Lemma 3.6.12 are satisfied and that $k$ is the subsequence identified in that lemma. Then $f \in C^\infty(C_{1/2}(0,0))$ and in particular the $\mathbb{C}^{2+\alpha,1+\alpha}$ norm of $f$ in $C_{1/4}(0,0)$ is bounded by an absolute constant.*

*Proof.* Extend $w$ by reflection across $\{x_n = 0\}$. By Lemma 3.6.14 this new $w$ satisfies the adjoint heat equation in all of $C_{1/2}(0,0)$ (recall a continuous weak solution to the adjoint heat equation in the cylinder is actually a classical solution to the adjoint heat equation). Since $\|w\|_{L^\infty(C_{3/4}(0,0))} \leq 2$, standard regularity theory yields the desired results about $-f = w|_{x_n=0}$. □

We can use this regularity to prove Lemma 3.6.6.

**Lemma** (Lemma 3.6.6)**.** *Let $\theta \in (0,1)$ and assume that $u \in HCF(\sigma,\sigma,\kappa)$ in $C_\rho(Q,\tau)$ in the direction $\nu$. There exists a constant $0 < \sigma_\theta < 1/2$ such that if $\sigma < \sigma_\theta$ and $\kappa \leq \sigma_\theta \sigma^2$ then $u \in \widetilde{HCF}(\theta\sigma,\theta\sigma,\kappa)$ in $C_{c(n)\rho\theta}(Q,\tau)$ in the direction $\overline{\nu}$ where $|\overline{\nu}-\nu| \leq C(n)\sigma$. Here $\infty > C(n), c(n) > 0$ are constants depending only on dimension.*

*Proof of Lemma 3.6.6.* Without loss of generality, let $(Q,\tau) = (0,0)$ and we will assume that the conclusions of the lemma do not hold. Choose a $\theta \in (0,1)$ and, by assumption, there exists $\rho_k, \sigma_k \downarrow 0$ and $\kappa_k/\sigma_k^2 \to 0$ such that $u \in HCF(\sigma_k,\sigma_k,\kappa_k)$ in $C_{\rho_k}(0,0)$ in the direction $\nu_k$ (which after a harmless rotation we can set to be $e_n$) but so that $u$ is not in $\widetilde{HCF}(\theta\sigma_k,\theta\sigma_k,\kappa_k)$ in $C_{c(n)\theta\rho_k}(0,0)$ in any direction $\nu$ with $|\nu_k - \nu| \leq C\sigma_k$ and for any constant $c(n)$. Let $u_k(X,t) = \frac{u(\rho_k X, \rho_k^2 t)}{\rho_k}$. It is clear that $u_k$ is adjoint caloric, that its zero set is a parabolic regular domain and that it is associated to an $h_k$ which satisfies $\|\log(h_k)\|_{\mathbb{C}^{1,1/2}} \leq C\|\log(h)\|_{\mathbb{C}^{1,1/2}}$ and $h_k(0,0) \equiv h(0,0)$.

By Lemma 3.6.12 we know that there exists a continuous function $f$ such that $\partial\{u_k > 0\} \to \{(X,t) \mid x_n = f(x,t)\}$ in the Hausdorff distance sense. Corollary 3.6.15 implies that there is a universal constant, call it $K$, such that

$$f(x,t) \leq f(0,0) + \nabla_x f(0,0)\cdot x + K(|t| + |x|^2) \tag{3.6.33}$$

for $(x,t) \in C_{1/4}(0,0)$. Since $(0,0) \in \partial\{u_k > 0\}$ for all $k$, $f(0,0) = 0$. If $\theta \in (0,1)$, then there exists a $k$ large enough (depending on $\theta$ and the dimension) such that

$$f_k^+(x,t) \leq \nabla_x f(0,0)\cdot x + \theta^2/4K,\ \forall (x,t) \in C_{\theta/(4K)}(0,0),$$

where $f_k^+$ is as in Lemma 3.6.12.

Let

$$\nu_k := \frac{(-\sigma_k \nabla_x f(0,0), 1)}{\sqrt{1 + |\sigma_k \nabla_x f(0,0)|^2}}$$

and compute

$$x \cdot \nu_k \geq \theta^2 \sigma_k / 4K \Rightarrow x_n \geq \sigma_k x' \cdot \nabla_x f(0,0) + \theta^2 \sigma_k / 4K \geq f_k^+(x,t). \tag{3.6.34}$$

Therefore, if $(X,t) \in C_{\theta/(4K)}(0,0)$ and $x \cdot \nu_k \geq \theta^2 \sigma_k / 4K$, then $u_k(X,t) > 0$. Arguing similarly for $f_k^-$ we can see that $u_k \in \widetilde{HCF}(\theta\sigma_k, \theta\sigma_k, \kappa_k)$ in $C_{\theta/4K}(0,0)$ in the direction $\nu_k$. It is easy to see that $|\nu_k - e_n| \leq C\sigma_k$ and so we have the desired contradiction. □

## 3.7 Higher Regularity

We begin by recalling the partial hodograph transform (see (KS80), Chapter 7 for a short introduction in the elliptic case). Here, and throughout the rest of the paper, we assume that $(0,0) \in \partial\Omega$ and that, at $(0,0)$, $e_n$ is the inward pointing normal to $\partial\Omega \cap \{t = 0\}$. Before we can use the hodograph transform, we must prove that $\nabla u$ extends smoothly to the boundary.

**Lemma 3.7.1.** *Let $s \in (0,1)$ and $\partial\Omega$ be a $\mathbb{C}^{1+s,(1+s)/2}$ domain with $\log(h) \in \mathbb{C}^{s,s/2}(\mathbb{R}^{n+1})$. Then $u \in \mathbb{C}^{1+s,(1+s)/2}(\overline{\Omega})$.*

*Proof.* We will show that $u$ has the desired regularity in a neighborhood of $(0,0)$. For any $R > 0$ let $H_R$ be as in Corollary 3.3.5: $H_R(X,t) = \varphi_R(X,t)\nabla u(X,t) - w_R(X,t)$. Therefore, we may estimate:

$$\begin{aligned}|\nabla u(X,t) - h(0,0)e_n| =& |H_R(X,t) - h(0,0)e_n| + |w_R(X,t)| \\ &\leq \int_{\partial\Omega} |h(Q,\tau)\hat{n}(Q,\tau)\varphi_R(Q,\tau) - h(0,0)e_n| d\hat{\omega}^{X,t} + |w_R(X,t)|.\end{aligned} \tag{3.7.1}$$

Since $\partial\Omega$ is locally given by a $\mathbb{C}^{1+s,(1+s)/2}$ graph, for any $\delta > 0$ there exists an $R_\delta > 0$ such that $\partial\Omega \cap C_r(Q,\tau)$ is $\delta$-flat for any $(Q,\tau) \in C_{100}(0,0)$ and any $r < R_\delta$. In particular, we can ensure that Lemma 3.2.1 applies at all $r < R_\delta/4$ for a $\varepsilon > 0$ such that $1 - \varepsilon > \frac{1+s}{2}$.

Arguing in the same way as in the proof of Lemma 3.3.4, we can deduce that

$$|w_R(X,t)| \leq C\frac{\|(X,t)\|^{(1+s)/2}}{R^{1/2}}. \tag{3.7.2}$$

Furthermore, we may deduce (in much the same manner as (3.6.2)), that if $\|(X,t)\| \leq r$ and $k_0 \in \mathbb{N}$ is such that $2^{-k_0-1} \leq r < 2^{-k_0}$, then

$$1 - \widehat{\omega}^{(X,t)}(C_{2^{-j}}(0,0)) \leq C2^{j(1+s)/2}r^{(1+s)/2}, \forall j < k_0 - 2. \tag{3.7.3}$$

In light of the estimates (3.7.1), (3.7.2) and (3.7.3), we may conclude

$$\begin{aligned}|\nabla u(X,t) - h(0,0)e_n| \leq{} & C\int_{\Delta_{4r}(0,0)} (4r)^s d\hat{\omega}^{(X,t)} + C\frac{r^{(1+s)/2}}{R^{1/2}} + C\hat{\omega}^{(X,t)}(\partial\Omega\setminus C_R(0,0)) \\ & + C\left(\int_{\Delta_R(0,0)\setminus\Delta_1(0,0)} d\hat{\omega}^{(X,t)} + \sum_{j=0}^{k_0-2}\int_{\Delta_{2^{-j}}(0,0)\setminus\Delta_{2^{-(j+1)}}(0,0)} 2^{-js}d\hat{\omega}^{(X,t)}\right),\end{aligned} \tag{3.7.4}$$

where $C$ depends on the Hölder norm of $h$, of $\hat{n}$ and on $R$. Also as above, $k_0$ is such that $2^{-k_0-1} \leq r < 2^{-k_0}$.

We may bound

$$\begin{aligned}\hat{\omega}^{(X,t)}(\Delta_R(0,0)\setminus\Delta_1(0,0)) \leq 1 - \widehat{\omega}^{(X,t)}(\Delta_1(0,0)) &\overset{eq.(3.7.3)}{\leq} C_R r^{(1+s)/2} \\ \hat{\omega}^{(X,t)}(\partial\Omega\setminus\Delta_R(0,0)) \leq 1 - \widehat{\omega}^{(X,t)}(\Delta_R(0,0)) &\overset{eq.(3.7.3)}{\leq} C_R r^{(1+s)/2} \\ \hat{\omega}^{(X,t)}(\Delta_{2^{-j}}(Q,\tau)\setminus\Delta_{2^{-(j+1)}}(0,0)) \leq 1 - \widehat{\omega}^{(X,t)}(\Delta_{2^{-j}}(0,0)) &\overset{eq.(3.7.3)}{\leq} C2^{j(1+s/2}r^{(1+s)/2}.\end{aligned}$$

Plug these estimates into equation (3.7.4) to obtain

$$\begin{aligned}|\nabla u(X,t) - h(0,0)e_n| \leq{} & Cr^s + Cr^{(1+s)/2} + Cr^{(1+s)/2}\sum_{j=0}^{[\log_2(r^{-1})]} 2^{j(1-s)/2)} \\ \leq{} & Cr^s\left(1 + r^{(1-s)/2}\frac{(2^{[\log_2(r^{-1})]})^{(1-s)/2} - 1}{2^{(1-s)/2} - 1}\right) \leq C_s r^s.\end{aligned} \tag{3.7.5}$$

Since $r = \|(X,t)\|$ we have proven that $\nabla u \in \mathbb{C}^{s,s/2}(\overline{\Omega})$. $\square$

With this regularity in hand, we may define $F : \Omega \to \mathbb{H}^+ := \{(x, x_n, t) \mid x_n > 0\}$ by $(x, x_n, t) = (X,t) \mapsto (Y,t) = (x, u(x,t), t)$. In a neighborhood of 0 we know that $u_n \neq 0$ and so $DF$ is invertible on each time slice (since $\nabla u \in C(\overline{\Omega})$). By the inverse function theorem, there is some neighborhood, $\mathcal{O}$, of $(0,0)$ in $\Omega$ that is mapped diffeomorphically to $U$, a neighborhood of $(0,0)$ in the upper half plane. Furthermore, this map extends in a $\mathbb{C}^{s,s/2}$ fashion from $\overline{\mathcal{O}^+}$ to $\overline{U}$ (by Proposition 3.6.1 and Lemma 3.7.1). Let $\psi : \overline{U} \to \mathbb{R}$ be given by $\psi(Y,t) = x_n$, where $F(X,t) = (Y,t)$. Because $F$ is locally one-to-one, $\psi$ is well defined.

If $\nu_{Q,\tau}$ denotes the spatial unit normal pointing into $\Omega$ at $(Q,\tau)$ then $u$ satisfies

$$\begin{aligned} u_t(X,t) + \Delta u(X,t) &= 0, \ (X,t) \in \Omega^+ \\ u_{\nu_{Q,\tau}}(Q,\tau) &= h(Q,\tau), \ (Q,\tau) \in \partial\Omega. \end{aligned}$$

After our change of variables these equations become

$$0 = -\frac{\psi_t}{\psi_n} + \frac{1}{2}\left(\frac{1}{\psi_n^2}\right)_n + \sum_{i=1}^{n-1}\left(-\left(\frac{\psi_i}{\psi_n}\right)_i + \frac{1}{2}\left(\frac{\psi_i^2}{\psi_n^2}\right)_n\right) \tag{3.7.6}$$

on $U$ and

$$\psi_n(y,0,t)h(y,\psi(y,0,t),t) = \sqrt{1 + \sum_{i=1}^{n-1}\psi_i^2(y,0,t)} \tag{3.7.7}$$

on the boundary.

**Remark 3.7.2.** *The following are true of $\psi$:*

- *Let $k \geq 1$. If $\partial\Omega$ is a $\mathbb{C}^{k+s,(k+s)/2}$ graph and $\log(h) \in \mathbb{C}^{k-1+s,(k-1+s)/2}$ then $u \in \mathbb{C}^{k+s,(k+s)/2}(\overline{\Omega})$.*

- *Let $k \geq 0$ be such that $h \in \mathbb{C}^{k+\alpha,(k+\alpha)/2}(\partial\Omega)$ and $\psi|_{\{y_n=0\}} \in \mathbb{C}^{k+1+s,(k+1+s)/2}$ for $0 < s \leq \alpha$. If $\tilde{h}(y,t) = h(y,\psi(y,0,t),t)$, then $\tilde{h} \in \mathbb{C}^{k+\alpha,(k+\alpha)/2}(\{y_n = 0\})$.*

- $\psi_n > 0$ *in* $\overline{U}$.

*Justification.* Let us address the first claim. When $k \geq 2$ we note that $\overline{u}(x, x_n, t) \equiv u(x, \psi(x, 0, t) + x_n, t)$ is the strong solution of an adjoint parabolic equation in the upper half space with zero boundary values and coefficients in $\mathbb{C}^{k-1+s,(k+s)/2-1}$. Standard parabolic regularity theory then gives the desired result. When $k = 1$, this is simply Lemma 3.7.1.

When $k = 0$ the second claim follows from a difference quotient argument and the fact that $|\psi(x + y, t + s) - \psi(x, t)| \leq C(|y| + |s|^{1/2})$. We $k \geq 1$ take a derivative and note $\partial_i \tilde{h} = \partial_i h + \partial_n h \partial_i \psi$. As $\psi$ has one more degree of differentiability than $h$ it is clear that $\partial_n h \partial_i \psi$ is just as regular as $\partial_n h$. We can argue similarly for higher spatial derivatives and for difference quotients or derivatives in the time direction.

Our third claim follows from the assumption that $e_n$ is the inward pointing normal at $(0, 0)$ and that $\partial_n u(0, 0) > 0$ in $\mathcal{O}$. $\square$

To prove higher regularity we will use two weighted Schauder-type estimates due to Lieberman (Lie86) for parabolic equations in a half space. Before we state the theorems, let us introduce weighted Hölder spaces (the reader should be aware that our notation here is non-standard).

**Definition 3.7.3.** *Let $\mathcal{O} \subset \mathbb{R}^{n+1}$ be a bounded open set. For $a, b \notin \mathbb{Z}$ define*

$$\|u\|_{\mathbb{C}_b^{a,a/2}(\mathcal{O})} = \sup_{\delta > 0} \delta^{a+b} \|u\|_{\mathbb{C}^{a,a/2}(\mathcal{O}_\delta)}$$

*where $\mathcal{O}_\delta = \{(X, t) \in \mathcal{O} \mid \mathrm{dist}((X, t), \partial\mathcal{O}) \geq \delta\}$. It should be noted that $\mathbb{C}_{-a}^{a,a/2} \equiv \mathbb{C}^{a,a/2}$.*

For the sake of brevity, the following are simplified versions of (Lie86), Theorems 6.1 and 6.2 (the original theorems deal with a more general class of domains, operators and boundary values that we do not need here).

**Theorem 3.7.4.** *Let $v$ be a solution to the boundary value problem*

$$
\begin{aligned}
v_t(X,t) - \sum_{ij} p_{ij}(X,t) D_{ij} v(X,t) &= f(X,t),\ (X,t) \in U \\
v(x,0,t) &= g(x,0,t),\ (x,0,t) \in \overline{U},
\end{aligned} \tag{3.7.8}
$$

*where $P = (p_{ij})$. Let $a > 2, b > 1$ be non-integral real numbers such that $P \in \mathbb{C}^{a-2,a/2-1}(U)$, $f \in \mathbb{C}^{a-2,a/2-1}_{2-b}(U)$ and $g \in \mathbb{C}^{b,b/2}(\overline{U} \cap \{x_n = 0\})$. If $v \in \mathbb{C}^{a,a/2}_{-b}(U)$ with $v|_{\partial U \setminus \{x_n=0\}} = 0$, there exists a constant $C > 0$ (depending on the ellipticity, the Hölder norms of $p_{ij}, m$ and the dimension) such that*

$$
\|v\|_{\mathbb{C}^{a,a/2}_{-b}(U)} \leq C \left( \|g\|_{\mathbb{C}^{b,b/2}} + \|v\|_{L^\infty} + \|f\|_{\mathbb{C}^{a-2,a/2-1}_{2-b}} \right).
$$

**Theorem 3.7.5.** *Let $v$ be a solution to the boundary value problem*

$$
\begin{aligned}
v_t(X,t) - \sum_{ij} p_{ij}(X,t) D_{ij} v(X,t) &= f(X,t),\ (X,t) \in U \\
\vec{m}(x,0,t) \cdot \nabla_x v(x,0,t) &= g(x,0,t),\ (x,0,t) \in \overline{U},
\end{aligned} \tag{3.7.9}
$$

*where $P = (p_{ij})$ is uniformly elliptic and $m_n(x,0,t) \geq c > 0$. Let $a > 2, b > 1$ be non-integral real numbers with $p_{ij} \in \mathbb{C}^{a-2,a/2-1}(U), m(x,0,t) \in \mathbb{C}^{b-1,(b-1)/2}(\overline{U} \cap \{x_n = 0\}), f \in \mathbb{C}^{a-2,a/2-1}_{2-b}(U)$ and $g \in \mathbb{C}^{b-1,b/2-1/2}(\overline{U} \cap \{x_n = 0\})$. Then, if $v \in \mathbb{C}^{a,a/2}_{-b}(U)$ and $v|_{\partial U \setminus \{x_n=0\}} = 0$, there exists a constant $C > 0$ (depending on the ellipticity, the Hölder norms of $p_{ij}, m$ and the dimension) such that*

$$
\|v\|_{\mathbb{C}^{a,a/2}_{-b}(U)} \leq C \left( \|g\|_{\mathbb{C}^{b-1,b/2-1/2}} + \|v\|_{L^\infty} + \|f\|_{\mathbb{C}^{a-2,a/2-1}_{2-b}} \right).
$$

With this theorem in hand we can use an iterative argument (modeled after one in (Jer90) to prove optimal Hölder regularity. To reduce clutter we define, for $f$ a function and

$\vec{v} \in \mathbb{R}^{n+1}$ with $v_n = 0$,

$$\delta_{\vec{v}}^2 f(X,t) = f((X,t) + \vec{v}) + f((X,t) - \vec{v}) - 2f(X,t). \tag{3.7.10}$$

It will also behoove us to define the parabolic length of a vector $\vec{v} = (v_x, v_n, v_t)$ by

$$\|\vec{v}\|_p = |(v_x, v_n)| + |v_t|^{1/2}. \tag{3.7.11}$$

Recall that

$$\|f\|_{L^\infty} + \sup_{\vec{v},(X,t)\in\mathbb{R}^{n+1}} \frac{|\delta_{\vec{v}}^2 f(X,t)|}{\|\vec{v}\|_p^\alpha} = \|f\|_{\mathbb{C}^{\alpha,\alpha/2}},$$

(see, e.g. (Ste70), Chapter 5, Proposition 8).

**Proposition 3.7.6.** *Let* $\Omega \subset \mathbb{R}^{n+1}$ *be a parabolic regular domain, and* $k \in \mathbb{N}, \alpha \in (0,1)$ *such that* $\log(h) \in \mathbb{C}^{k+\alpha,(k+\alpha)/2}(\mathbb{R}^{n+1})$*. There is a* $\delta_n > 0$ *such that if* $\delta_n \geq \delta > 0$ *and* $\Omega$ *is* $\delta$*-Reifenberg flat then* $\Omega$ *is a* $\mathbb{C}^{k+1+\alpha,(k+1+\alpha)/2}(\mathbb{R}^{n+1})$ *domain.*

*Proof.* Let us first prove the theorem for $k = 0$. By Proposition 3.6.1 and Remark 3.7.2 we know that $\psi \in \mathbb{C}^{1+s,(1+s)/2}(\overline{U})$. For any vector $\vec{v} \in \mathbb{R}^{n+1}$ with $v_n = 0$ define $w^\varepsilon(X,t) = \psi((X,t) + 2\varepsilon\vec{v}) - \psi((X,t))$. It is then easy to check that $w^\varepsilon(X,t)$ satisfies the following oblique derivative problem:

$$\begin{aligned} -\partial_t w^\varepsilon(X,t) - \sum_{ij} p_{ij}((X,t) + \varepsilon\vec{v}) D_{ij} w^\varepsilon(X,t) = f^\varepsilon(X,t),\ (X,t) \in U \\ \vec{m}(y,0,t) \cdot \nabla w^\varepsilon(y,0,t) = g^\varepsilon(y,0,t),\ (x,0,t) \in \overline{U}. \end{aligned} \tag{3.7.12}$$

Where

$$
\begin{aligned}
(p_{ij}(X,t)) &= \begin{pmatrix} -1 & 0 & 0 & \dots & \frac{\psi_1(X,t)}{\psi_n(X,t)} \\ 0 & -1 & 0 & \dots & \frac{\psi_2(X,t)}{\psi_n(X,t)} \\ \vdots & 0 & \ddots & \dots & \vdots \\ \frac{\psi_1(X,t)}{\psi_n(X,t)} & \dots & \frac{\psi_i(X,t)}{\psi_n(X,t)} & \dots & -\left(\frac{\sqrt{1+\sum_{i=1}^{n-1}\psi_i(X,t)^2}}{\psi_n(X,t)}\right)^2 \end{pmatrix}, \\
\vec{m}_j(y,0,t) &= \int_0^1 \frac{\partial}{\partial \psi_j} G(\nabla_X \psi(y,0,t) + s(\nabla_X \psi((y,0,t)+\varepsilon\vec{v}) - \nabla_X \psi(y,0,t)))ds, \\
G(\nabla_X \psi) &= \frac{\sqrt{1+\sum_{i=1}^{n-1}\psi_i^2}}{\psi_n}, \\
f^\varepsilon(X,t) &= \sum_{ij}(p_{ij}(X,t) - p_{ij}((X,t)+\varepsilon\vec{v}))D_{ij}\psi(X,t), \\
g^\varepsilon(y,0,t) &= \tilde{h}((y,0,t)+\varepsilon\vec{v}) - \tilde{h}(y,0,t).
\end{aligned}
$$

It is a consequence of $\psi_n > 0$ (see Remark 3.7.2) that $(p_{ij})$ is uniformly elliptic and $m_n(y,0,t) \geq c > 0$ is uniformly in $(y,t)$. Furthermore, $p_{ij}(X+\varepsilon v, t)$ and $m(y+\varepsilon\vec{v}, 0, t)$ are $C^{s,s/2}$ Hölder continuous, uniformly in $\varepsilon, \vec{v}$.

To apply Theorem 3.7.5 we need that $w^\varepsilon$ is in a weighted Hölder space. Apply Theorem 3.7.4 with $a = 2+s, b = 1+s$ to $\psi$ in the space $U \cap \{x_n \geq \delta\}$ and let $\delta \downarrow 0$ to obtain the *a priori* estimate that $\psi \in \mathbb{C}^{2+s,1+s/2}_{-1-s}(U)$ (note that $\psi|_{x_n=\delta}$ is uniformly in $\mathbb{C}^{1+s,(1+s)/2}$ by Remark 3.7.2). As such, $w^\varepsilon \in \mathbb{C}^{2+\eta,1+\eta/2}_{-1-s}$, uniformly in $\varepsilon > 0$ for any $0 < \eta << s$.

We will now compute the Hölder norm of $g^\varepsilon$ and the weighted Hölder norm of $f^\varepsilon$. For any $(x,0,t),(y,0,r) \in \overline{U}$,

$$
\begin{aligned}
2\|\tilde{h}\|_{\mathbb{C}^{\alpha,\alpha/2}}(|x-y|^s + |t-r|^{s/2})\varepsilon^{\alpha-s} \geq \\
\min\{2(|x-y|^\alpha + |t-r|^{\alpha/2})\|\tilde{h}\|_{\mathbb{C}^\alpha}, 2\varepsilon^\alpha\|\tilde{h}\|_{\mathbb{C}^\alpha}\} \geq \\
|\tilde{h}(x,0,t) - \tilde{h}((x,0,t)+\varepsilon\vec{v}) - \tilde{h}(y,0,r) + \tilde{h}((y,0,r)+\varepsilon\vec{v})|.
\end{aligned}
\tag{3.7.13}
$$

Thus $\|\tilde{h}(-) - \tilde{h}(-+\varepsilon\vec{v})\|_{\mathbb{C}^{s,s/2}} \leq C\varepsilon^{\alpha-s}$.

For $d > 0$ we have that

$$|f^{\varepsilon}(x,d,t) - f^{\varepsilon}(y,d,r)| \leq |p_{ij}((y,d,r)+\varepsilon\vec{v}) - p_{ij}(y,d,r)||D_{ij}\psi(x,d,t) - D_{ij}\psi(y,d,r)|$$

$$+|p_{ij}((x,d,t)+\varepsilon\vec{v}) - p_{ij}(x,d,t) - p_{ij}((y,d,r)+\varepsilon\vec{v}) + p_{ij}(y,d,r)||D_{ij}\psi(x,d,t)|.$$

Arguing as in equation (3.7.13) we can bound the first term on the right hand side by

$$d^{s-\eta-1}\|\psi\|_{\mathbb{C}^{2+\eta,1+\eta/2}_{-1-s}(U)}(|x-y|^{\eta} + |t-r|^{\eta/2})\|\nabla\psi\|_{\mathbb{C}^{s,s/2}}\varepsilon^{s-\eta}.$$

The second term is more straightforward and can be bounded by

$$\|\nabla\psi\|_{\mathbb{C}^{s,s/2}}\varepsilon^{s}d^{s-\eta-1}\|\psi\|_{\mathbb{C}^{2+\eta,1+\eta/2}_{-1-s}(U)}(|x-y|^{\eta} + |t-r|^{\eta/2}).$$

Therefore $\|f^{\varepsilon}\|_{\mathbb{C}^{\eta,\eta/2}_{1-s}} \leq C(\varepsilon^{s-\eta} + \varepsilon^{s})$.

We may apply Theorem 3.7.5 to obtain

$$\|w^{\varepsilon}\|_{\mathbb{C}^{2+\eta,1+\eta/2}_{-1-s}} \leq C(\varepsilon^{s-\eta} + \varepsilon^{s} + \varepsilon^{\alpha-s} + \varepsilon) \tag{3.7.14}$$

(as $\|w^{\varepsilon}\|_{L^{\infty}} \leq C\varepsilon$). The reader may be concerned that $w^{\varepsilon}$ is not zero on the boundary of $U$ away from $\{x_n = 0\}$. However, we can rectify this by multiplying with a cutoff function and hiding the resulting error on the right hand side (see, e.g. the proof of Theorem 6.2 in (ADN59)).

We claim that if $w^{\varepsilon} \in \mathbb{C}^{2+\eta,1+\eta/2}_{-1-s}$ then in fact $w_{\varepsilon}|_{\{x_n=0\}} \in \mathbb{C}^{1+s,(1+s)/2}$ for any $\eta > 0$. For any $i = 1,...,n-1$, the fundamental theorem of calculus gives (recall the notation in

(3.7.10) and (3.7.11))

$$\begin{aligned}
|\delta^2_{\vec{v}} D_i w^\varepsilon(x,0,t)| &\leq |\delta^2_{\vec{v}} D_i w^\varepsilon(x,\|\vec{v}\|_p,t)| + \int_0^{\|\vec{v}\|_p} |\delta^2_{\vec{v}} w^\varepsilon_{in}(x,r,t)| dr \\
&\leq C\|w^\varepsilon\|_{\mathbb{C}^{2+\eta,1+\eta/2}_{-1-s}} \|\vec{v}\|_p^{-1+s-\eta} \|\vec{v}\|_p^{1+\eta} + C\|w^\varepsilon\|_{\mathbb{C}^{2+\eta,1+\eta/2}_{-1-s}} \int_0^{\|\vec{v}\|_p} \frac{\|\vec{v}\|_p^\eta}{r^{1-(s-\eta)}} dr \\
&\leq C_{s,\eta} \|w^\varepsilon\|_{\mathbb{C}^{2+\eta,1+\eta/2}_{-1-s}} \|\vec{v}\|_p^s.
\end{aligned} \tag{3.7.15}$$

Therefore, for $i = 1, ..., n-1$ we have

$$\begin{aligned}
|\delta^2_{\varepsilon\vec{v}} D_i \psi(x,0,t)| =& |D_i w^\varepsilon((x,0,t)) - D_i w^\varepsilon((x,0,t) - \varepsilon\vec{v})| \\
&\leq \|D_x w^\varepsilon\|_{\mathbb{C}^{s,s/2}} \|\varepsilon \vec{v}_x\|_p^s \\
&\overset{\text{eqn (3.7.15)}}{\leq} C_{s,\eta} \|w^\varepsilon\|_{\mathbb{C}^{2+\eta,1+\eta/2}_{-1-s}} \|\varepsilon \vec{v}_x\|_p^s \\
&\overset{\text{eqn (3.7.14)}}{\leq} C_{s,\eta}(\varepsilon^{s-\eta} + \varepsilon^{\alpha-s} + \varepsilon + \varepsilon^s) \|\varepsilon\vec{v}\|_p^s.
\end{aligned} \tag{3.7.16}$$

If $\|\vec{v}\|_p = 1$ and points either completely in a spacial or the time direction then we can conclude that $\psi \in \mathbb{C}^{1+\beta,(1+\beta)/2}(\overline{U} \cap \{x_n = 0\})$, where $\beta = \min\{\alpha, 2s - \eta\}$. As such, $\partial\Omega$ is a $\mathbb{C}^{1+\beta,(1+\beta)/2}$ domain and, invoking Remark 3.7.2, we can conclude that $\psi \in \mathbb{C}^{1+\beta,(1+\beta)/2}(\overline{U})$. Repeat this argument until $\beta = \alpha$ to get optimal regularity.

When $k = 1$ (that is $\log(h) \in \mathbb{C}^{1+\alpha,(1+\alpha)/2}$) we want to show that $\psi \in \mathbb{C}^{2+s,1+s/2}$ for some $s$. Then we will invoke classical Schauder theory. We can argue almost exactly as above, except that equation (3.7.13) cannot detect regularity in $\log(h)$ above $\mathbb{C}^{1,1/2}$. The argument above tells us that $\psi \in \mathbb{C}^{1+s,(1+s)/2}$ for any $s < 1$, thus, $\tilde{h} \in \mathbb{C}^{1+\alpha,(1+\alpha)/2}$.

Let $\vec{\xi}, \vec{v} \in \mathbb{R}^{n+1}$ be such that $\xi_n, v_n = 0$. Then, in the same vein as equation (3.7.13), we can estimate

$$\begin{aligned}
|\delta^2_{\vec{\xi}} \tilde{h}((x,0,t) + \vec{v}) - \delta^2_{\vec{\xi}} \tilde{h}((x,0,t))| \leq& \|h\|_{\mathbb{C}^{1+\alpha,(1+\alpha)/2}} \min\{2\|\vec{\xi}\|_p^{1+\alpha}, 3\|\vec{v}\|_p\} \\
\leq& 3\|h\|_{\mathbb{C}^{1+\alpha,(1+\alpha)/2}} \|\vec{\xi}\|_p^s \|\vec{v}\|_p^{1-\frac{s}{1+\alpha}}.
\end{aligned} \tag{3.7.17}$$

Consequently, $\|\tilde{h}(-) - \tilde{h}(- + \vec{v})\|_{\mathbb{C}^{s,s/2}} \leq C\|\vec{v}\|_p^{1-\frac{s}{1+\alpha}}$. We may then repeat the argument above until we reach equation (3.7.14), which now reads

$$\|w^{\varepsilon}\|_{\mathbb{C}^{2+\eta,1+\eta/2}_{-1-s}} \leq C(\varepsilon^{s-\eta} + \varepsilon^{s} + \varepsilon^{1-\frac{s}{1+\alpha}} + \varepsilon). \tag{3.7.18}$$

Resume the argument until equation (3.7.16), which is now,

$$|\delta^2_{\varepsilon\vec{v}} D_i\psi(x,0,t)| \leq C_{s,\eta}(\varepsilon^{s-\eta} + \varepsilon^{s} + \varepsilon^{1-\frac{s}{1+\alpha}} + \varepsilon)\|\varepsilon\vec{v}\|_p^{s}. \tag{3.7.19}$$

If $\|\vec{v}\|_p = 1$ and points either completely in the spacial or time direction then we can conclude that $\psi \in \mathbb{C}^{1+\beta,(1+\beta)/2}(\{x_n = 0\})$ where $\beta = \min\{2s-\eta, 1+s, 1+s-\frac{s}{1+\alpha}\}$. Pick $\eta, s$ such that $2s - \eta > 1$ and we have that there is some $\gamma \in (0,1)$ such that $\psi \in \mathbb{C}^{2+\gamma,1+\gamma/2}(\{x_n = 0\})$. Remark 3.7.2 ensures that

$$\psi \in \mathbb{C}^{2+\gamma,1+\gamma/2}(\overline{U}), \tag{3.7.20}$$

for some $\gamma \in (0,1)$.

Now that we have the *a priori* estimate (3.7.20), we may apply Theorem 3.7.5 to $w^{\varepsilon}$, the solution of (3.7.12), but with $a = 2+\beta, b = 2+\beta$. In this form, Theorem 3.7.5 comports with classical Schauder theory. An iterative argument one similar to the above, but substantially simpler, yields optimal regularity. In fact, we can use the same iterative argument to prove that $\psi \in \mathbb{C}^{k+\alpha,(k+\alpha)/2}$ given $\psi \in \mathbb{C}^{k-1+\alpha,(k-1+\alpha)/2}$. Thus the full result follows. □

We have almost completed a proof of Theorem 3.1.11–we need only to discuss what happens when $\log(h)$ is analytic. However, the anisotropic nature of the heat equation makes analyticity the wrong notion of regularity to consider.

**Definition 3.7.7.** *We say that a function $F(X,t)$ is analytic in $X$ and of the second Gevrey class in $t$ if there are constants $C, \kappa$ such that*

$$|D_X^{\ell}\partial_t^m F| \leq C\kappa^{|\ell|+2m}(|\ell| + 2m)!.$$

Let us recall Theorem 3.1.11:

**Theorem** (Theorem 3.1.11)**.** *Let $\Omega \subset \mathbb{R}^{n+1}$ be a parabolic regular domain with* $\log(h) \in \mathbb{C}^{k+\alpha,(k+\alpha)/2}(\mathbb{R}^{n+1})$ *for $k \geq 0$ and $\alpha \in (0,1)$. There is a $\delta_n > 0$ such that if $\delta_n \geq \delta > 0$ and $\Omega$ is $\delta$-Reifenberg flat then $\Omega$ is a $\mathbb{C}^{k+1+\alpha,(k+1+\alpha)/2}(\mathbb{R}^{n+1})$ domain.*

*Furthermore, if* $\log(h)$ *is analytic in $X$ and in the second Gevrey class in $t$ then, under the assumptions above, we can conclude that $\Omega$ is the region above the graph of a function which is analytic in the spatial variables and in the second Gevrey class in $t$. Similarly, if* $\log(h) \in C^\infty$*, then $\partial\Omega$ is locally the graph of a $C^\infty$ function.*

It is clear that Proposition 3.7.6 implies the above theorem except for the statement when $\log(h)$ analytic in $X$ and second Gevrey class in $t$. This follows from a theorem of Kinderlehrer and Nirenberg:

**Theorem 3.7.8.** *[Modified Theorem 1 in (KN78)] Let $v \in C^\infty(\overline{U})$ be a solution to*

$$-F(D^2 v, Dv, v, X, t) + v_t = 0 \; (X,t) \in U$$

$$v_n = \Phi(\partial_1 v, ...., \partial_{n-1} v, v, x, t), \; (x, 0, t) \in \overline{U} \cap \{x_n = 0\}$$

*where $(F_{v_{ij}})$ is a positive definite form. Assume that $F$ is analytic in $D^2 v, Dv$ and $\Phi$ is analytic in $\partial_x v$ and $v$. If $F, \Phi$ are analytic in $X$ and in the second Gevrey class in $t$ then $v$ is analytic in $X$ and in the second Gevrey class in $t$.*

More precisely, Theorem 1 in (KN78) is stated for Dirichlet boundary conditions. But the remarks after the theorem (and, especially, equation 2.11 there) show that the result applies to Neumann conditions also. Finally, it is easy to see that if $\log(h)$ is analytic in $X$ and in the second Gevrey class in $t$, then $\psi$ satisfies the hypothesis of Theorem 3.7.8. This finishes the proof of Theorem 3.1.11.

# APPENDIX A

# APPENDIX FOR CHAPTER 2

## A.1 Proof of Theorem 2.8.4 for $h < h_0$

Let us recall the statement we are trying to prove:

**Theorem 2.8.4** *Let $u^k, k = 1, 2$ satisfy a system of coercive and elliptic equations with proper weights. Suppose the coefficients in*(2.8.3) *and the $B_{rk\gamma}$ satisfy the $h - \mu$-conditions on a domain $\Gamma \supset U$, where $0 < \mu < 1$. Additionally, assume the following regularity: $f_j^\alpha \in C^{\rho,\mu}(U)$, $\rho = \max\{0, h - s_j + |\alpha|\}, g_{r\gamma} \in C^{\tau,\mu}(U)$ with $\tau = \max\{0, h + h_r + |\gamma|\}$ and $u^k \in C^{t_k+h,\mu}(U)$. Then*

$$\sum_k \|u^k\|_{C^{t_k+h,\mu}(U)} \leq C \left( \sum_{j,\alpha} \|f_j^\alpha\|_{C^{\rho,\mu}(U)} + \sum_{r,\gamma} \|g_{r\gamma}\|_{C^{\tau,\mu}(U)} + \sum_k \|u^k\|_{C^0(U)} \right). \quad \text{(A.1.1)}$$

*Here $C$ is independent of the $u^k$'s, the $f$'s and the $g$'s.*

For simplicity's sake, we establish the above in the special case where $h_0 = 0, h = -1, t_1 = t_2 = 2, s_1 = s_2 = 0$ and $p_1 = p_2 = 0$ (which is the case that is applied in the proof of Proposition 2.8.5). However, our techniques work for $h_0 \geq 0, h \geq h_0 - 1$ and any proper assignment of weights. To further simplify the proof, we will make the assumptions that $U$ is bounded and that $u^k \in C^\infty(\overline{U} \backslash \{y_n = 0\})$, i.e. that $u^k$ is infinitely smooth away from $\{y_n = 0\}$. In the context of the paper, these assumptions are clearly satisfied. This simplification can be avoided through the use of cutoff functions (e.g. in the proof of Theorem 6.2 in (ADN59)).

Here we will follow closely the work of Agmon, Douglis and Nirenberg (see (ADN59), (ADN64)). Our proof has three steps; first, we present a representation formula for solutions to constant coefficient systems and show how this formula implies the desired result in that circumstance. Second, we analyze the variable coefficient case. Finally, we will justify the representation formula introduced in the first step.

### *A.1.1 The constant coefficient case*

We present a formula for solutions to constant-coefficient systems of the form (2.8.3) with boundary conditions (2.8.4).

If every function involved is $C^\infty$ with compact support, then integration by parts and (ADN64) Theorem 6.1 tell us

$$D_i u^k(y', y_n) + C_i^k = D_i v^k(y', y_n) + D_i \int_{\mathbb{R}^{n-1}} \sum_{r=1}^{2} K_{kr}(y' - x', y_n)(\tilde{g}^r(x') - \phi^r(x'))dx' \quad \text{(A.1.2)}$$

for any $i = 1, ..., n-1$ (this is essentially equation 6.7 in (ADN64) with the addition of a constant to compensate for $h = h_0 - 1$). We need to define some of the above terms:

- The $C_i^k$s are constants.
- Let $\Gamma$ be the fundamental solution to the linear operator $(-1)^\chi a_{\chi\gamma}^k D^{\gamma+\chi}$. We define

$$v^k(Y) = \int_{\mathbb{R}^n} \sum_{|\chi| \le m_k} (-1)^\chi \Gamma_k(Y - X) D_X^\chi \tilde{f}_\chi^k(X) dX.$$

  Here $\tilde{f}_\chi^k$ is a smooth, compactly supported extension of $f_\chi^k$ to all of $\mathbb{R}^n$. How the extension is created is not particularly important.

- Similarly $\tilde{g}^r(x)$ is a smooth, compactly supported extension of $g^r$ to all of $\mathbb{R}^{n-1}$. We will abuse notation and refer to $\tilde{g}$ as $g$ (similarly with $\tilde{f}$).
- $\phi^r(x') := \sum_{k=1}^2 B_k^r(D_{x'}, D_{x_n}) v^k(x', 0)$.
- $K_{kr}$ are kernels so that if the $\psi^r$s have sufficient smoothness/growth properties and

$$U^k(y', y_n) := \int_{\mathbb{R}^{n-1}} \sum_r K_{kr}(y' - x', y_n) \psi^r(x') dx'$$

then $(-1)^\chi a^k_{\chi\gamma} D^{\gamma+\chi} U^k = 0$ and

$$\sum_{k=1}^{2} B^r_k(D_{y'}, D_{y_n}) U^k(y', 0) = \psi^r(y').$$

Classical results imply that $\Gamma(Z), D\Gamma(Z)$ are integrable (at zero) and that $D^\chi\Gamma$ (for any $|\chi| = 2$) is a Calderon-Zygmund kernel which integrates to zero on $\mathbb{R}^n$. For the Poisson kernels, $K$, we turn to (ADN59), Sections 2 and 3. When $s = \text{ord } B^r_k = t_k - h_r - p_r = 2 - h_r$, we can deduce that $D^s K_{kr}$ is homogenous of degree $-(n-1)$ (see (ADN59), equation (2.13)'). In this case, we can write

$$D^s K_{kr}(y', y_n) = \frac{\Omega(\frac{y'}{|Y|}, \frac{y_n}{|Y|})}{|Y|^{-n+1}}, \ Y = (y', y_n).$$

As $D^s K_{kr}$ satisfies the same differential equation as $u^k$ we conclude that

$$\int_{|y'|=1} \Omega(y', 0) d\sigma(y') = 0$$

(see the corollary on pg 645 of (ADN59)). Furthermore, $D^s K$ has bounded first derivatives away from zero, so $\Omega$ is smooth. In particular, $D^s K_{kr}(y', 0)$ is a Calderon-Zygmund kernel.

As the $u$'s, $f$'s and $g$'s are assumed to be $C^\infty_c$, we can differentiate under the integral sign and rewrite (A.1.2) as

$$\begin{aligned} D_i u^k(y', y_n) + C_k = \int_{\mathbb{R}^n} \sum_{|\chi| \le m_k} \tilde{\Gamma}_{k\chi}(Y - X) f^k_\chi(X) dX + \\ \sum_{r=1}^{2} \int_{\mathbb{R}^{n-1}} D^{2-h_r}_Y K_{kr}(y' - x', y_n) D^{h_r - 1}_{x'} (g^r(x') - \phi^r(x')) dx'. \end{aligned} \tag{A.1.3}$$

Where we define

$$\tilde{\Gamma}_{k\chi}(Y - X) := D^{e_i + \chi}_Y \Gamma(Y - X)$$

(depending on the parity of $t_k - h_r - p_r$ the above equation may be missing some minus signs, these omissions are irrelevant to future analysis). It should also be noted all the kernels above are either integrable or Calderon-Zygmund kernels. We now make a crucial claim:

**Claim:** The above (A.1.3) holds for weak solutions of the constant-coefficient system (2.8.3) and (2.8.4) under the regularity assumptions $f_j^\alpha \in C^{\rho,\mu}(U)$ where $\rho = \max\{0, |\alpha| - 1\}$, $g_{r\gamma} \in C^{\tau,\mu}(U)$ with $\tau = \max\{0, h_r - 1\}$, and $u^k \in C^{1,\mu}(U)$.

From this one can conclude:

**Lemma A.1.1.** *Let $u^k, k = 1, 2$ satisfy a system of constant-coefficient coercive and elliptic equations with proper weights. Additionally, assume that for some for $0 < \mu < 1$: $f_j^\chi \in C^{\rho,\mu}(U)$ where $\rho = \max\{0, |\chi| - 1\}$, $g_r \in C^{\tau,\mu}(U)$ with $\tau = \max\{0, h_r - 1\}$ and $u^k \in C^{1,\mu}(U)$. Then*

$$\sum_k \|u^k\|_{C^{1,\mu}(U)} \leq C_1 \left( \sum_{j,\chi} \|f_j^\chi\|_{C^{\rho,\mu}(U)} + \sum_r \|g_r\|_{C^{\tau,\mu}(U)} + \sum_k \|u^k\|_{C^0(U)} \right). \quad \text{(A.1.4)}$$

*Here, $C$ is independent of the $u^k$'s, the $f$'s and the $g$'s.*

*Proof assuming the Claim.* It suffices to estimate the $C^{1,\mu}$ norm of $u^k|_{\{y_n=0\}}$ (as each $u^k$ satisfies an elliptic equation in $U$, the full estimate can be obtained using weighted Schauder estimates. See, e.g., (GH80) Theorem 5.1 or (ADN59) Theorem 9.1).

We use the classical fact that

$$\|f\|_{C^1} \leq \varepsilon [Df]_\alpha + C_{\varepsilon,\alpha} \sup |f| \quad \text{(A.1.5)}$$

where $f \in C^{1,\alpha}(\mathbb{R}^{n-1})$ and $[f]_\alpha = \sup_{x \neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}$ (see equations 7.4, 7.5 in (ADN59)). From here it follows that we need only estimate $[D_i u^k|_{\{y_n=0\}}]_\mu, i = 1, ..., n-1$ in terms of the norms on the right hand side. That such an estimate exists, follows immediately from the theory of singular integrals and the fact that the kernels in (A.1.3) are either Calderon-Zygmund kernels or integrable at 0. □

### A.1.2 The variable coefficient case

Given Lemma A.1.1, the standard way to handle variable coefficients is to "freeze" the coefficients at a point. For any $y_0 = (y_0', 0) \in \overline{U}$, we write:

$$\begin{aligned}
\int_U \sum_{\substack{|\chi|\leq m_1 \\ |\gamma|\leq 2-m_1}} a^1_{\chi\gamma}(y_0) D^\gamma u^1 D^\chi \zeta dx &= \int_U \sum (f^1_\chi + [a^1_{\chi\gamma}(y_0) - a^1_{\chi\gamma}(x)] D^\gamma u^1) D^\chi \zeta dx \\
\int_U \sum_{\substack{|\chi|\leq m_2 \\ |\gamma|\leq 2-m_2}} a^2_{\chi\gamma}(y_0) D^\gamma u^2 D^\chi \zeta dx &= \int_U \sum (f^2_\chi + [a^2_{\chi\gamma}(y_0) - a^2_{\chi\gamma}(x)] D^\gamma u^2) D^\chi \zeta dx
\end{aligned} \tag{A.1.6}$$

for all $\zeta \in C_0^\infty(U)$. On the boundary

$$\begin{aligned}
\int_{\{y_n=0\}} \left( \sum_{k=1}^2 B^1_k(D_{x'}, D_{x_n}, y_0') u^k \right) \xi dx' &= \int_{\{y_n=0\}} (g^1 + G^1) \xi dx' \\
\int_{\{y_n=0\}} \left( \sum_{k=1}^2 B^2_k(D_{x'}, D_{x_n}, y_0') u^k \right) \xi dx' &= \int_{\{y_n=0\}} (g^2 + G^2) \xi dx'
\end{aligned} \tag{A.1.7}$$

for all $\xi \in C_0^\infty(\partial U \cap \{y_n = 0\})$. Here $G^r := \sum_{k=1}^2 (B^r_k(D_{x'}, D_{x_n}, y_0') - B^r_k(D_{x'}, D_{x_n}, x'))u^k$.

However, naïve application of Lemma A.1.1 will not work as the semi-norms $[Du^k]_\mu$ may appear with large coefficients on the wrong side of the inequality.

(A.1.5) allows us to argue

$$[Du^k]_\mu \leq \frac{1}{2} \|u^k\|_{C^{1,\mu}} \Rightarrow \|u^k\|_{C^{1,\mu}} \leq C \|u^k\|_{C^0},$$

for $k = 1, 2$ (which renders our desired estimate trivially true). So, without loss of generality, it suffices to consider the case

$$\exists k = 1, 2 \text{ s.t. } \exists P, Q \in U \text{ with } \frac{|Du^k(P) - Du^k(Q)|}{|P - Q|^\mu} > \frac{1}{2} \|u^k\|_{C^{1,\mu}}. \tag{A.1.8}$$

Let $\lambda > 0$ be determined later and assume, without loss of generality, $P = (0, t), k = 1$. We have three cases:

**Case 1:** $|P - Q| \geq \lambda$. This easily implies $2 \sup |Du^1| \geq \lambda^\mu \frac{1}{2} \|u^1\|_{C^{1,\mu}}$. From here, if $\lambda$ is sufficiently small, use (A.1.5) to get

$$C_\lambda \|u^1\|_{C^0} \geq \|u^1\|_{C^{1,\mu}}$$

which, as stated above, yields the desired estimate.

**Case 2:** $|P - Q| < \lambda$ but $t \geq 2\lambda$. In this case $u^k, k = 1, 2$ are solutions to an elliptic system of equations in $B_{3\lambda/2}(P) \subset U$. Interior Schauder estimates for weak solutions (see e.g. (Mor66), Theorem 6.4.3 or (GT98) Chapter 8) give

$$\sum_k \|u^k\|_{C^{1,\mu}(B_{5\lambda/4}(P))} \leq C_\lambda \left( \sum_{j,\alpha} \|f_j^\alpha\|_{C^{0,\mu}(B_{3\lambda/2}(P))} + \sum_k \|u^k\|_{C^0(B_{3\lambda/2}(P))} \right).$$

By assumption,

$$\frac{1}{2} \|u^1\|_{C^{1,\mu}(U)} < \frac{|Du^1(P) - Du^1(Q)|}{|P - Q|^\mu} \leq \|u^1\|_{C^{1,\mu}(B_{5\lambda/4}(P))}$$

and so, once we have fixed $\lambda$, we have the desired result.

**Case 3:** $|P - Q| < \lambda$ and $t < 2\lambda$. Consider a smooth cutoff function, $\eta \in C^\infty(\mathbb{R}^n)$, such that $\eta(Y) \equiv 1$ when $|Y| \leq 3\lambda$ and $\eta(Y) \equiv 0$ when $|Y| \geq 5\lambda$. Additionally, $\eta$ can be chosen such that $|D^\ell \eta| \leq C\lambda^{-\ell}$. Now consider $V^k := \eta u^k$. $V^k$ satisfies equations similar to (A.1.6) and (A.1.7) but with different right hand sides.

We can use the representation (A.1.3) and thus Lemma A.1.1 on the $V^k$s. We need to estimate each term on the right. The term that comes from the interior equations is dominated by

$$\| \sum_{\chi,\gamma} \eta(f^k_\chi + [a^k_{\chi\gamma}(y_0) - a^k_{\chi\gamma}(x)] D^\gamma u^1) \|_{C^{0,\mu}} + \| \sum_{|\gamma|=1} \sum_{\chi} [a^k_{\chi\gamma}(y_0) - a^k_{\chi\gamma}(x)] u^k D^\gamma \eta \|_{C^{0,\mu}}.$$

Note that $\eta$ is supported on $B_{5\lambda}$ so $\sup |a^k_{\chi\gamma}(y_0) - a^k_{\chi\gamma}(x)| < C\lambda^\mu$. Also recall that the $h-\mu$-conditions imply the $a^k_{\chi\gamma}$ are Hölder continuous. Thus, the first term in the offset equation above can be dominated by $\sum_{\chi,k} \|f^k_\chi\|_{C^{0,\mu}} + C\lambda^\mu [Du^k]_\mu + C \sup |Du^k|$, where the constants above are independent of $\lambda$. Similarly, the second term can be bounded by $\sum_{\chi,k} C\lambda^{\mu-1}[u]_\mu + C \sup |u^k| \lambda^{-2} + C\lambda^{-1} \sup |u^k|$.

From the boundary terms we get

$$\sum_r \left( \| \sum_{k=1}^{2} (B^r_k(D_{x'}, D_{x_n}, y'_0) - B^r_k(D_{x'}, D_{x_n}, x'))\eta u^k \|_{C^{h_r-1,\mu}} + \|\eta g^r\|_{C^{h_r-1,\mu}} \right).$$

As we have seen above, we need not worry when the derivatives in the boundary operators land on $\eta$ (as these terms will all be bounded by the $C^{0,\mu}$ norms of the $f$s, $g$s and $u$s and the $C^1$ norm of the $u$s). When the derivatives all land on the $u^k$ term, we argue just as above (recalling that that $h-\mu$ conditions imply that the $B$s are Hölder continuous in position) and conclude that the coefficient of $[Du^k]_\mu$ contains a positive power of $\lambda$.

We can then pick $\lambda$ small enough so that the coefficient of $[Du^k]_\mu$ on the right hand side is less than $1/4$. This yields the estimate

$$\sum_k \|V^k\|_{C^{1,\mu}(U)} \leq \frac{1}{4} \sum_k [Du^k]_\mu + C \left( \sum_{j,\chi} \|f^\chi_j\|_{C^{\rho,\mu}(U)} + \sum_r \|g_r\|_{C^{\tau,\mu}(U)} + \sum_k \|u^k\|_{C(U)} \right).$$

But $V^k = u^k$ on $P, Q$ so we have that

$$\frac{1}{2}\|u^1\|_{C^{1,\mu}(U)} < \frac{|Du^1(P) - Du^1(Q)|}{|P-Q|^\mu} \leq \|V^1\|_{C^{1,\mu}(U)}$$

$$\Rightarrow \frac{1}{2}\|u^1\|_{C^{1,\mu}(U)} \leq \frac{1}{4}\sum_k [Du^k]_\mu + C\left(\sum_{j,\chi}\|f_j^\chi\|_{C^{\rho,\mu}(U)} + \sum_r \|g_r\|_{C^{\tau,\mu}(U)} + \sum_k \|u^k\|_{C(U)}\right).$$

From here the desired estimate follows immediately. As such, we are done modulo the proof that (A.1.3) holds for non-$C^\infty$ functions.

### *A.1.3 Justifying* (A.1.3)

It remains to prove our claim above: namely, that the representation in (A.1.3) is valid without the *a priori* assumption of $C^\infty$ regularity. Here we follow closely the discussion on pages 673-674 of (ADN59). It should first be noted that the integrals on the right hand side of (A.1.3) converge if $f_j^\alpha \in C^{\rho,\mu}(U)$ and $g_{r\gamma} \in C^{\tau,\mu}(U)$.

Let $j(r)$ be an approximation to the identity and then define

$$J_\varepsilon u(y', y_n) := \varepsilon^{-n+1} \int \prod_{i=1}^{n-1} j\left(\frac{y_i - x_i}{\varepsilon}\right) u(x_1, ..., x_{n-1}, x_n) dx'.$$

Similarly, we can define

$$J_{\varepsilon,\tilde{\varepsilon}} u(y', y_n) := \frac{1}{\tilde{\varepsilon}} \int_0^\infty j\left(\frac{y_n + \tilde{\varepsilon} - s}{\tilde{\varepsilon}}\right) J_\varepsilon u(y', s) ds.$$

For any $u$ it is clear that $J_{\varepsilon,\tilde{\varepsilon}} u$ is a $C^\infty$ function in the closed upper half plane.

Now assume the $u^k$'s satisfy a coercive and elliptic system with constant coefficients and let the $f$'s and $g$'s be as in Definition 2.8.2. Then (as the system has constant coefficients) it is true that $J_{\varepsilon,\tilde{\varepsilon}} u^k$ satisfies (2.8.3) with $J_{\varepsilon,\tilde{\varepsilon}} f_\chi^k$ on the right hand side. So, with $v^k$ defined as above, (A.1.3) becomes

$$\begin{aligned} J_{\varepsilon,\tilde{\varepsilon}} D_i u^k(y', y_n) + C_i^k(\varepsilon, \tilde{\varepsilon}) = \int_{\mathbb{R}^n} \sum_{|\chi| \leq m_k} \tilde{\Gamma}_{k\chi}(Y - X) J_{\varepsilon,\tilde{\varepsilon}} f_\chi^k(X) dX + \\ \sum_{r=1}^{2} \int_{\mathbb{R}^{n-1}} D_Y^{2-h_r} K_{kr}(y' - x', y_n) G_{\varepsilon,\tilde{\varepsilon}}(x', 0) dx' \end{aligned} \quad \text{(A.1.9)}$$

where

$$G_{\varepsilon,\tilde{\varepsilon}}(x',0) := (J_{\varepsilon,\tilde{\varepsilon}} D_{x'}^{h_r-1} \sum_k B_k^r(D_{x'}, D_{x_n})(u^k(x',x_n) - v^k(x',x_n)))_{x_n=0}.$$

Note that, for $H \in C^{0,\mu}$, $J_{\varepsilon,\tilde{\varepsilon}} H \overset{\tilde{\varepsilon}\downarrow 0}{\to} J_\varepsilon H$ uniformly (by Arzelà-Ascoli). By assumption $f_\chi^k$ is Hölder continuous. To analyze the boundary terms, note first that $D_{x'}^{h_r-1} B_k^r$ is an operator of order 1 and, as such, $D_{x'}^{h_r-1} \sum_k B_k^r(D_{x'}, D_{x_n})(u^k(x',x_n) - v^k(x',x_n))$ is at least as regular as $C^{0,\mu}$. So $J_{\varepsilon,\tilde{\varepsilon}} D_{x'}^{h_r-1} \sum_k B_k^r(D_{x'}, D_{x_n})(u^k(x',x_n) - v^k(x',x_n))$ (and thus its restriction to $\{x_n = 0\}$) converges in the uniform topology.

Let $\tilde{\varepsilon} \downarrow 0$ to obtain

$$\begin{gathered} J_\varepsilon D_i u^k(y', y_n) + C_k(\varepsilon) = \int_{\mathbb{R}^n} \sum_{|\chi| \le m_k} \tilde{\Gamma}_{k\chi}(Y - X) J_\varepsilon f_\chi^k(X) dX + \\ \sum_{r=1}^{2} \int_{\mathbb{R}^{n-1}} D_Y^{2-h_r} K_{kr}(y' - x', y_n) G_\varepsilon(x', 0) dx' \end{gathered} \tag{A.1.10}$$

where

$$G_\varepsilon(x',0) := (J_\varepsilon D_{x'}^{h_r-1} \sum_k B_k^r(D_{x'}, D_{x_n})(u^k(x',x_n) - v^k(x',x_n)))_{x_n=0}.$$

Since $J_\varepsilon$ is a convolution in only the $\mathbb{R}^{n-1}$ directions, we can set $x_n = 0$ to obtain

$$G_\varepsilon(x',0) = J_\varepsilon D_{x'}^{h_r-1}(g^r(x') - \phi^r(x')).$$

We note, by assumption, that $g^r$ is at least Hölder continuous. As such, we can use the same argument as above to justify taking $\varepsilon \downarrow 0$; the validity of our claim follows.

# APPENDIX B

# APPENDIX FOR CHAPTER 3

## B.1 Classification of "Flat Blowups"

A key piece of the blowup argument is a classification of "flat blowups"; Theorem 3.1.10. The proof of this theorem follows from two lemmas which are modifications of results in Andersson and Weiss (AW09). Before we begin, let us try to clarify the relationship between this work and that of (AW09).

As was mentioned in the introduction, the results in (AW09) are for solutions in the sense of domain variations to a problem arising in combustion. Although this is the natural class of solutions to consider when studying their problem (see the introduction in (Wei03)), the definition of these solutions is quite complex and it is unclear whether the parabolic Green function at infinity satisfies it. For example, neither the integral bounds on the growth of time and space derivatives (see the first condition in Definition 6.1 in (Wei03)) nor the monotonicity formula (see the second condition in Definition 6.1 in (Wei03)), clearly hold *a priori* for functions $u$ which satisfy the conditions of Theorem 3.1.10.

Upon careful examination of (AW09) we identified which properties of solutions in the sense of domain variations were crucial to the proof. The first of these was the following "representation" formula which holds in the sense of distributions for almost every time, $t_0$, (see Theorem 11.1 in (Wei03))

$$\Delta u - \partial_t u|_{t_0} = \mathcal{H}^{n-1}|_{R(t_0)} + 2\theta(t_0,-)\mathcal{H}^{n-1}|_{\Sigma_{**}(t_0)} + \lambda(t)|_{\Sigma_z(t_0)}. \tag{B.1.1}$$

Without going too deep into details, we should think of $R(t_0) \subset \partial\{u > 0\}$ as boundary points which are best behaved. In particular, blowups at these points are plane solutions with slope 1. $\Sigma_{**}(t_0)$ points are also regular in the sense that the set is countable rectifiable and the blowups are planes, but the slope of the blowup solution is $2\theta(t_0,-) < 1$. $\Sigma_z(t_0)$ consists of singular points at which blowups may not be planes. We observe that this situation is much more complicated than the elliptic one in (AC81), but, as is pointed out in the introduction (Wei03), this is an unavoidable characteristic of solutions to the combustion problem.

Lemma B.1.6 (which corresponds to Lemma 5.1 in (AW09) and is a parabolic version of Lemma 4.10 in (AC81)) bounds from below the slope of blowup solutions at points which have an exterior tangent ball. For solutions in the sense of domain variation, points in $\Sigma_{**}(t_0)$ complicate matters and thus more information than is given by (B.1.1) is needed. Indeed, it is mentioned in (AW09) that defining solutions merely as those which satisfy (B.1.1) would not be sufficient to implement their approach. However, in our setting, Lemma 3.4.24 says that $k \geq 1$ at almost every point on the free boundary which suffices to show that any blowup at a regular point must have slope at least 1.

Another property of solutions in the sense of domain variations which is critical to (AW09) is that $|\nabla u(X,t)|^2$ always approaches a value less than or equal to 1 as $(X,t)$ approaches the free boundary (Lemma 8.2 in (AW09)). In our setting we know that $|\nabla u| \leq 1$ everywhere (by Proposition 3.4.4) and so we need not worry. That $|\nabla u| \leq 1$ (along with Lemma 3.6.9, proven in Section 3.6 above), implies that blowups of $u$ are precompact in the $\mathrm{Lip}(1,1/2)$ norm–which is another property of domain variations that is used in (AW09). Finally, it is important to the arguments in (AW09) that the set $R(t_0)$ is rectifiable (e.g. in order to apply integration by parts). In the setting of Proposition 3.1.10, Ahlfors regularity lets us apply integration by parts as well (albeit, we must be more careful. See, e.g., the proof of Lemma B.1.10 below).

By finding appropriate substitutes for the relevant properties of domain variations (as described above) we were able to prove that the results of (AW09) apply mostly unchanged to $u, k, \Omega$ which satisfy the hypothesis of Proposition 3.1.10. However, we needed to make an additional modification, as the conclusions of Proposition 3.1.10 are global whereas the main theorem in (AW09) is a local regularity result (see Corollary 8.5 there). In particular, Theorem 5.2 and Lemma 8.1 in (AW09) roughly state that if a solution is flat in a certain sense in a cylinder, then it is even flatter in another sense in a smaller cylinder *whose center has been translated in some direction.* This translation occurs because the considered cylinders contain a free boundary point that is centered in the space variables but in the "past" in the time coordinate. We want to improve flatness at larger and larger scales, so we cannot allow our cylinders to move in this manner (otherwise our cylinder might drift off to infinity).

To overcome this issue, we introduce the concept of "current flatness" (see Definition B.1.1). However, the parabolic equation is anisotropic, so centering our cylinder in time means that we have to accept weaker results. This leads to the notion of "weak current flatness" (Definition B.1.3). Unfortunately, the qualitative nature of weak flatness is not always sufficient so we still need to prove some results for "past flatness" (as introduced in (AW09)). We also remark that this idea of "current flatness" could be used in the setting of domain variations to analyze the global properties of those solutions.

Let us recall Theorem 3.1.10;

**Theorem** (Theorem 3.1.10)**.** *Let $\Omega_\infty$ be a $\delta$-Reifenberg flat parabolic regular domain with Green function at infinity $u_\infty$ and associated parabolic Poisson kernel $h_\infty$ (i.e. $h_\infty = \frac{d\omega_\infty}{d\sigma}$). Furthermore, assume that $|\nabla u_\infty| \leq 1$ in $\Omega_\infty$ and $|h_\infty| \geq 1$ for $\sigma$-almost every point on $\partial\Omega_\infty$. There exists a $\delta_n > 0$ such that if $\delta_n \geq \delta > 0$ we may conclude that, after a potential rotation and translation, $\Omega_\infty = \{(X,t) \mid x_n > 0\}$.*

We define three notions of "flatness" for solutions. The definition of "past flatness" is taken from Andersson and Weiss (AW09) (who in turn adapted it from the corresponding elliptic definitions in (AC81)). As mentioned above, we also introduce two types of "current flatness". The first type is quantitative and we call it "strong current flatness."

**Definition B.1.1.** *For $0 < \sigma_i \leq 1/2$ we say that $U \in CF(\sigma_1, \sigma_2)$ in $C_\rho(Q,\tau)$ in the direction $\nu \in \mathbb{S}^{n-1}$ if*

- $(Q,\tau) \in \partial\{U > 0\}$

- $U((Y,s)) = 0$ *whenever* $(Y-Q)\cdot\nu \le -\sigma_1\rho$ *and* $(Y,s) \in C_\rho(Q,\tau)$.

- $U((Y,s)) \ge (Y-Q)\cdot\nu - \sigma_2\rho$ *whenever* $(Y-Q)\cdot\nu - \sigma_2\rho \ge 0$ *and* $(Y,s) \in C_\rho(Q,\tau)$.

The parabolic nature of our problem means that it behooves us to introduce a "past" version of this flatness:

**Definition B.1.2.** *For* $0 < \sigma_i \le 1$ *we say that* $u \in PF(\sigma_1,\sigma_2)$ *in* $C_\rho(X,t)$ *in the direction* $\nu \in \mathbb{S}^{n-1}$ *if for* $(Y,s) \in C_\rho(X,t)$

- $(X, t-\rho^2) \in \partial\{U > 0\}$

- $U((Y,s)) = 0$ *whenever* $(Y-X)\cdot\nu \le -\sigma_1\rho$

- $U((Y,s)) \ge (Y-X)\cdot\nu - \sigma_2\rho$ *whenever* $(Y-X)\cdot\nu - \sigma_2\rho \ge 0$.

Our final notion of flatness is qualitative and weaker than strong current flatness. We call it "weak current flatness."

**Definition B.1.3.** *For* $0 < \sigma_i \le 1/2$ *we say that* $U \in \widetilde{CF}(\sigma_1,\sigma_2)$ *in* $C_\rho(Q,\tau)$ *in the direction* $\nu \in \mathbb{S}^{n-1}$ *if*

- $(Q,\tau) \in \partial\{U > 0\}$

- $U((Y,s)) = 0$ *whenever* $(Y-Q)\cdot\nu \le -\sigma_1\rho$

- $U((Y,s)) > 0$ *whenever* $(Y-Q)\cdot\nu > \sigma_2\rho$.

We may now state our two main lemmas, the first of which allows us to conclude flatness on the positive side of $\partial\{u_\infty > 0\}$ given flatness on the zero side.

**Lemma B.1.4.** *["Current" version of Theorem 5.2 in (AW09)] Let* $\Omega, u_\infty$ *satisfy the assumptions of Proposition 3.1.10. Furthermore, assume* $u_\infty \in \widetilde{CF}(\sigma, 1/2)$ *in* $C_\rho(Q,\tau)$ *in the direction* $\nu$. *Then there is a constant* $K_1 > 0$ *(depending only on dimension) such that* $u_\infty \in CF(K_1\sigma, K_1\sigma)$ *in* $C_{\rho/2}(Q,\tau)$ *in the direction* $\nu$.

The second lemma provides greater flatness on the zero-side under the assumption of flatness on both sides.

**Lemma B.1.5.** *["Current" version of Lemma 8.1 in (AW09)] Let $u_\infty, \Omega$ satisfy the assumptions of Proposition 3.1.10 and assume, for some $\sigma, \rho > 0$ that $u_\infty \in CF(\sigma,\sigma)$ in $C_\rho(Q,\tau)$ in the direction $\nu$. For $\theta \in (0,1)$, there exists a constant $1/2 > \sigma_\theta > 0$ which depends only on $\theta, n$, such that if $0 < \sigma < \sigma_\theta$ then $u_\infty \in \widetilde{CF}(\theta\sigma, \theta\sigma)$ in $C_{K_2\theta\rho}(0,0)$ in the direction $\tilde{\nu}$, where $\tilde{\nu}$ satisfies $|\tilde{\nu} - \nu| < K_3\sigma$. Here $0 < K_2, K_3 < \infty$ are constants depending only on the dimension.*

Our proof that these two lemmas imply that $\Omega_\infty$ is a half-space is based on the analogous result in the elliptic setting proven by Kenig and Toro in (KT04).

*Proof of the Proposition 3.1.10 assuming the two lemmas.* Pick $\theta' \in (0,1)$ small enough so that $\max\{\theta', K_1^2\theta', K_2\theta'/4\} < 1/2$. Then let $\delta_n < \min\{1/2, \sigma_{\theta'}/K_1\}$. Here, and through the rest of this proof, $K_1, K_2, K_3$ and $\sigma_{\theta'}$ are as in Lemmas B.1.4 and B.1.5.

Assume, without loss of generality, that $(0,0) \in \partial\Omega$. For every $\rho > 0$, there is an $n$-plane, $V(\rho)$, containing a line parallel to the $t$-direction, such that $\frac{1}{\rho}D[V(\rho)\cap C_\rho(0,0), \partial\Omega \cap C_\rho(0,0)] < \delta$. Let $\nu_\rho$ be the unit normal vector to $V(\rho)$ correctly oriented so that if $(X,t) \in C_\rho(0,0)$ and $\langle X, \nu_\rho\rangle \leq -\delta\rho$ then $(X,t) \in \Omega^c$ (similarly, $(X,t) \in C_\rho(0,0), \langle X,\nu_\rho\rangle \geq \delta\rho$ implies $(X,t) \in \Omega$). Translated into the language of weak current flatness, $u_\infty \in \widetilde{CF}(\delta,\delta)$ in $C_\rho(0,0)$ in the direction $\nu_\rho$.

Apply Lemma B.1.4 so that $u_\infty \in CF(K_1\delta, K_1\delta)$ in $C_{\rho/2}(0,0)$ in the direction $\nu_\rho$. Then Lemma B.1.5 implies that $u_\infty \in \widetilde{CF}(K_1\theta'\delta, K_1\theta'\delta)$ in $C_{K_2\theta'\rho/2}(0,0)$ in the direction $\nu_\rho^{(1)}$ where $|\nu_\rho^{(1)} - \nu_\rho| < K_1K_3\delta$. Returning to Lemma B.1.4 yields $u_\infty \in CF(K_1^2\theta'\delta, K_1^2\theta'\delta)$ in $C_{K_2\theta'\rho/4}(0,0)$ in the direction $\nu_\rho^{(1)}$. Note that $\theta, \delta_n$ were chosen small enough to justify repeating this procedure arbitrarily many times. After $m$ iterations we have shown $u_\infty \in CF(\theta^m\delta, \theta^m\delta)$ in $C_{\eta^m\rho}(0,0)$ in the direction $\nu_\rho^{(m)}$ where $\eta \equiv K_2\theta'/4 < 1/2$. Additionally, for $m \geq 1$, $|\nu_\rho^{(m)} - \nu_\rho^{(m+1)}| < K_3\theta^m\delta$. From now on we will abuse notation and refer to all constants which only depend on the dimension by $C$.

Define $\overline{\nu}_\rho := \lim_{m\to\infty} \nu_\rho^{(m)}$ and compute

$$|\overline{\nu}_\rho - \nu_\rho^{(m)}| < C\delta\theta^m \frac{1}{1-\theta} < C\delta\theta^m. \tag{B.1.2}$$

For any $(P,\xi) \in C_\rho(0,0)$; there is some $m$ such that $(P,\xi) \in C_{\eta^m\rho}(0,0)$ but $(P,\xi) \notin C_{\eta^{m+1}\rho}(0,0)$. As $u_\infty \in CF(\theta^m\delta, \theta^m\delta)$ in the direction $\nu_\rho^{(m)}$ we can conclude

$$\begin{aligned} \left\langle P, \nu_\rho^{(m)} \right\rangle &\leq -\theta^m\delta\eta^m\rho \Rightarrow u_\infty(P,\xi) = 0 \\ \left\langle P, \nu_\rho^{(m)} \right\rangle &\geq \theta^m\delta\eta^m\rho \Rightarrow u_\infty(P,\xi) > 0. \end{aligned} \tag{B.1.3}$$

We may write $\left\langle P, \overline{\nu}_\rho \right\rangle = \left\langle P, \nu_\rho^{(m)} \right\rangle + \left\langle P, \overline{\nu}_\rho - \nu_\rho^{(m)} \right\rangle$ and estimate, using equation (B.1.2), $|\left\langle P, \overline{\nu}_\rho - \nu_\rho^{(m)} \right\rangle| < C\delta\theta^m\eta^m\rho$. Then equation (B.1.3) implies,

$$\begin{aligned} \left\langle P, \overline{\nu}_\rho \right\rangle &\leq -C\theta^m\delta\eta^m\rho \Rightarrow u_\infty(P,\xi) = 0 \\ \left\langle P, \overline{\nu}_\rho \right\rangle &\geq C\theta^m\delta\eta^m\rho \Rightarrow u_\infty(P,\xi) > 0. \end{aligned} \tag{B.1.4}$$

Hence,

$$\frac{1}{\eta^m\rho} D[\Lambda(\rho) \cap C_{\eta^m\rho}(0,0), \partial\Omega \cap C_{\eta^m\rho}(0,0)] < C\theta^m\delta,$$

where $\Lambda(\rho)$ is the $n$-hyperplane containing $(0,0)$ that is perpendicular to $\overline{\nu}_\rho$.

If $\eta^{m+1}\rho \leq s < \eta^m\rho$ one computes

$$\frac{1}{s} D[\Lambda(\rho) \cap C_s(0,0), \partial\Omega \cap C_s(0,0)] \leq \frac{1}{\eta} D[\Lambda(\rho) \cap C_{\eta^m\rho}(0,0), C_{\eta^m\rho}(0,0) \cap \partial\Omega] \leq C\delta\frac{\theta^m}{\eta}.$$

As $\theta, \eta < 1$ we can write $\theta = \eta^\beta$ for some $\beta > 0$ and the above becomes

$$\frac{1}{s} D[\Lambda(\rho) \cap C_s(0,0), \partial\Omega \cap C_s(0,0)] \leq C_\eta\delta \left(\frac{s}{\rho}\right)^\beta, \forall s < \rho. \tag{B.1.5}$$

Pick any $\rho_j \to \infty$. By compactness, $\Lambda(\rho_j) \to \Lambda_\infty$ in the Hausdorff distance (though we may need to pass to a subsequence that the limit plane may depend on the subsequence chosen). Then for any $s > 0$, equation (B.1.5) implies

$$\frac{1}{s} D[\Lambda_\infty \cap C_s(0,0), \partial\Omega \cap C_s(0,0)] = 0.$$

We conclude that $\partial\Omega = \Lambda_\infty$, and that $\Lambda_\infty$ is, in fact, independent of $\{\rho_j\}$. After a rotation and translation $\Omega = \{(X,t) \mid x_n > 0\}$. □

### *B.1.1 Flatness of the zero side implies flatness of the positive side: Lemma B.1.4*

Before we begin we need two technical lemmas. The first allows us to conclude regularity in the time direction given regularity in the spatial directions. We stated and proved this Lemma in Section 3.6 so we will just state it again here.

**Lemma.** *If $f$ satisfies the (adjoint)-heat equation in $\mathcal{O}$ and is zero outside $\mathcal{O}$ then*

$$\|f\|_{\mathbb{C}^{1,1/2}(\mathbb{R}^{n+1})} \leq c\|\nabla f\|_{L^\infty(\mathcal{O})},$$

*where $0 < c < \infty$ depends only on the dimension.*

This second lemma allows us to bound from below the normal derivative of a solution at a smooth point of $\partial\Omega_\infty$. For ease of notation we will drop the subscript $\infty$ from $u_\infty, \Omega_\infty$ and $h_\infty$. However, all these results are proven with the same assumptions as Theorem 3.1.10.

**Lemma B.1.6.** *Let $(Q,\tau) \in \partial\Omega$ be such that there exists a tangent ball (in the Euclidean sense) $B$ at $(Q,\tau)$ contained in $\overline{\Omega}^c$. Then*

$$\limsup_{\Omega \ni (X,t) \to (Q,\tau)} \frac{u(X,t)}{d((X,t),B)} \geq 1.$$

*Proof.* Without loss of generality set $(Q,\tau) = (0,0)$ and let $(X_k, t_k) \in \Omega$ be a sequence that achieves the supremum, $\ell$. Let $(Y_k, s_k) \in B$ be such that $d((X_k,t_k),B) = \|(X_k,t_k) - (Y_k,s_k)\| =: r_k$. Define $u_k(X,t) := \frac{u(r_kX+Y_k, r_k^2 t+s_k)}{r_k}$, $\Omega_k := \{(Y,s) \mid Y = (X-Y_k)/r_k, s = (t-s_k)/r_k^2, \text{ s.t. } (X,t) \in \Omega\}$ and $h_k(X,t) := h(r_kX + Y_k, r_k^2 t + s_k)$. Then

$$\int_{\mathbb{R}^{n+1}} u_k(\Delta\phi - \partial_t\phi)dXdt = \int_{\partial\Omega_k} h_k \phi d\sigma.$$

By assumption, the $u_k$ have uniform Lipschitz bound 1. Thus Lemma 3.6.9 implies that the $u_k$ are bounded uniformly in $\mathbb{C}^{1,1/2}$. Therefore, perhaps passing to a subsequence, $u_k \to u_0$ uniformly on compacta. In addition, as there exists a tangent ball at $(0,0)$, $\Omega_k \to \{x_n > 0\}$ in the Hausdorff distance norm (up to a rotation). We may assume, passing to a subsequence, that $\frac{X_k - Y_k}{r_k} \to Z_0, \frac{t_k - s_k}{r_k^2} \to t_0$ with $(Z_0, t_0) \in C_1(0,0) \cap \{x_n > 0\}$ and $u_0(Z_0,t_0) = \ell$. Furthermore, by the definition of supremum, for any $(Y,s) \in \{x_n > 0\}$ we have

$$\begin{aligned} u_0(Y,s) &= \lim_{k\to\infty} u(r_kY + y_k, r_k^2 s + s_k)/r_k \\ &\leq \lim_{k\to\infty} \ell \frac{\text{pardist}((r_kY + y_k, r_k^2 s + s_k), B)}{r_k} \\ &= \lim_{k\to\infty} \ell \text{pardist}((Y,s), B_k) \\ &= \ell y_n, \end{aligned} \tag{B.1.6}$$

where $B_k$ is defined like $\Omega_k$ above.

Let $\phi \in C_0^\infty(\mathbb{R}^{n+1})$ be positive, then

$$\begin{aligned} \int_{\{x_n>0\}} \ell x_n(\Delta\phi - \partial_t\phi)dXdt &\geq \int_{\{x_n>0\}} u_0(X,t)(\Delta\phi - \partial_t\phi)dXdt \\ &= \lim_{k\to\infty} \int_{\Omega_k} u_k(X,t)(\Delta\phi - \partial_t\phi)dXdt \\ &= \lim_{k\to\infty} \int_{\partial\Omega_k} h_k\phi d\sigma. \end{aligned} \tag{B.1.7}$$

Integrating by parts yields

$$\ell \int_{\{x_n=0\}} \phi dxdt = \int_{\{x_n>0\}} \ell x_n(\Delta\phi - \partial_t\phi)dXdt$$
$$\overset{eqn.\ (B.1.7)}{\geq} \lim_{k\to\infty}\int_{\partial\Omega_k} h_k\phi d\sigma$$
$$\overset{h_k\geq 1}{\geq} \lim_{k\to\infty}\int_{\{x_n=0\}} \phi dxdt$$

Hence, $\ell \geq 1$, the desired result. □

We will first show that for "past flatness", flatness on the positive side gives flatness on the zero side.

**Lemma B.1.7.** *[Compare with Lemma 5.2 in (AW09)] Let $0 < \sigma \leq \sigma_0$ where $\sigma_0$ depends only on dimension. Then if $u \in PF(\sigma, 1)$ in $C_\rho(\tilde{X}, \tilde{t})$ in the direction $\nu$, there is a constant $C$ such that $u \in PF(C\sigma, C\sigma)$ in $C_{\rho/2}(\tilde{X} + \alpha\nu, \tilde{t})$ in the direction $\nu$ for some $|\alpha| \leq C\sigma\rho$.*

*Proof.* Let $(\tilde{X}, \tilde{t}) = (0,0), \rho = 1$ and $\nu = e_n$. First we will construct a regular function which touches $\partial\Omega$ at one point.

Define

$$\eta(x,t) = e^{\frac{16(|x|^2+|t+1|)}{16(|x|^2+|t+1|)-1}}$$

for $16(|x|^2 + |t+1|) < 1$ and $\eta(x,t) \equiv 0$ otherwise. Let $D := \{(x, x_n, t) \in C_1(0,0) \mid x_n > -\sigma + s\eta(x,t)\}$. Now pick $s$ to be the largest such constant that $C_1(0,0) \cap \Omega \subset D$. As $(0,-1) \in \partial\{u > 0\}$, there must be a touching point $(X_0, t_0) \in \partial D \cap \partial\Omega \cap \{-1 \leq t \leq -15/16\}$ and $s \leq \sigma$.

Define the barrier function $v$ as follows:

$$\begin{aligned} \Delta v + \partial_t v &= 0 \text{ in } D, \\ v &= 0 \text{ in } \partial_p D \cap C_1(0,0) \\ v &= (\sigma + x_n) \text{ in } \partial_p D \cap \partial C_1(0,0). \end{aligned} \qquad \text{(B.1.8)}$$

Note that on $\partial_p D \cap C_1(0)$ we have $u = 0$ because $D$ contains the positivity set. Also, as $|\nabla u| \leq 1$, it must be the case that $u(X,t) \leq \max\{0, \sigma + x_n\}$ for all $(X,t) \in C_1(0,0)$. Since, $v \geq u$ on $\partial_p D$ it follows that $v \geq u$ on all of $D$ (by the maximum principle for subadjoint-caloric functions). We now want to estimate the normal derivative of $v$ at $(X_0, t_0)$. To estimate from below, apply Lemma B.1.6,

$$1 \leq \limsup_{(X,t)\to(X_0,t_0)} \frac{u(X,t)}{\text{pardist}((X,t),B)} \leq -\partial_\nu v(X_0,t_0) \tag{B.1.9}$$

where $\nu$ is the normal pointing out of $D$ at $(X_0, t_0)$ and $B$ is the tangent ball at $(X_0, t_0)$ to $D$ contained in $D^c$.

To estimate from above, first consider $F(X,t) := (\sigma + x_n) - v$. On $\partial_p D$,

$$-\sigma \leq x_n - v \leq \sigma$$

thus (by the maximum principle) $0 \leq F(X,t) \leq 2\sigma$. As $\partial D$ is piecewise smooth domain, standard parabolic regularity gives $\sup_D |\nabla F(X,t)| \leq K\sigma$. Note, since $s \leq \sigma$, that $-\sigma + s\eta(x,t)$ is a function whose $\text{Lip}(1,1)$ norm is bounded by a constant. Therefore, $K$ does not depend on $\sigma$.

Hence,

$$\begin{aligned} |\nabla v| - 1 \leq & |\nabla v - e_n| \leq K\sigma \\ \overset{\text{eqn (B.1.9)}}{\Rightarrow} \ 1 \leq & -\partial_\nu v(Z) \leq 1 + K\sigma. \end{aligned} \tag{B.1.10}$$

We want to show that $u \geq v - \tilde{K}\sigma x_n$ for some large constant, $\tilde{K}$, to be choosen later depending only on the dimension. Let $\tilde{Z} := (Y_0, s_0)$, where $s_0 \in (-3/4, 1), |y_0| \leq 1/2$ and $(Y_0)_n = 3/4$, and assume, in order to obtain a contradiction, that $u \leq v - \tilde{K}\sigma x_n$ at every point in $\{(Y,s_0) \mid |Y-Y_0| \leq 1/8\}$. We construct a barrier function, $w \equiv w_{\tilde{Z}}$, defined by

$$\Delta w + \partial_t w = 0 \text{ in } D \cap \{t < s_0\},$$
$$w = x_n \text{ on } \partial_p(D\{t < s_0\}) \cap \{(Y,s_0) \mid |Y-Y_0| < 1/8\},$$
$$w = 0 \text{ on } \partial_p(D \cap \{t < s_0\}) \backslash \{(Y,s_0) \mid |Y-Y_0| < 1/8\}.$$

By our initial assumption (and the definition of $w$), $u \leq v - \tilde{K}\sigma x_n$ on $\partial_p(D \cap \{t < s_0\})$ and, therefore, $u \leq v - \tilde{K}\sigma x_n$ on all of $D \cap \{t < s_0\}$. Since $t_0 \leq -15/16$ we know $(X_0, t_0) \in \partial_p(D \cap \{t < s_0\})$. Furthermore, the Hopf lemma gives an $\alpha > 0$ (independent of $\tilde{Z}$) such that $\partial_\nu w(X_0, t_0) \leq -\alpha$. With these facts in mind, apply Lemma B.1.6 at $(X_0, t_0)$ and recall equation (B.1.10) to estimate,

$$\begin{aligned} 1 = & \limsup_{(X,t)\to(X_0,t_0)} \frac{u(X,t)}{\text{pardist}((X,t),B)} \\ \leq & -\partial_\nu v(X_0,t_0) + \tilde{K}\sigma\partial_\nu w(X_0,t_0) \\ \leq & (1+K\sigma) - \tilde{K}\alpha\sigma. \end{aligned} \tag{B.1.11}$$

Setting $\tilde{K} \geq K/\alpha$ yields the desired contradiction.

Hence, there exists a point, call it $(\overline{Y}, s_0)$, such that $|\overline{Y} - Y_0| \leq 1/8$ and

$$(u-v)(\overline{Y}, s_0) \geq -\tilde{K}\sigma(\overline{Y})_n \overset{(\overline{Y})_n \leq 1}{\geq} -\tilde{K}\sigma.$$

Apply the parabolic Harnack inequality to obtain,

$$\begin{aligned} \inf_{\{|X-Y_0|<1/8\}} (u-v)(X, s_0 - 1/32) \geq c_n \sup_{\{|\tilde{X}-Y_0|<1/8\}} (u-v)(\tilde{X}, s_0) \geq -c_n\tilde{K}\sigma \\ \overset{v \geq x_n - \sigma}{\Rightarrow} u(X, s_0 - 1/32) \geq x_n - \sigma - c_n\tilde{K}\sigma, \end{aligned}$$

for all $X$ such that $|X - Y_0| < 1/8$. Ranging over all $s_0 \in (-3/4, 1)$ and $|y_0| \leq 1/2$, the above implies

$$u(X,t) \geq x_n - C\sigma,$$

whenever $(X,t)$ satisfies $|x| < 1/2, |x_n - 3/4| < 1/8, t \in (-1/2, 1/2)$ and for some constant $C > 0$ which depends only on the dimension. As $|\nabla u| \leq 1$ we can conclude, for any $(X,t)$ such that $|x| < 1/2, t \in (-1/2, 1/2)$ and $3/4 \geq x_n \geq C\sigma$, that

$$u(X,t) \geq u(x, 3/4, t) - (3/4 - x_n) \geq (x_n - C\sigma). \tag{B.1.12}$$

We now need to find an $\alpha$ such that $(0, \alpha, -1/4) \in \partial\Omega$. By the initial assumed flatness, and equation (B.1.12), $\alpha \in \mathbb{R}$ exists and $-\sigma \leq \alpha \leq C\sigma$ (here we pick $\sigma_0$ is small enough such that $C\sigma_0 < 1/4$).

In summary we have shown that,

- $(0, \alpha, -1/4) \in \partial\Omega,\ |\alpha| < C\sigma$
- $x_n - \alpha \leq -3C\sigma/2 \Rightarrow x_n \leq -\sigma \Rightarrow u(X,t) = 0.$
- When $x_n - \alpha \geq 2C\sigma \Rightarrow x_n \geq C\sigma$, hence equation (B.1.12)) implies $u(X,t) \geq (x_n - C\sigma) \geq (x_n - \alpha - 2C\sigma)$.

Therefore $u \in PF(2C\sigma, 2C\sigma)$ in $C_{1/2}(0, \alpha, 0)$ which is the desired result. □

Lemma B.1.4 is the current version of the above and follows almost identically. Thus we will omut the full proof in favor of briefly pointing out the ways in which the argument differs.

**Lemma** (Lemma B.1.4)**.** *Let $\Omega, u_\infty$ satisfy the assumptions of Proposition 3.1.10. Furthermore, assume $u_\infty \in \widetilde{CF}(\sigma, 1/2)$ in $C_\rho(Q,\tau)$ in the direction $\nu$. Then there is a constant $K_1 > 0$ (depending only on dimension) such that $u_\infty \in CF(K_1\sigma, K_1\sigma)$ in $C_{\rho/2}(Q,\tau)$ in the direction $\nu$.*

*Proof of Lemma B.1.4.* We begin in the same way; let $(Q, \tau) = (0, 0), \rho = 1$ and $\nu = e_n$. Then we recall the smooth function

$$\eta(x,t) = e^{\frac{16(|x|^2+|t+1|)}{16(|x|^2+|t+1|)-1}}$$

for $16(|x|^2 + |t + 1|) < 1$ and $\eta(x, t) \equiv 0$ otherwise. Let $D := \{(x, x_n, t) \in C_1(0, 0) \mid x_n > -\sigma + s\eta(x, t)\}$. Now pick $s$ to be the largest such constant that $C_1(0, 0) \cap \Omega \subset D$. Since $|x_n| > 1/2$ implies that $u(X, t) > 0$ there must be some touching point $(X_0, t_0) \in \partial D \cap \partial\Omega \cap \{-1 \leq t \leq -15/16\}$. Furthermore, we can assume that $s < \sigma + 1/2 < 2$.

The proof then proceeds as above until equation (B.1.12). In the setting of "past flatness" we need to argue further; the boundary point is at the bottom of the cylinder, so after the cylinder shrinks we need to search for a new boundary point. However, in current flatness the boundary point is at the center of the cylinder, so after equation (3.6.16) we have completed the proof of Lemma B.1.4. □

### *B.1.2 Flatness on Both Sides Implies Greater Flatness on the Zero Side: Lemma B.1.5*

In this section we prove Lemma B.1.5. The outline of the argument is as follows: arguing by contradiction, we obtain a sequence $u_k$ whose free boundaries $\partial\{u_k > 0\}$ approaches the graph of a function $f$. Then we prove that this function $f$ is $C^\infty$ and this smoothness leads to a contradiction.

Throughout this subsection, $\{u_k\}$ is a sequence of adjoint caloric functions such that $\partial\{u_k > 0\}$ is a parabolic regular domain and such that, for all $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$,

$$\int_{\{u_k>0\}} u_k(\Delta\varphi - \partial_t\varphi)dXdt = \int_{\partial\{u_k>0\}} h_k\varphi d\sigma.$$

We will also assume the $h_k \geq 1$ at $\sigma$-a.e. point on $\partial\{u_k > 0\}$ and $|\nabla u_k| \leq 1$. While we present these arguments for general $\{u_k\}$ it suffices to think of $u_k(X,t) := \frac{u(r_kX, r_k^2 t)}{r_k}$ for some $r_k \downarrow 0$.

**Lemma B.1.8.** *[Compare with Lemma 6.1 in (AW09)] Suppose that* $u_k \in CF(\sigma_k, \sigma_k)$ *in* $C_{\rho_k}(0,0)$*, in direction* $e_n$*, with* $\sigma_k \downarrow 0$*. Define* $f_k^+(x,t) = \sup\{d \mid (\rho_k x, \sigma_k \rho_k d, \rho_k^2 t) \in \{u_k = 0\}\}$ *and* $f_k^-(x,t) = \inf\{d \mid (\rho_k x, \sigma_k \rho_k d, \rho_k^2 t) \in \{u_k > 0\}\}$*. Then, passing to subsequences,* $f_k^+, f_k^- \to f$ *in* $L^\infty_{\text{loc}}(C_1(0,0))$ *and* $f$ *is continuous.*

*Proof.* By scaling each $u_k$ we may assume $\rho_k \equiv 1$. Then define

$$D_k := \{(y,d,t) \in C_1(0,0) \mid (y, \sigma_k d, t) \in \{u_k > 0\}\}.$$

Let

$$f(x,t) := \liminf_{\substack{(y,s)\to(x,t) \\ k\to\infty}} f_k^-(y,s),$$

so that, for every $(y_0,t_0)$, there exists a $(y_k,t_k) \to (y_0,t_0)$ such that $f_k^-(y_k,t_k) \overset{k\to\infty}{\to} f(y_0,t_0)$.

Fix a $(y_0,t_0)$ and note, as $f$ is lower semicontinuous, for every $\varepsilon > 0$, there exists a $\delta > 0, k_0 \in \mathbb{N}$ such that

$$\{(y,d,t) \mid |y-y_0| < 2\delta, |t-t_0| < 4\delta^2, d \leq f(y_0,t_0) - \varepsilon\} \cap \overline{D}_k = \emptyset, \ \forall k \geq k_0.$$

Consequently

$$x_n - f(y_0,t_0) \leq -\varepsilon \Rightarrow u_k(x, \sigma_k x_n, s) = 0, \ \forall (X,s) \in C_{2\delta}(Y_0,t_0). \tag{B.1.13}$$

Together with the definition of $f$, equation (B.1.13) implies that there exist $\alpha_k \in \mathbb{R}$ with $|\alpha_k| < 2\varepsilon$ such that $(y_0, \sigma_k(f(y_0,t_0)+\alpha_k), t_0 - \delta^2) \in \partial\{u_k > 0\}$. This observation, combined with (B.1.13) allows us to conclude, $u_k(\cdot, \sigma_k \cdot, \cdot) \in PF(3\sigma_k \frac{\varepsilon}{\delta}, 1)$ in $C_\delta(y_0, \sigma_k(f(y_0,t_0) + \alpha_k), t_0)$, for $k$ large enough.

As $\tau_k/\sigma_k^2 \to 0$ the conditions of Lemma B.1.7 hold for $k$ large enough. Thus, $u_k(\cdot, \sigma_k \cdot, \cdot) \in$

$PF(C\sigma_k \frac{\varepsilon}{\delta}, C\sigma_k \frac{\varepsilon}{\delta})$ in $C_{\delta/2}(y_0, \sigma_k f(y_0,t_0) + \tilde{\alpha}_k, t_0)$ where $|\tilde{\alpha}_k| \leq C\sigma_k \varepsilon$. Thus, whenever $z_n - (\sigma_k f(y_0,t_0) + \tilde{\alpha}_k) \geq C\varepsilon\sigma_k/2$, we have $u_k(z, \sigma_k z_n, t) > 0$ for $(Z,t) \in C_{\delta/2}(y_k, \sigma_k f_k^-(y_k,t_k) + \alpha, t_k + \delta^2)$. In other words

$$\sup_{(Z,s) \in C_{\delta/2}(y_0, \sigma_k f(y_0,t_0) + \tilde{\alpha}_k, t_0)} f_k^+(z,s) \leq f(y_0,t_0) + 3C\varepsilon. \tag{B.1.14}$$

As $f_k^+ \geq f_k^-$, if

$$\tilde{f}(y_0,t_0) := \limsup_{\substack{(y,s)\to(y_0,t_0)\\ k\to\infty}} f_k^+(y,s),$$

it follows (in light of (B.1.14) that $\tilde{f} = f$. Consequently, $f$ is continuous and $f_k^+, f_k^- \to f$ locally uniformly on compacta. □

We now show that $f$ is given by the boundary values of $w$, a solution to the adjoint heat equation in $\{x_n > 0\}$.

**Lemma B.1.9.** *[Compare with Proposition 6.2 in (AW09)] Suppose that $u_k \in CF(\sigma_k, \sigma_k)$ in $C_{\rho_k}(0,0)$, in direction $e_n$ with $\rho_k \geq 0, \sigma_k \downarrow 0$. Further assume that, after relabeling, $k$ is the subsequence given by Lemma B.1.8. Define*

$$w_k(x,d,t) := \frac{u_k(\rho_k x, \rho_k d, \rho_k^2 t) - \rho_k d}{\sigma_k}.$$

*Then, $w_k$ is bounded on $C_1(0,0) \cap \{x_n > 0\}$ (uniformly in $k$) and converges, in the $\mathbb{C}^{2,1}$-norm, on compact subsets of $C_1(0,0) \cap \{x_n > 0\}$ to $w$. Furthermore, $w$ is a solution to the adjoint-heat equation and $w(x,d,t)$ is non-increasing in $d$ when $d > 0$. Finally $w(x,0,t) = -f(x,t)$ and $w$ is continuous in $\overline{C_{1-\delta}(0,0) \cap \{x_n > 0\}}$ for any $\delta > 0$.*

*Proof.* As before we rescale and set $\rho_k \equiv 1$. Since $|\nabla u_k| \leq 1$ and $x_n \leq -\sigma_k \Rightarrow u_k = 0$ it follows that $u_k(X,t) \leq (x_n + \sigma_k)$. Which implies $w_k(X,t) \leq 1 + \tau_k$. On the other hand when $0 < x_n \leq \sigma_k$ we have $u_k(X,t) - x_n \geq -x_n \geq -\sigma_k$ which means $w_k \geq -1$. Finally, if $x_n \geq \sigma_k$ we have $u_k(X,t) - x_n \geq (x_n - \sigma_k) - x_n \Rightarrow w_k \geq -1$. Thus, $|w_k| \leq 1$ in $C_1(0,0) \cap \{x_n > 0\}$.

By definition, $w_k$ is a solution to the adjoint-heat equation in $C_1(0,0) \cap \{x_n > \sigma_k\}$. So for any $K \subset\subset \{x_n > 0\}$ the $\{w_k\}$ are, for large enough $k$, a uniformly bounded sequence of solutions to the adjoint-heat equation on $K$. As $|w_k| \leq 1$, standard estimates for parabolic equations tell us that $\{w_k\}$ is uniformly bounded in $\mathbb{C}^{2+\alpha,1+\alpha/2}(K)$. Therefore, perhaps passing to a subsequence, $w_k \to w$ in $\mathbb{C}^{2,1}(K)$. Furthermore, $w$ is also be a solution to the adjoint heat equation in $K$ and $|w| \leq 1$. A diagonalization argument allows us to conclude that $w$ is adjoint caloric on all of $\{x_n > 0\}$.

Compute that $\partial_n w_k = (\partial_n u_k - 1)/(\sigma_k) \leq 0$, which implies, $\partial_n w \leq 0$ on $\{x_n > 0\}$. As such, $w(x,0,t) := \lim_{d\to 0^+} w(x,d,t)$ exists. We will now show that this limit is equal to $-f(x,t)$. If true, then regularity theory for adjoint-caloric functions implies that $w$ is continuous is $\overline{C_{1-\delta} \cap \{x_n > 0\}}$.

First we show that the limit is less than $-f(x,t)$. Let $\varepsilon > 0$ and pick $0 < \alpha \leq 1/2$ small enough so that $|w(x,\alpha,t) - w(x,0,t)| < \varepsilon$. For $k$ large enough we have $\alpha/\sigma_k > f(x,t)+1 > f_k^-(x,t)$, therefore,

$$\begin{aligned}
&w(x,0,t) \leq w(x,\alpha,t) + \varepsilon = w_k(x, \sigma_k \frac{\alpha}{\sigma_k}, t) + \varepsilon + o_k(1) \\
&= \left( w_k(x, \sigma_k \frac{\alpha}{\sigma_k}, t) - w_k(x, \sigma_k f_k^-(x,t), t) \right) + w_k(x, \sigma_k f_k^-(x,t), t) + \varepsilon + o_k(1) \\
&\overset{\partial_n w_k \leq 0}{\leq} w_k(x, \sigma_k f_k^-(x,t), t) + o_k(1) + \varepsilon.
\end{aligned} \tag{B.1.15}$$

By definition, $w_k(x, \sigma_k f_k^-(x,t), t) = -f_k^-(x,t) \to -f(x,t)$ uniformly in $C_{1-\delta}(0,0)$. In light of equation (B.1.15), this observation implies $w(x,0,t) \leq -f(x,t) + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have $w(x,0,t) \leq -f(x,t)$.

To show $w(x,0,t) \geq -f(x,t)$ we first define, for $S > 0, k \in \mathbb{N}$,

$$\tilde{\sigma}_k = \frac{1}{S} \sup_{(Y,s) \in C_{2S\sigma_k}(x, \sigma_k f_k^-(x, t - S^2\sigma_k^2), t - S^2\sigma_k^2)} (f_k^-(x, t - S^2\sigma_k^2) - f_k^-(y,s)).$$

Observe that if $k$ is large enough (depending on $S, \delta$), then $(x, t - S^2\sigma_k^2) \in C_{1-\delta}(0,0)$. By construction, $\forall (Y,s) \in C_{2S\sigma_k}(x, \sigma_k f_k^-(x, t - S^2\sigma_k^2), t - S^2\sigma_k^2)$,

$$y_n - \sigma_k f_k^-(x, t - S^2\sigma_k^2) \le -S\sigma_k\tilde{\sigma}_k \Rightarrow y_n \le \sigma_k f_k^-(y,s) \Rightarrow u_k(Y,s) = 0.$$

In other words, $u_k \in PF(\tilde{\sigma}_k, 1)$ in $C_{S\sigma_k}(x, \sigma_k f_k^-(x, t - S^2\sigma_k^2), t)$. Note, by Lemma B.1.8, $\tilde{\sigma}_k \to 0$.

Apply Lemma B.1.7, to conclude that

$$u_k \in PF(C\tilde{\sigma}_k, C\tilde{\sigma}_k) \text{ in } C_{S\sigma_k/2}(x, \sigma_k f_k^-(x, t - S^2\sigma_k^2) + \alpha_k, t) \text{ where } |\alpha_k| \le CS\sigma_k\tilde{\sigma}_k. \tag{B.1.16}$$

Define $D_k \equiv f_k^-(x, t - S^2\sigma_k^2) + \alpha_k/\sigma_k + S/2$. Pick $S > 0$ large such that $D_k \ge 1$ and then, for large enough $k$, we have $D_k - \alpha_k/\sigma_k - f_k^-(x, t - S^2\sigma_k^2) - CS\tilde{\sigma}_k > 0$ and $(x, \sigma_k D_k, t) \in C_{S\sigma_k/2}(x, \sigma_k f_k^-(x, t - S^2\sigma_k^2) + \alpha_k, t)$. As such, the flatness condition, (B.1.16), gives

$$\begin{aligned} u_k(x, \sigma_k D_k, t) &\ge \left(\sigma_k D_k - \sigma_k f_k^-(x, t - S^2\sigma_k^2) - \alpha_k - CS\tilde{\sigma}_k\sigma_k/2\right)^+ \\ &= \frac{S\sigma_k}{2}\left(1 - C\tilde{\sigma}_k\right). \end{aligned} \tag{B.1.17}$$

Plugging this into the definition of $w_k$,

$$\begin{aligned} w_k(x, \sigma_k D_k, t) \ge& \frac{S}{2}(1 - C\tilde{\sigma}_k) - D_k \\ =& \frac{S}{2}(1 - C\tilde{\sigma}_k) - (f_k^-(x, t - S^2\sigma_k^2) + \alpha_k/\sigma_k + S/2) \\ =& - f_k^-(x, t - S^2\sigma_k^2) + o_k(1) = -f(x,t) + o_k(1). \end{aligned} \tag{B.1.18}$$

We would like to replace the left hand side of equation (B.1.18) with $w_k(x, \alpha, t)$ where $\alpha$ does not depend on $k$. We accomplish this by means of barriers; for $\varepsilon > 0$ define $z_\varepsilon$ to be the unique solution to

$$\begin{aligned} \partial_t z_\varepsilon + \Delta z_\varepsilon = 0, \text{ in } C_{1-\delta}(0,0) \cap \{x_n > 0\} \\ z_\varepsilon = g_\varepsilon, \text{ on } \partial_p(C_{1-\delta}(0,0) \cap \{x_n > 0\}) \cap \{x_n = 0\} \\ z_\varepsilon = -2, \text{ on } \partial_p(C_{1-\delta}(0,0) \cap \{x_n > 0\}) \cap \{x_n > 0\}, \end{aligned} \tag{B.1.19}$$

where $g_\varepsilon \in C^\infty(C_{1-\delta}(0,0))$ and $-f(x,t) - 2\varepsilon < g_\varepsilon(x,t) < -f(x,t) - \varepsilon$. By standard parabolic theory for any $\varepsilon > 0$ there exists an $\alpha > 0$ (which depends on $\varepsilon > 0$) such that $|x_n| < \alpha$ implies $|z_\varepsilon(x, x_n, t) - z_\varepsilon(x, 0, t)| < \varepsilon/2$. Pick $k$ large enough so that $\sigma_k < \alpha$. We know $w_k$ solves the adjoint heat equation on $\{x_n \geq \sigma_k\}$ and, by equations (B.1.19) and (B.1.18), $w_k \geq z_\varepsilon$ on $\partial_p(C_{1-\delta}(0,0) \cap \{x_n > \sigma_k\})$. Therefore, $w_k \geq z_\varepsilon$ on all of $C_{1-\delta}(0,0) \cap \{x_n > \sigma_k\}$.

Consequently,

$$w_k(x, \alpha, t) \geq z_\varepsilon(x, \alpha, t) \geq z_\varepsilon(x, 0, t) - \varepsilon/2 \geq -f(x,t) - 3\varepsilon.$$

As $k \to \infty$ we know $w_k(x, \alpha, t) \to w(x, \alpha, t) \leq w(x, 0, t)$. This gives the desired result. □

The next step is to prove that the normal derivative of $w$ on $\{x_n = 0\}$ is zero. This will allow us to extend $w$ smoothly over $\{x_n = 0\}$ and obtain regularity for $f$.

**Lemma B.1.10.** *Suppose the assumptions of Lemma B.1.8 are satisfied and that $k$ is the subsequence identified in that lemma. Further suppose that $w$ is the limit function identified in Lemma B.1.9. Then $\partial_n w = 0$, in the sense of distributions, on $\{x_n = 0\} \cap C_{1/2}(0,0)$.*

*Proof.* Rescale so $\rho_k \equiv 1$ and define $g(x,t) = 5 - 8(|x|^2 + |t|)$. For $(x, 0, t) \in C_{1/2}(0,0)$ we observe $f(x, 0, t) \leq 1 \leq g(x, 0, t)$. We shall work in the following set

$$Z := \{(x, x_n, t) \mid |x|, |t| \leq 1, x_n \in \mathbb{R}\}.$$

For any $\phi(x,t)$, define $Z^+(\phi)$ to be the set of points in $Z$ above the graph $\{(X,t) \mid x_n =$

$\phi(x,t)\}$, $Z^-(\phi)$ as set of points below the graph and $Z^0(\phi)$ as the graph itself. Finally, let $\Sigma_k := \{u_k > 0\} \cap Z^0(\sigma_k g)$.

Recall, for any Borel set $A$, we define the "surface measure", $\mu(A) = \int_{-\infty}^{\infty} \mathcal{H}^{n-1}(A \cap \{s = t\})dt$. If $k$ is sufficiently large, and potentially adding a small constant to $g$, $\mu(Z^0(\sigma_k g) \cap \partial\{u_k > 0\} \cap C_{1/2}(0,0)) = 0$.

There are three claims, which together prove the desired result.

**Claim 1:**

$$\mu(\partial\{u_k > 0\} \cap Z^-(\sigma_k g)) \leq \int_{\Sigma_k} \partial_n u_k - 1 dx dt + \mu(\Sigma_k) + C\sigma_k^2$$

*Proof of Claim 1*: For any positive $\phi \in C_0^\infty(C_1(0,0))$ we have

$$\begin{aligned}\int_{\partial\{u_k>0\}} \phi d\mu \leq \int_{\partial\{u_k>0\}} \phi d\mu &= \int_{\{u_k>0\}} u_k(\Delta\phi - \partial_t\phi) dX dt \\ &= -\int_{\{u_k>0\}} \nabla u_k \cdot \nabla\phi + u_k \partial_t \phi\end{aligned} \tag{B.1.20}$$

(we can use integration by parts because, for almost every $t$, $\{u_k > 0\} \cap \{s = t\}$ is a set of finite perimeter). Let $\phi \to \chi_{Z^-(\sigma_k g)}\chi_{C_1}$ (as functions of bounded variation) and, since $|t| > 3/4$ or $|x|^2 > 3/4$ implies $u(x, \sigma_k g(x,t), t) = 0$, equation (3.6.25) becomes

$$\mu(\partial\{u_k > 0\} \cap Z^-(\sigma_k g)) \leq -\int_{\Sigma_k} \frac{\nabla u_k \cdot \nu + \sigma_k u_k \operatorname{sgn}(t)}{\sqrt{1 + \sigma_k^2(|\nabla_x g(x,t)|^2 + 1)}} d\mu, \tag{B.1.21}$$

where $\nu(x,t) = (\sigma_k \nabla g(x,t), -1)$ points outward spatially in the normal direction.

We address the term with $\operatorname{sgn}(t)$ first; the gradient bound on $u_k$ tells us that $|u_k| \leq C\sigma_k$ on $\Sigma_k$, so

$$\left|\sigma_k \int_{\Sigma_k} \frac{u_k \operatorname{sgn}(t)}{\sqrt{1 + \sigma_k^2(|\nabla_x g(x,t)|^2 + 1)}} d\mu\right| \leq C\sigma_k^2. \tag{B.1.22}$$

To bound the other term, note that $\frac{d\mu}{\sqrt{1+\sigma_k^2(|\nabla_x g(x,t)|^2+1)}} = dxdt$ where the latter integration takes place over $E_k = \{(x,t) \mid (x, \sigma_k g(x,t), t) \in \Sigma_k\} \subset \{x_n = 0\}$. Then integrate by parts in $x$ to obtain

$$\int_{E_k} (\sigma_k \nabla g(x,t), -1) \cdot \nabla u_k(x, \sigma_k g(x,t), t) dx dt = \int_{\partial E_k} \sigma_k u_k(x, \sigma_k g(x,t), t) \partial_\eta g d\mathcal{H}^{n-2} dt$$
$$- \int_{E_k} \sigma_k u_k(x, \sigma_k g(x,t), t) \Delta g(x,t) + \sigma_k^2 \partial_n u_k(x, \sigma_k g(x,t), t) |\nabla g|^2 dx dt$$
$$- \int_{E_k} \partial_n u_k(x, \sigma_k g(x,t), t) - 1 dx dt + \mathcal{L}^n(E_k), \tag{B.1.23}$$

where $\eta$ is the outward space normal on $\partial E_k$. Since $u_k(x, \sigma_k g(x,t), t) = 0$ for $(x,t) \in \partial E_k$ the first term zeroes out.

The careful reader may object that $E_k$ may not be a set of finite perimeter and thus our use of integration by parts is not justified. However, for any $t_0$, we may use the coarea formula with the $L^1$ function $\chi_{\{u(x,\sigma_k g(x,t_0),t_0)>0\}}$ and the smooth function $\sigma_k g(-,t_0)$ to get

$$\begin{aligned} \infty &> \int \sigma_k |\nabla g(x,t_0)| \chi_{\{u(x,\sigma_k g(x,t_0),t_0)>0\}} dx \\ &= \int_{-\infty}^{\infty} \int_{\{(x,t_0) | \sigma_k g(x,t_0) = r\}} \chi_{\{u(x,r,t_0)>0\}} d\mathcal{H}^{n-2}(x) dr. \end{aligned}$$

Thus $\{(x,t_0) \mid \sigma_k g(x,t_0) > r\} \cap \{(x,t) \mid u(x, \sigma_k g(x,t_0), t_0) > 0\}$ is a set of finite perimeter for almost every $r$. Equivalently, $\{(x,t_0) \mid \sigma_k(g(x,t_0) + \varepsilon) > 0\} \cap \{(x,t) \mid u(x, \sigma_k(g(x,t_0) + \varepsilon), t_0) > 0\}$ is a set of finite perimeter for almost every $\varepsilon \in \mathbb{R}$. Hence, there exists a $\varepsilon > 0$ aribtrarily small such that if we replace $g$ by $g + \varepsilon$ then $E_k \cap \{t = t_0\}$ will be a set of finite perimeter for almost every $t_0$. Since we can perturb $g$ slightly without changing the above arguments, we may safely assume that $E_k$ is a set of finite perimeter for almost every time slice.

Observe that $\Delta g$ is bounded above by a constant, $|u_k| \leq C\sigma_k$ on $\Sigma_k$, $|\partial_n u_k| \leq 1$ and finally $\mu(\Sigma_k) \geq \mathcal{L}^n(E_k)$. Hence,

$$\mu(\partial\{u_k > 0\} \cap Z^-(\sigma_k g)) \leq \int_{E_k} \partial_n u_k - 1 dx dt + \mu(\Sigma_k) + C\sigma_k^2.$$

Integrating over $E_k$ is the same as integrating over $\Sigma_k$ modulo a factor of $\sqrt{1+|\sigma_k \nabla_x g|^2}$ (which is comparable to $1+\sigma_k^2$). As $|\partial_n u_k - 1|$ is bounded we may rewrite the above as

$$\mu(\partial\{u_k > 0\} \cap Z^-(\sigma_k g)) \leq \int_{\Sigma_k} \partial_n u_k - 1 d\mu + \mu(\Sigma_k) + C\sigma_k^2,$$

which is the desired inequality.

Before moving on to Claim 2, observe that arguing as in equations (B.1.22) and (B.1.23),

$$\int_{\Sigma_k} \frac{(\sigma_k \nabla_x g(x,t), 0, \sigma_k \mathrm{sgn}(t))}{\sqrt{1+\|(\sigma_k \nabla_x g(x,t), 0, \sigma_k \mathrm{sgn}(t))\|^2}} \cdot (\nabla_x w_k, 0, w_k) d\mu \overset{k\to\infty}{\to} 0, \tag{B.1.24}$$

which will be useful to us later.

**Claim 2:**

$$\mu(\Sigma_k) - C_2\sigma_k^2 \leq \mu(\partial\{u_k > 0\} \cap Z^-(\sigma_k g)).$$

*Proof of Claim 2:* Let $\nu_k(x,t)$ the inward pointing measure theoretic space normal to $\partial\{u_k > 0\} \cap \{s = t\}$ at the point $x$. For almost every $t$ it is true that $\nu_k$ exists $\mathcal{H}^{n-1}$ almost everywhere. Defining $\nu_{\sigma_k g}(X,t) = \frac{1}{\sqrt{1+256\sigma_k^2|x|^2}}(-\sigma_k 16x, 1, 0)$, we have

$$\mu(\partial\{u_k > 0\} \cap Z^-(\sigma_k g)) = \int_{\partial\{u_k>0\}\cap Z^-(\sigma_k g)} \nu_k \cdot \nu_k d\mu \geq \int_{\partial^*\{u_k>0\}\cap Z^-(\sigma_k g)} \nu_k \cdot \nu_k d\mu \geq$$

$$\int_{\partial^*\{u_k>0\}\cap Z^-(\sigma_k g)} \nu_k \cdot \nu_{\sigma_k g} d\mu \overset{\text{div thm}}{=} -\int_{Z^-(\sigma_k g)\cap\{u_k>0\}} \text{div } \nu_{\sigma_k g} dX dt + \int_{\Sigma_k} 1 d\mu.$$

We compute $|\text{div } \nu_{\sigma_k g}| = \left| \frac{-16\sigma_k(n-1)}{\sqrt{1+256\sigma_k^2|x|^2}} + \frac{3\sigma_k^3(16*256)|x|^2}{\sqrt{1+256\sigma_k^2|x|^2}^3} \right| \leq C\sigma_k$. Since the "width" of $Z^-(\sigma_k g) \cap \{u_k > 0\}$ is of order $\sigma_k$, the claim follows.

**Claim 3:**

$$\int_{\Sigma_k} |\partial_n w_k| \overset{k\to\infty}{\to} 0.$$

*Proof of Claim 3:* We first notice that $\partial_n u_k \leq 1$ and, therefore, $\partial_n w_k \leq 0$. To show the limit above is at least zero we compute

$$\begin{aligned}\int_{\Sigma_k} \partial_n w_k d\mu &= \int_{\Sigma_k} \frac{\partial_n u_k - 1}{\sigma_k} d\mu \\ &\overset{\text{Claim1}}{\geq} \frac{\mu(\partial\{u_k > 0\} \cap Z^-(\sigma_k g))}{\sigma_k} - \frac{\mu(\Sigma_k)}{\sigma_k} - C\sigma_k \\ &\overset{\text{Claim2}}{\geq} \frac{\mu(\Sigma_k)}{\sigma_k} - \frac{\mu(\Sigma_k)}{\sigma_k} - C\sigma_k \to 0.\end{aligned} \tag{B.1.25}$$

We can now combine these claims to reach the desired conclusion. We say that $\partial_n w = 0$, in the sense of distributions on $\{x_n = 0\}$, if, for any $\zeta \in C_0^\infty(C_{1/2}(0,0))$,

$$\int_{\{x_n=0\}} \partial_n w \zeta = 0.$$

By claim 3

$$0 = \lim_{k\to\infty} \int_{\Sigma_k} \zeta \partial_n w_k. \tag{B.1.26}$$

On the other hand equation (B.1.24) (and $\zeta$ bounded) implies

$$\lim_{k\to\infty} \int_{\Sigma_k} \zeta \partial_n w_k = \lim_{k\to\infty} \int_{\Sigma_k} \zeta \nu_{\Sigma_k} \cdot (\nabla w_k, w_k), \tag{B.1.27}$$

where $\nu_{\Sigma_K}$ is the unit normal to $\Sigma_k$ (thought of as a Lispchitz graph in $(x,t)$) pointing upwards. Together, equations (B.1.26), (B.1.27) and the divergence theorem in the domain $Z^+(\sigma_k g) \cap C_{1/2}(0,0)$ have as a consequence

$$\begin{aligned}0 &= \lim_{k\to\infty} \int_{Z^+(\sigma_k g)} \operatorname{div}_{X,t}(\zeta(\nabla_X w_k, w_k)) dX dt \\ &= \lim_{k\to\infty} \int_{Z^+(\sigma_k)} \nabla_X \zeta \cdot \nabla_X w_k + (\partial_t \zeta) w_k + \zeta(\Delta_X w_k + \partial_t w_k) dX dt \\ &\overset{\Delta w + \partial_t w = 0}{=} \int_{\{x_n>0\}} \nabla_X w \cdot \nabla_X \zeta + (\partial_t \zeta) w_k dX dt \\ &\overset{\text{integration by parts}}{=} \int_{\{x_n=0\}} w_n \zeta dx dt - \int_{\{x_n>0\}} \zeta(\Delta_X w + \partial_t w).\end{aligned}$$

As $w$ is a solution to the adjoint heat equation this implies that $\int_{\{x_n=0\}} \partial_n w \zeta = 0$ which is the desired result. □

From here it is easy to conclude regularity of $f$.

**Corollary B.1.11.** *Suppose the assumptions of Lemma B.1.8 are satisfied and that $k$ is the subsequence identified in that lemma. Then $f \in C^\infty(C_{1/2}(0,0))$ and in particular the $\mathbb{C}^{2+\alpha,1+\alpha}$ norm of $f$ in $C_{1/4}$ is bounded by an absolute constant.*

*Proof.* Extend $w$ by reflection across $\{x_n = 0\}$. By Lemma B.1.10 this new $w$ satisfies the adjoint heat equation in all of $C_{1/2}(0,0)$ (recall a continuous weak solution to the adjoint heat equation in the cylinder is actually a classical solution to the adjoint heat equation). Since $\|w\|_{L^\infty(C_{3/4}(0,0))} \leq 2$, standard regularity theory yields the desired results about $-f = w|_{x_n=0}$. □

We can use this regularity to prove Lemma B.1.5.

*Proof of Lemma B.1.5.* Without loss of generality let $(Q,\tau) = (0,0)$ and we will assume that the conclusions of the lemma do not hold. Choose a $\theta \in (0,1)$ and, by assumption, there exists $\rho_k$ and $\sigma_k \downarrow 0$ such that $u \in CF(\sigma_k, \sigma_k)$ in $C_{\rho_k}(0,0)$ in the direction $\nu_k$ (which after a harmless rotation we can set to be $e_n$) but so that $u$ is not in $\widetilde{CF}(\theta\sigma_k, \theta\sigma_k)$ in $C_{c(n)\theta\rho_k}(0,0)$ in any direction $\nu$ with $|\nu_k - \nu| \leq C\sigma_k$ and for any constant $c(n)$. Let $u_k = \frac{u(\rho_k X, \rho_k^2 t)}{\rho_k}$. It is clear that $u_k$ is adjoint caloric, that its zero set is a parabolic regular domain, that $|\nabla u_k| \leq 1$ and $h_k \geq 1$.

By Lemma B.1.8 we know that there exists a continuous function $f$ such that $\partial\{u_k > 0\} \to \{(X,t) \mid x_n = f(x,t)\}$ in the Hausdorff distance sense. Corollary B.1.11 implies that there is a universal constant, call it $K$, such that

$$f(x,t) \leq f(0,0) + \nabla_x f(0,0) \cdot x + K(|t| + |x|^2) \tag{B.1.28}$$

for $(x,t) \in C_{1/4}(0,0)$. Since $(0,0) \in \partial\{u_k > 0\}$ for all $k$, $f(0,0) = 0$. If $\theta \in (0,1)$, then there exists a $k$ large enough (depending on $\theta$ and the dimension) such that

$$f_k^+(x,t) \leq \nabla_x f(0,0) \cdot x + \theta^2/4K, \ \forall (x,t) \in C_{\theta/(4K)}(0,0),$$

where $f_k^+$ is as in Lemma B.1.8. Let

$$\nu_k := \frac{(-\sigma_k \nabla_x f(0,0), 1)}{\sqrt{1 + |\sigma_k \nabla_x f(0,0)|^2}}$$

and compute

$$x \cdot \nu_k \geq \theta^2 \sigma_k/4K \Rightarrow x_n \geq \sigma_k x' \cdot \nabla_x f(0,0) + \theta^2 \sigma_k/4K \geq f_k^+(x,t). \tag{B.1.29}$$

Therefore, if $(X,t) \in C_{\theta/(4K)}(0,0)$ and $x \cdot \nu_k \geq \theta^2 \sigma_k/4K$, then $u_k(X,t) > 0$. Arguing similarly for $f_k^-$ we can see that $u_k \in \widetilde{CF}(\theta\sigma_k, \theta\sigma_k)$ in $C_{\theta/4K}(0,0)$ in the direction $\nu_k$. It is easy to see that $|\nu_k - e_n| \leq C\sigma_k$ and so we have the desired contradiction. □

## B.2 Non-tangential limit of $F(Q,\tau)$: proof of Lemma 3.3.6

If $\partial\Omega$ is smooth and $F(Q,\tau)$ is the non-tangential limit of $\nabla u(Q,\tau)$, then $F(Q,\tau) = h(Q,\tau)\hat{n}(Q,\tau)$. In this section we prove this is true when $\partial\Omega$ is a parabolic chord arc domain; this is Lemma 3.3.6. Easy modifications of our arguments will give the finite pole result, Lemma 3.3.10. Before we begin, we establish two geometric facts about parabolic regular domains which will be useful. The first is on the existence of "tangent planes" almost everywhere.

**Lemma B.2.1.** *Let $\Omega$ be a parabolic regular domain. For $\sigma$-a.e. $(Q,\tau) \in \partial\Omega$ there exists a $n$-plane $P \equiv P(Q,\tau)$ such that $P$ contains a line parallel to the $t$ axis and such that for any*

$\varepsilon > 0$ *there exists a* $r_\varepsilon > 0$ *where*

$$0 < r < r_\varepsilon \Rightarrow |\langle \hat{n}(Q,\tau), Z - Q\rangle| < r\varepsilon,\ \forall (Z,t) \in \partial\Omega \cap C_r(Q,\tau).$$

*Proof.* For $(Q,\tau) \in \partial\Omega, r > 0$ define

$$\gamma_\infty(Q,\tau,r) = \inf_P \sup_{(Z,\eta)\in C_r(Q,\tau)\cap\partial\Omega} \frac{d((Z,\eta),P)}{r}$$

where the infinum is taken over all $n$-planes through $(Q,\tau)$ with a line parallel to the $t$-axis. Let $P(Q,\tau,r)$ be a plane which achieves the minimum. For $(Q,\tau) \in \partial\Omega, r > 0$ we have

$$\gamma_\infty(Q,\tau,r)^{n+3} \leq 16^{n+3}\gamma(Q,\tau,2r)$$

(see (HLN03), equation (2.2)).

Parabolic uniform rectifiability demands that $\frac{\gamma(Q,\tau,2r)}{2r}$ be integrable at zero for $\sigma$-a.e. $(Q,\tau)$. Thus it is clear that $\gamma_\infty(Q,\tau,r) \overset{r\downarrow 0}{\to} 0$ for $\sigma$-a.e. $(Q,\tau)$. Let $r_j \downarrow 0$ and $P_j := P(Q,\tau,r_j)$, which approximate $\partial\Omega$ near $(Q,\tau)$ increasingly well. Passing to a subsequence, compactness implies $P_j \to P_\infty(Q,\tau)$. Our lemma is proven if $P_\infty(Q,\tau)$ does not depend on the sequence (or subsequence) chosen.

If $K$ is any compact set then for $\mathcal{L}^1$-a.e. $t$, $\partial\Omega \cap K \cap \{s = t\}$ is a set of locally finite perimeter. The theory of sets of locally finite perimeter (see e.g. (EG92)) says that for $\mathcal{H}^{n-1}$-a.e. point on this time slice there exists a unique measure theoretic space-normal. Therefore, for $\sigma$-a.e. $(Q,\tau) \in \partial\Omega$ there is a measure theoretic space normal $\hat{n} := \hat{n}((Q,\tau))$ (this vector is normal to the time slice as opposed to the whole surface). Let $(Q,\tau)$ be a point both with a measure theoretic normal and such that $\gamma_\infty(Q,\tau,r) \to 0$. We claim $P_\infty(Q,\tau) = \hat{n}(Q,\tau)^\perp$, and thus is independent of the sequence $r_j \downarrow 0$.

Restricting the equality, $\lim_j \sup_{(Z,\eta)\in\Delta_{r_j}(Q,\tau)} \frac{d((Z,\eta),P(Q,\tau,r_j))}{r_j} = 0$, to the time-slice, $\{\eta \equiv \tau\}$, we get $\lim_j \sup_{(Z,\tau)\in\Delta_{r_j}(Q,\tau)} \frac{d((Z,\tau),P(Q,\tau,r_j))}{r_j} = 0$. Since a point-wise tangent plane must also be a measure theoretic tangent plane, $P_j \to \hat{n}^{\perp}$. $\square$

A consequence of the above lemma is a characterization of the infinitesimal behavior of $\sigma$.

**Corollary B.2.2.** *Let $\Omega$ be a parabolic chord regular domain. For $\sigma$-a.e. $(Q,\tau) \in \partial\Omega$ and any $\varepsilon > 0$, we can choose an $R \equiv R(\varepsilon, Q, \tau) > 0$ such that for $r < R$,*

$$\left| \frac{\sigma(\Delta_r(Q,\tau))}{r^{n+1}} - 1 \right| < \varepsilon.$$

*Proof.* Let $\Omega_{r,(Q,\tau)} = \{(P,\eta) \mid (Q + rP, r^2\eta + \tau) \in \Omega\}$. Lemma B.2.1 tells us that, for any compact set $K$ containing 0, after a rotation which depends on $(Q,\tau)$, we have $K \cap \Omega_{r,(Q,\tau)} \to K \cap \{(X,t) \mid x_n > 0\}$ in the Hausdorff distance sense. In particular $\chi_{\Omega_{r,(Q,\tau)}} =: \chi_r \to \chi_{\{x_n > 0\}}$ in $L^1_{\mathrm{loc}}$. This convergence immediately gives, using the divergence theorem (on each time slice), that $\hat{n}_r\sigma_r := \hat{n}_r\sigma|_{\partial\Omega_{r,(Q,\tau)}}$ converges weakly to $e_n\sigma|_{\{x_n > 0\}}$ (here $\hat{n}_r$ is the measure theoretic space normal to $\partial\Omega_{r,(Q,\tau)}$).

$\hat{n}(Q,\tau) \in L^1_{\mathrm{loc}}(d\sigma)$, therefore, $\sigma$-a.e. $(Q,\tau) \in \partial\Omega$ is a Lebesgue point for $\hat{n}(Q,\tau)$. Assume that $(Q,\tau)$ is a Lesbesgue point and that the tangent plane at $(Q,\tau)$ is $\{x_n = 0\}$. Then,

$$\lim_{r \downarrow 0} \fint_{\Delta_r(Q,\tau)} \hat{n}\, d\sigma = e_n \Leftrightarrow \lim_{r \downarrow 0} \fint_{C_1(0,0) \cap \partial\Omega_{r,(Q,\tau)}} \hat{n}_r\, d\sigma_r = e_n.$$

Weak convergence implies (recall that $\sigma(C_1(0,0) \cap \{x_n = 0\}) = 1$)

$$\liminf_{r \downarrow 0} \frac{1}{\sigma_r(C_1(0,0) \cap \partial\Omega_{r,(Q,\tau)})} \leq 1 \leq \limsup_{r \downarrow 0} \frac{1}{\sigma_r(C_1(0,0) \cap \partial\Omega_{r,(Q,\tau)})}.$$

As $C_1(0,0)$ is a set of continuity for $\sigma|_{\{x_n = 0\}}$ we can conclude that $\lim_{r \downarrow 0} \sigma_r(C_1(0,0) \cap \partial\Omega_{r,(Q,\tau)}) = \sigma(C_1(0,0) \cap \{x_n = 0\}) = 1$. Recall $\sigma_r(C_1(0,0) \cap \partial\Omega_{r,(Q,\tau)}) = \frac{1}{r^{n+1}}\sigma(\Delta_r(Q,\tau))$ (see (EG92), Chapter 5.7, pp. 202-204 for more details) and the result follows. $\square$

Our proof of Lemma 3.3.6 is in two steps; first, we show that $F(Q,\tau)$ points in the direction of the measure theoretic space normal $\hat{n}(Q,\tau)$. Here we follow very closely the proof of Lemma 3.2 in (KT06).

**Lemma B.2.3.** *For $\sigma$-a.e. $(Q,\tau) \in \partial\Omega$ we have $F(Q,\tau) = \langle F(Q,\tau), \vec{n}(Q,\tau)\rangle\, \vec{n}(Q,\tau)$.*

*Proof.* Let $(Q,\tau) \in \partial\Omega$ be a point of density for $F$ and $M_1(h)$, be such that there is a tangent plane at $(Q,\tau)$ (in the sense of Lemma B.2.1), satisfy $F(Q,\tau), M_1(h)(Q,\tau) < \infty$ and be such that $\nabla u$ converges non-tangentially to $F$ at $(Q,\tau)$.

In order to discuss the above conditions in a quantative fashion we introduce, for $\varepsilon, \eta > 0$:

$$\begin{aligned} \delta_\varepsilon(r) &:= \frac{1}{r^{n+1}}\sigma(\{(P,\zeta) \in C_{2r}(Q,\tau) \cap \partial\Omega \mid |F(P,\zeta) - F(Q,\tau)| > \varepsilon\}) \\ \delta'(r) &:= \frac{1}{r^{n+1}}\sigma(\{(P,\zeta) \in C_{2r}(Q,\tau) \cap \partial\Omega \mid M_1(h)(P,\zeta) \geq 2M_1(h)(Q,\tau)\}) \\ \delta''_\eta(r) &:= \frac{1}{r^{n+1}}\sigma((P,\zeta) \in C_{2r}(Q,\tau) \cap \partial\Omega\backslash E(\eta)). \end{aligned} \tag{B.2.1}$$

To define $E(\eta)$, first, for any $\varepsilon > 0$ and $\lambda > 0$, let

$$H(\lambda,\varepsilon) := \{(P,\zeta) \in \partial\Omega \mid |F(P,\zeta) - \nabla u(X,t)| < \varepsilon,\ \forall (X,t) \in \Gamma^{\lambda}_{10}(P,\zeta)\}.$$

By Corollary 3.3.5, for each $\varepsilon > 0$ and $\sigma$-a.e. $(P,\zeta) \in \partial\Omega$ there is some $\lambda$ such that $(P,\zeta) \in H(\lambda,\varepsilon)$. Arguing as in the proof of Egoroff's theorem (see, e.g., Theorem 3, Chapter 1.2 in (EG92)), for any $\eta > 0$ we can find a $\lambda(\varepsilon,\eta)$ such that $\sigma(\partial\Omega\backslash H(\lambda(\varepsilon,\eta),\varepsilon)) < \eta$. Let $\varepsilon_n = 2^{-n}$ and $\eta_n = 2^{-n-1}\eta$ to obtain $\lambda_n := \lambda(\varepsilon_n,\eta_n)$ as above. Then define $E(\eta) = \bigcap_{n\geq 0} H(\lambda_n,\varepsilon_n)$. Note, that $\sigma(\partial\Omega\backslash \bigcup_{\eta>0} E(\eta)) = 0$, as such, for $\sigma$-a.e. $(Q,\tau) \in \partial\Omega$ there is some $\eta > 0$ such that $(Q,\tau)$ is a density point for $E(\eta)$. At those points (for the relevant $\eta$) we have $\delta''_\eta(r) \to 0$.

$(Q,\tau)$ is a point of density for $M_1(h), F$, hence $\delta'(r), \delta_\varepsilon(r) \to 0$ for any $\varepsilon > 0$. Additionally, $\partial\Omega$ has a tangent plane at $(Q,\tau)$, so there exists an $n$-plane $V$ (containing a line parallel to the $t$-axis), a function $\ell(r)$ and $R, \eta > 0$ such that

$$\lim_{r \downarrow 0} \frac{\ell(r)}{r} = 0$$

$$\sup_{(P,\zeta) \in C_{2r}(Q,\tau) \cap \partial\Omega} d((P,\zeta), V \cap C_{2r}(Q,\tau)) \leq \ell(r) \tag{B.2.2}$$

$$\sigma(C_{\ell(r)}(P,\zeta)) \geq 2[\delta_\varepsilon(r) + \delta'(r) + \delta''_\eta(r)]r^{n+1}, \forall (P,\zeta) \in \partial\Omega \cap C_{2r}(Q,\tau).$$

For ease of notation, assume that $(Q,\tau) = (0,0)$ and $V$ (the tangent plane at $(Q,\tau)$) is $\{x_n = 0\}$. Let $D(r) := \{(x, x_n, \tau) \mid |x| < r, x_n = \frac{1}{2} C_0 \ell(r)\}$ where $C_0$ is a large constant satisfying the following constraint:

If $(Y,0) \in D(r)$ and $(P,\zeta) \in C_{2r}(0,0)$ are such that $\|(y,0,0) - (p,0,\zeta)\| \leq 2\ell(r)$ then

$$D(r) \cap C_{\ell(r)}(Y,0) \subset \Gamma_{10}^{C_0 \ell(r)}(P,\zeta).$$

We make the following claim, whose proof, for the sake of continuity, will be delayed until later.

**Claim 1:** *Under the assumptions above, if $r > 0$ is small enough, $(Y,0) \in D(r)$, we have:*

$$|u(Y,0)| \leq C M_1(h)(0,0)\ell(r) \tag{B.2.3}$$

Given two points $(Y_1,0), (Y_2,0) \in D(r)$, we want to estimate $\langle F(0,0), Y_2 - Y_1 \rangle$ in terms of $u(Y_2,0) - u(Y_1,0)$. Define $R(Y_1,Y_2) = u(Y_2,0) - u(Y_1,0) - \langle F(0,0), Y_2 - Y_1 \rangle$. Equations (B.2.2) and (B.2.3) imply that, for $r > 0$ small, we have $|u(Y_1,0)|, |u(Y_2,0)| < CM_1(h)(0,0)\ell(r) < C\varepsilon r$. Therefore, in order to show that $\langle F(0,0), Y_2 - Y_1 \rangle$ is small, it suffices to show that $R(Y_1,Y_2)$ is small.

Write $u(Y_2,0) - u(Y_1,0) = \int_0^1 \langle \nabla u(Y_1 + \theta(Y_2 - Y_1), 0), Y_2 - Y_1 \rangle \, d\theta$ which implies

$$|R(Y_1,Y_2)| \leq |Y_2 - Y_1| \int_0^1 |\nabla u(Y_1 + \theta(Y_2 - Y_1), 0) - F(0,0)| d\theta. \tag{B.2.4}$$

If we define $I(r) = \frac{1}{r} \fint_{D(r)} \fint_{D(r)} |R(Y_1, Y_2)| dY_1 dY_2$, then Fubini's theorem and equation (B.2.4) yields

$$I(r) \leq C_n \fint_{D(r)} |\nabla u(X,0) - F(0,0)| dX \tag{B.2.5}$$

(for more details see (KT03), Appendix A.2). We now arrive at our second claim:

**Claim 2:** *For $(Y,0) \in D(r)$ we have $|\nabla u(Y,0) - F(0,0)| < 2\varepsilon$.*

Claim 2 immediately implies that $I(r) < C\varepsilon$, which, as $|u| \leq C\varepsilon r$ on $D(r)$, gives

$$\fint_{D(r)} \fint_{D(r)} | \langle F(0,0), Y_2 - Y_1 \rangle | dY_1 dY_2 \leq rI(r) + 2C\varepsilon r < C\varepsilon r. \tag{B.2.6}$$

Pick any direction $e \perp e_n$ such that $e \in \mathbb{R}^n$ (i.e. has no time component) and let $M := | \langle F(0,0), e \rangle |$. Now consider the cone of directions $\tilde{\Gamma}$ in $\mathbb{S}^{n-2}$ (i.e. perpendicular to both the time direction and $e_n$) such that $| \langle F(0,0), \tilde{e} \rangle | \geq M/2$ for $\tilde{e} \in \tilde{\Gamma}$. A simple calculation reveals that $\mathcal{H}^{n-2}(\tilde{\Gamma})/\mathcal{H}^{n-2}(\mathbb{S}^{n-2}) = c_n$ a constant depending only on dimension. Thus (B.2.6) implies

$$C\varepsilon r > \fint_{D(r)} \fint_{D(r)} | \langle F(0,0), Y_2 - Y_1 \rangle | dY_1 dY_2 \geq$$

$$\frac{c(n)}{r^{n-1}} \fint_{D(r) \cap \{(Y,t) || y| < r/2\}} \int_0^{r/2} \int_{\theta \in \tilde{\Gamma}} \langle F(0,0), \rho\theta \rangle \rho^{n-2} d\mathbb{S}^{n-2} d\rho \geq \tilde{C} M r$$

which of course implies that $M \leq C\varepsilon$. In other words $\left\langle F(0,0), x' \right\rangle \leq C\varepsilon |x'|$ for any $x' \in \mathbb{R}^{n-1}$, which is the desired result. $\square$

*Proof of Claim 1.* For $(Y,0) \in D(r)$, we want to show that $|u(Y,0)| \leq CM_1(h)(0,0)\ell(r)$. As $\partial\Omega$ is well approximated by $\{x_n = 0\}$ in $C_{2r}(0,0)$, there must be a $(P,0) \in \partial\Omega \cap C_{3r/2}(0,0)$ such that $p = y$, and, hence, $|p_n| < \ell(r)$. Let $C(Y) := C_{\ell(r)}(P,0)$. Note that equation (B.2.2) implies $\sigma(C(Y)) \geq 2\delta'(r) r^{n+1}$. Given the definition of $\delta'(r)$ we can conclude the existence of $(\tilde{P}, \tilde{\zeta}) \in C(Y) \cap \partial\Omega$ such that $M_1(h)(\tilde{P}, \tilde{\zeta}) < 2M_1(h)(0,0)$. Furthermore $(\tilde{P}, \tilde{\zeta}) \in C_{2r}(0,0)$ and $\|(\tilde{p}, 0, \tilde{\zeta}) - (y, 0, 0)\| < 2\ell(r)$.

By the condition on $C_0$ and the aperture of the cone we can conclude that $(Y,0) \in$

$\Gamma_{10}^{C_0\ell(r)}(\tilde{P},\tilde{\zeta})$. Then, arguing as in Lemma 3.3.2 (i.e. using Lemma 3.2.2, the backwards Harnack inequality for the Green's function, $\|(Y,0)-(\tilde{P},\tilde{\zeta})\| \sim C_0\ell(r)$, Lemma B.3.4 and that $\omega$ is doubling) we can conclude

$$u((Y,0)) \leq Cu(A^{-}_{C\ell(r)}(\tilde{P},\tilde{\zeta})) \leq$$

$$\frac{C}{\ell(r)^n}\omega(C_{C\ell(r)}(\tilde{P},\tilde{\zeta})) \leq C\ell(r) \fint_{C_{c\ell(r)}(\tilde{P},\tilde{\zeta})} h(Z,t)d\sigma(Z,t) \leq C\ell(r)M_1(h)(\tilde{P},\tilde{\zeta}).$$

As $M_1(h)(\tilde{P},\tilde{\zeta}) \leq 2M_1(h)(0,0)$ we are done. □

*Proof of Claim 2.* We want to show that for $(Y,0) \in D(r)$ we have $|\nabla u(Y,0) - F(0,0)| < 2\varepsilon$. Arguing exactly like in Claim 1 produces a $(P,0) \in \partial\Omega$ and then $C(Y)$. This time we use equation (B.2.2) to give the bound $\sigma(C(Y)) \geq 2(\delta''(r) + \delta_\varepsilon(r))r^{n+1}$. We can then conclude that there exists a $(\tilde{P},\tilde{\zeta}) \in C(Y)\cap\partial\Omega$ such that $(\tilde{P},\tilde{\zeta}) \in E(\eta,R)$ and $|F(\tilde{P},\tilde{\zeta}) - F(0,0)| < \varepsilon$.

Recall that $(\tilde{P},\tilde{\zeta}) \in E(\eta) \Rightarrow (\tilde{P},\tilde{\zeta}) \in H(\lambda_n, 2^{-n})$ for all $n$. So pick $n$ large enough that $2^{-n} < \varepsilon$ and then $r$ small enough so that $C_0\ell(r) < \lambda_n$. Thus $|\nabla u(Y,0) - F(0,0)| < |\nabla u(Y,0) - F(\tilde{P},\tilde{\zeta})| + |F(\tilde{P},\tilde{\zeta}) - F(0,0)| < 2\varepsilon$. □

We now want to show that $|F(Q,\tau)| = h(Q,\tau)$ $d\sigma$-almost everywhere. Here, again, we follow closely the approach of Kenig and Toro ((KT06), Lemma 3.4) who prove the analogous elliptic result. One difference here is that the time and space directions scale differently. To deal with this difficulty, we introduce a technical lemma.

**Lemma B.2.4.** *Let $1 < p < \infty$ and $g \in L^p_{\text{loc}}(d\sigma)$. Then*

$$\frac{1}{\sigma(\Delta((Q,\tau),r,s))}\int_{\Delta((Q,\tau),r,s)\cap\partial\Omega} g d\sigma \overset{s,r\downarrow 0}{\to} g(Q,\tau)$$

*for $\sigma$-a.e. $(Q,\tau) \in \partial\Omega$. (Here, and from now on, $C((Q,\tau),r,s) = \{(X,t) \mid |X-Q| \leq r, |t-\tau| < s^{1/2}\}$ and $\Delta((Q,\tau),r,s) = C((Q,\tau),r,s) \cap \partial\Omega$.)*

*Proof.* Follows from the work of Zygmund, (Zyg34), and the fact that $(\partial\Omega, \sigma)$ is a space of homogenous type. $\square$

*Proof of Lemma 3.3.6.* We will prove the theorem for all $(Q, \tau) \in \partial\Omega$ for which Proposition B.2.3 holds, such that there is a tangent plane to $\partial\Omega$ at $(Q, \tau)$ and such that

$$\begin{aligned}
\lim_{r\downarrow 0} \frac{\sigma(\Delta_r(Q,\tau))}{r^{n+1}} &= 1 \\
\lim_{r,s\downarrow 0} \fint_{C((Q,\tau),r,s)\cap\partial\Omega} h d\sigma &= h(Q,\tau) \\
\lim_{r\downarrow 0} \fint_{C_r(Q,\tau)\cap\partial\Omega} F d\sigma &= F(Q,\tau) \\
\lim_{r\downarrow 0} \fint_{C_r(Q,\tau)\cap\partial\Omega} M_1(h) d\sigma &= M_1(h)(Q,\tau) \\
\lim_{\substack{(X,t)\to(Q,\tau) \\ (X,t)\in\Gamma(Q,\tau)}} \nabla u(X,t) &= F(Q,\tau) \\
M_1(h)((Q,\tau)), F(Q,\tau), h(Q,\tau) &< \infty.
\end{aligned} \tag{B.2.7}$$

That this is $\sigma$-a.e. point follows from Proposition B.2.3, Lemmas B.2.1, B.2.2, B.2.4 and 3.3.5, and $F, h, M_1(h) \in L^2_{\text{loc}}(d\sigma)$.

$\partial\Omega \cap \{s = t\}$ is a set of locally finite perimeter for almost every $t$, hence, for any $\phi \in C_c^\infty(\mathbb{R}^{n+1})$,

$$\int_{\partial\Omega} \phi h d\sigma = \int_\Omega u(\Delta\phi - \phi_t) dX dt = -\int_\Omega \nabla\phi \cdot \nabla u + u\phi_t dX dt. \tag{B.2.8}$$

Let $\rho_1, \rho_2 > 0$ and set $\phi(X,t) = \zeta(|X-Q|/\rho_1)\xi(|t-\tau|/\rho_2^2)$. We calculate that $\nabla\phi(X,t) = \xi(|t-\tau|/\rho_2^2)\zeta'(|X-Q|/\rho_1)\frac{X-Q}{|X-Q|\rho_1}$ and also that $\frac{d}{d\rho_1}\zeta(|X-Q|/\rho_1) = -\frac{|X-Q|}{\rho_1^2}\zeta'(|X-Q|/\rho_1)$. Together this implies

$$-\nabla\phi(X,t) = \xi(|t-\tau|/\rho_2^2)\rho_1 \frac{X-Q}{|X-Q|^2}\frac{d}{d\rho_1}\zeta(|X-Q|/\rho_1). \tag{B.2.9}$$

Similarly $\partial_t\phi(X,t) = \zeta(|X-Q|/\rho_1)\xi'(|t-\tau|/\rho_2^2)\frac{\text{sgn}(t-\tau)}{\rho_2^2}$ and $\frac{d}{d\rho_2}\xi(|t-\tau|/\rho_2^2) = -\frac{2|t-\tau|}{\rho_2^3}\xi'(|t-$

$\tau|/\rho_2^2)$, therefore,

$$-\partial_t\phi(X,t) = \zeta(|X-Q|/\rho_1)\frac{\rho_2}{2(t-\tau)}\frac{d}{d\rho_2}\xi(|t-\tau|/\rho_2^2). \tag{B.2.10}$$

Plugging equations (B.2.9), (B.2.10) into equation (B.2.8) and letting $\xi, \zeta$ approximate $\chi_{[0,1]}$ we obtain

$$\int_{\partial\Omega\cap C((Q,\tau),\rho_1,\rho_2)} h d\sigma = \rho_1 \frac{d}{d\rho_1}\int_{\Omega\cap C((Q,\tau),\rho_1,\rho_2)} \left\langle \nabla u(X,t), \frac{X-Q}{|X-Q|^2}\right\rangle dXdt+$$

$$\rho_2\frac{d}{d\rho_2}\int_{\Omega\cap C((Q,\tau),\rho_1,\rho_2)} \frac{u(X,t)}{2(t-\tau)} dXdt.$$

Differentiating under the integral and then integrating $\rho_1, \rho_2$ from 0 to $\rho > 0$ yields

$$\underbrace{\int_0^\rho\int_0^\rho\int_{\partial\Omega\cap C((Q,\tau),\rho_1,\rho_2)} h d\sigma d\rho_1 d\rho_2}_{(I)} = \underbrace{\int_0^\rho\int_{\Omega\cap C((Q,\tau),\rho,\rho_2)}\left\langle \nabla u, \frac{X-Q}{|X-Q|}\right\rangle dXdtd\rho_2}_{(II)} + \underbrace{\int_0^\rho\int_{\Omega\cap C((Q,\tau),\rho_1,\rho)} udXdtd\rho_1}_{(III)}. \tag{B.2.11}$$

For any $\varepsilon > 0$ we will prove that there is a $\delta > 0$ such that if $\rho < \delta$ we have

$$|(\mathrm{III})| < \varepsilon\rho^{n+3}$$
$$|(\mathrm{I}) - \frac{1}{3n}h(Q,\tau)\rho^{n+3}| < \varepsilon\rho^{n+3}$$
$$|(\mathrm{II}) - \langle F(Q,\tau), \hat{n}(Q,\tau)\rangle \frac{1}{3n}\rho^{n+3}| < \varepsilon\rho^{n+3},$$

which implies the desired result. Note that throughout the proof the constants may seem a little odd due to our initial normalization of Hausdorff measure.

**Analysis of (III):** $u$ is continuous in $\overline{\Omega}$, hence for any $\varepsilon' > 0$ there is a $\delta = \delta(\varepsilon') > 0$ such that if $\delta > \rho$ then $(X,t) \in C_\rho(Q,\tau) \Rightarrow u(X,t) < \varepsilon'$. It follows that, $|(\mathrm{III})| \leq C\varepsilon' \int_0^\rho \rho_1^n \rho^2 d\rho_1 = \varepsilon\rho^{n+3}$, choosing $\varepsilon'$ small enough.

**Analysis of (I):** $(Q,\tau)$ is a point of density for $h$, so for any $\varepsilon' > 0$ there exists a $\delta > 0$ such that if $\rho < \delta$ then

$$|(I) - h(Q,\tau)\int_0^\rho \int_0^\rho \sigma(\Delta((Q,\tau),\rho_1,\rho_2))d\rho_1 d\rho_2| < \varepsilon' \int_0^\rho \int_0^\rho \sigma(\Delta((Q,\tau),\rho_1,\rho_2))d\rho_1 d\rho_2.$$

Switching the order of integration, $\int_0^\rho \int_0^\rho \sigma(\Delta((Q,\tau),\rho_1,\rho_2))d\rho_1 d\rho_2 = \int_{\Delta_\rho(Q,\tau)}(\rho - |X - Q|)(\rho - \sqrt{|t-\tau|})d\sigma$. Consider the change of coordinates $X = \rho Y + Q$ and $t = s\rho^2 + \tau$. As $\partial\Omega$ has a tangent plane, $V$, at $(Q,\tau)$ the set $\{(Y,s) \mid (X,t) \in C_\rho(Q,\tau) \cap \partial\Omega\}$ converges (in the Hausdorff distance sense) to $C_1(0,0) \cap V$. Therefore,

$$\frac{1}{\rho^{n+3}}\int_{\Delta_\rho(Q,\tau)}(\rho - |X-Q|)(\rho - \sqrt{|t-\tau|})d\sigma \overset{\rho\downarrow 0}{\to} \int_{|y|<1,|s|<1}(1-|y|)(1-\sqrt{s})dyds = \frac{1}{3n}.$$

Which, together with the above arguments, yields the desired inequality.

**Analysis of (II):** Writing $\nabla u(X,t) = (\nabla u(X,t) - F(Q,\tau)) + F(Q,\tau)$ we obtain

$$\text{(II)} = \underbrace{\int_0^\rho \int_{\Omega\cap C((Q,\tau),\rho,\rho_2)} \left\langle \nabla u(X,t) - F(Q,\tau), \frac{X-Q}{|X-Q|}\right\rangle dXdtd\rho_2}_{\text{(E)}}$$

$$+|F(Q,\tau)|\int_0^\rho \int_{\Omega\cap C((Q,\tau),\rho,\rho_2)} \left\langle \hat{n}(Q,\tau), \frac{X-Q}{|X-Q|}\right\rangle dXdtd\rho_2.$$

In the second term above, divide the domain of integration into points within $\varepsilon'\rho$ of the tangent plane $V$ at $(Q,\tau)$ and those distance more than $\varepsilon'\rho$ away. By the Ahlfors regularity of $\partial\Omega$, the former integral will have size $< C\varepsilon'\rho^{n+3}$. The latter (without integrating in $\rho_2$) is

$$\int_{\Omega\cap C((Q,\tau),\rho,\rho_2)\cap\{(X,t)|\langle X-Q,\hat{n}(Q,\tau)\rangle \geq \varepsilon'\rho\}} \left\langle \hat{n}(Q,\tau), \frac{X-Q}{|X-Q|}\right\rangle dXdt.$$

Again we change variables so that $X = \rho Y + Q$ and $t = s\rho^2 + \tau$, and recall that, under this change of variables, our domain $\Omega$ converges to a half space. Arguing as in our analysis of

(I), a simple computation (see equation 3.72 in (KT06)) yields

$$\left| \int_{\Omega \cap C((Q,\tau),\rho,\rho_2) \cap \{(x,t) | \langle x-Q, \hat{n}(Q,\tau)\rangle \geq \varepsilon' \rho\}} \left\langle \hat{n}(Q,\tau), \frac{x-Q}{|x-Q|} \right\rangle dxdt - \frac{\rho_2^2 \rho^n}{n} \right| < C\varepsilon' \rho^n \rho_2^2.$$

Integration in $\rho_2$ gives

$$|(\mathrm{II}) - |F(Q,\tau)| \frac{1}{3n} \rho^{n+3}| < C\varepsilon' \rho^{n+3} + |(\mathrm{E})|.$$

The desired result then follows if we can show $|(E)| < \varepsilon' \rho^{n+3}$. To accomplish this we argue exactly as in proof of Lemma 3.4, (KT06) but include the arguments here for completeness.

We first make the simple estimate

$$|(E)| \leq \rho \left| \int_{\Omega \cap C_\rho(Q,\tau)} \left\langle \nabla u(X,t) - F(Q,\tau), \frac{X-Q}{|X-Q|} \right\rangle dXdt \right|.$$

If $(X,t) \in C_\rho(Q,\tau) \cup \{(X,t) \mid \langle X-Q, \hat{n}(Q,\tau)\rangle \geq 4\varepsilon'\rho\}$ then $\delta(X,t) > 2\varepsilon'\rho$ for small enough $\rho$ (because we have a tangent plane at $(Q,\tau)$). On the other hand we have $\varepsilon' \|(X,t) - (Q,\tau)\| \leq 2\varepsilon'\rho < \delta(X,t)$ which implies that $(X,t)$ is in some fixed non-tangential region of $(Q,\tau)$.

By the definition of non-tangential convergence (which says we have convergence for all cones of all apertures), for any $\eta > 0$ if we make $\rho > 0$ even smaller we have $|\nabla u(X,t) - F(Q,\tau)| < \eta$. Therefore,

$$(\mathrm{E}) \leq C\eta\rho^{n+3} + \rho \int_{C_\rho(Q,\tau) \cap \Omega \cap \{(X,t) || \langle X-Q, \hat{n}(Q,\tau)\rangle| \leq 4\varepsilon'\rho\}} |\nabla u(X,t)| + |F(Q,\tau)| dXdt.$$

Standard parabolic regularity results imply that $|\nabla u(X,t)| \leq C\frac{u(X,t)}{\delta(X,t)}$. As the closest point to $(X,t)$ on $\partial\Omega$ is in $C_{2\rho}(Q,\tau)$ we may apply Lemma 3.2.1 and then Lemma 3.2.2 to get $|\nabla u(X,t)| \leq C\frac{u(A^-_{8\rho}(Q,\tau))}{\sqrt{\delta(X,t)\rho}}$. Continue arguing as in Lemma 3.3.2 to conclude

$$(X,t) \in C_\rho(Q,\tau) \cap \Omega \cap \{\frac{\rho}{2^{i+1}} < \delta(X,t) \leq \frac{\rho}{2^i}\} \Rightarrow |\nabla u(X,t)| \leq C2^{i/2}M_1(h)(Q,\tau).$$

Let $i_0 \geq 1$ be such that $\frac{1}{2^{i_0+1}} < 4\varepsilon' < \frac{1}{2^{i_0}}$ and recall $|F(Q,\tau)| < \infty$ to obtain,

$$\rho \int_{C_\rho(Q,\tau)\cap\Omega\cap\{(X,t)||\langle X-Q,\hat{n}(Q,\tau)\rangle|\leq 4\varepsilon'\rho\}} |\nabla u(X,t)| + |F(Q,\tau)|dXdt$$

$$< C\varepsilon'\rho^{n+3} + \rho\sum_{i=i_0}^{\infty}\int_{C_\rho(Q,\tau)\cap\Omega\cap\{\frac{\rho}{2^{i+1}}<\delta(X,t)\leq\frac{\rho}{2^i}\}} |\nabla u(X,t)|dXdt$$

$$< C\varepsilon'\rho^{n+3} + C\rho M_1(h)(Q,\tau)\sum_{i=i_0}^{\infty} 2^{i/2}|C_\rho(Q,\tau)\cap\Omega\cap\{\frac{\rho}{2^{i+1}} < \delta(X,t) \leq \frac{\rho}{2^i}\}|.$$

A covering argument (using the Ahlfors regularity of $\partial\Omega$) yields

$$|C_\rho(Q,\tau)\cap\Omega\cap\{\frac{\rho}{2^{i+1}} < \delta(X,t) \leq \frac{\rho}{2^i}\}| \leq C2^{i(n+1)}\left(\frac{\rho}{2^i}\right)^{n+2} \leq C\frac{\rho^{n+2}}{2^i}.$$

As $\sum_{i=i_0}^{\infty} 2^{-i/2} < C\sqrt{\varepsilon'}$ the desired result follows. □

## B.3 Caloric Measure at $\infty$

We recall the existence and uniquness of the Green function and caloric measure with pole at infinity (done by Nyström (Nys06b) ). We also establish some estimates in the spirit of Section 3.2.

**Lemma B.3.1.** *[Lemma 14 in (Nys06b)] Let $\Omega \subset \mathbb{R}^{n+1}$ be an unbounded $\delta$-Reifenberg flat domain, with $\delta > 0$ small enough (depending only on dimension), and $(Q,\tau) \in \partial\Omega$. There* *exists a unique function $u$ such that,*

$$\begin{aligned} u(X,t) > 0,\ (X,t) \in \Omega \\ u(X,t) = 0,\ (X,t) \in \mathbb{R}^{n+1} \backslash \Omega \\ \Delta u(X,t) + u_t(X,t) = 0,\ (X,t) \in \Omega \\ u(A_1^-(Q,\tau)) = 1. \end{aligned} \tag{B.3.1}$$

*Furthermore $u$ satisfies a backwards Harnack inequality at any scale with constant $c$ depending only on dimension and $\delta > 0$ (see Lemma 3.2.6).*

*Proof.* Without loss of generality let $(Q,\tau) = (0,0)$ and for ease of notation write $A \equiv A_1^-(0,0)$. Any $u$ which satisfies equation (B.3.1) also satisfies a backwards Harnack inequality at any scale with constant $c$ depending only on dimension and $\delta > 0$ (see the proof of Lemma 3.11 in (HLN04)). The proof then follows as in (Nys06b), Lemma 14. □

**Corollary B.3.2.** *[Lemma 15 in (Nys06b)] Let $\Omega, u$ be as in Lemma B.3.1. There exists a unique Radon measure $\omega$, supported on $\partial\Omega$ satisfying*

$$\int_{\partial\Omega} \varphi d\omega = \int_{\Omega} u(\Delta\varphi - \partial_t \varphi),\ \forall \varphi \in C_c^\infty(\mathbb{R}^{n+1}). \tag{B.3.2}$$

*Proof.* Uniqueness is immediate from equation (B.3.2); for any $(Q,\tau) \in \partial\Omega$ and $r > 0$ let $\varphi$ approximate $\chi_{C_r(Q,\tau)}(X,t)$. Existence follows as in Lemma 15 in (Nys06b). □

**Lemma B.3.3.** *[Lemma 3.2.7 for the caloric measure at infinity] Let $u, \Omega, \omega$, be as in Corollary B.3.2. There is a universal constant $c > 0$ such that for all $(Q,\tau) \in \partial\Omega, r > 0$ we have*

$$\omega(\Delta_{2r}(Q,\tau)) \leq c\omega(\Delta_r(Q,\tau)).$$

*Proof.* Follows as in Lemma 15 in (Nys06b). □

**Corollary B.3.4.** *[Lemma 3.2.4 for caloric measure and Green's function at infinity] Let $u, \Omega, \omega$ be as in Corollary B.3.2. There is a universal constant $c > 0$ such that for all $(Q,\tau) \in \partial\Omega, r > 0$ we have*

$$c^{-1}r^n u(A_r^+(Q,\tau)) \leq \omega(\Delta_{r/2}(Q,\tau)) \leq cr^n u(A_r^-(Q,\tau)) \tag{B.3.3}$$

*Proof.* The inequality on the right hand side follows from (B.3.2); let $\chi_{C_{r/2}(Q,\tau)}(X,t) \leq \varphi(X,t) \leq \chi_{C_r(Q,\tau)}(X,t)$ and $|\Delta\varphi|, |\partial_t\varphi| < C/r^2$. Inequality then follows from Lemma 3.2.2.

The left hand side is more involved: as in the proof of Lemmas B.3.1 and B.3.2 we can write $u$ as the uniform limit of $u_j$s (multiples of Green's functions with finite poles) and $\omega$ as the weak limit of $\omega_j$s (multiplies of caloric measures with finite poles) which satisfy Lemma 3.2.4 at $(Q,\tau)$ for larger and larger scales. Taking limits gives that

$$c^{-1}r^n u(A_r^+(Q,\tau)) \leq \omega(\overline{\Delta_{r/2}(Q,\tau)}).$$

That $\omega$ is doubling implies the desired result. □

**Proposition B.3.5.** *[see Theorem 1 in (HLN04)] Let $\Omega$ be a parabolic regular domain with Reifenberg constant $\delta > 0$. There is some $\overline{\delta} = \overline{\delta}(M, \|\nu\|_+) > 0$ such that if $\delta < \overline{\delta}$ then $\omega \in A^\infty(d\sigma)$. That is to say, there exists a $p > 1$ and a constant $c = c(n,p) > 0$ such that $\omega$ satisfies a reverse Harnack inequality with exponent $p$ and constant $c$ at any $(Q,\tau) \in \partial\Omega$ and at any scale $r > 0$.*

*Proof Sketch.* The proof follows exactly as in (HLN04), with Lemma B.3.3, Corollary B.3.4 and the fact that the Green function at infinity satisfies the strong Harnack inequality substituting for the corresponding facts for the Green function with a finite pole. □

## B.4 Boundary behaviour of caloric functions in parabolic Reifenberg flat domains

In this appendix we prove some basic facts about the boundary behviour of caloric functions in parabolic Reifenberg flat domains, culminating in an analogue of Fatou's theorem (Lemma B.4.1) and a representation formula for adjoint caloric functions with integrable non-tangential maximal functions (Proposition B.4.4). Often the theorems and proofs mirror those in the elliptic setting; in these cases we follow closely the presentation of Jerison and Kenig ((JK82)) and Hunt and Wheeden ((HW68) and (HW70)).

In the elliptic setting, the standard arguments rely heavily on the fact that harmonic measure (on, e.g. NTA domains) is doubling. Unfortunately, we do not know if caloric measure is doubling for parabolic NTA domains. However, in Reifenberg flat domains, Lemma 3.2.7 tells us that caloric measure is doubling in a certain sense. Using this, and other estimates in Section 3.2, we can follow Hunt and Wheeden's argument to show that the Martin boundary of a Reifenberg flat domain is equal to its topological boundary. The theory of Martin Boundaries (see Martin's original paper (Mar41) or Part 1 Chapter 19 in Doob (Doo84)) then allows us to conclude the following representation formula for bounded caloric functions.

**Lemma B.4.1.** *Let $\Omega$ be a parabolic $\delta$-Reifenberg flat domain with $\delta > 0$ small enough. Then for any $(X_0, t_0) \in \Omega$ the adjoint-Martin boundary of $\Omega$ relative to $(X_0, t_0)$ is all of $\partial\Omega \cap \{t > t_0\}$. Furthermore, for any bounded solution to the adjoint-heat equation, $u$, and any $s > t_0$ there exists a $g(P, \eta) \in L^\infty(\partial\Omega)$ such that*

$$u(Y, s) = \int_{\partial\Omega} g(P, \eta) K^{(X_0,t_0)}(P, \eta, Y, s) d\hat{\omega}^{(X_0,t_0)}(P, \eta). \tag{B.4.1}$$

*Here $K^{(X_0,t_0)}(P, \eta, Y, s) \equiv \frac{d\hat{\omega}^{(Y,s)}}{d\hat{\omega}^{(X_0,t_0)}}(P, \eta)$ which exists for all $s > t_0$ and $\omega^{(X_0,t_0)}$-a.e. $(P, \eta) \in \partial\Omega$ by the Harnack inequality.*

*Finally, if* $u(Y,s) = \int_{\partial\Omega} g(P,\eta) K^{(X_0,t_0)}(P,\eta,Y,s)\, d\hat{\omega}^{(X_0,t_0)}(P,\eta)$*, then* $u$ *has a non-tangential limit,* $g(Q,\tau)$*, for* $d\hat{\omega}^{(X_0,t_0)}$*-almost every* $(Q,\tau) \in \partial\Omega$*.*

*Proof.* Recall that the Martin boundary $\partial^M \Omega$ of $\Omega$ with respect to $(X_0, t_0)$ is the largest subset of $\partial\Omega$ such that the Martin kernel $V^{(X_0,t_0)}(X,t,Y,s) := \frac{G(X,t,Y,s)}{G(X,t,X_0,t_0)}$ has a continuous extension $V^{(X_0,t_0)} \in C(\partial^M \Omega \cap \{(X,t) \in \Omega \mid t > t_0\} \times \{(Y,s) \in \Omega \mid s > t_0\})$. Martin's representation theorem (see the theorem on page 371 of (Doo84)) states that for any bounded solution to the adjoint-heat equation, $u$, there exists a measure, $\mu_u$, such that $u(Y,s) = \int_{\partial^M \Omega} V^{(X_0,t_0)}(Q,\tau,Y,s)\, d\mu_u(Q,\tau)$ where $\partial^M \Omega$ is the Martin boundary of $\Omega$.

That $V^{(X_0,t_0)}(Q,\tau,Y,s)$ exists for all $\tau, s > t_0$ (and is, in fact, Hölder continuous in $(Q,\tau)$ for $\tau > s$) follows from Lemmas 3.2.8 and 3.2.9. When $s > \tau$ it is clear that $V^{(X_0,t_0)}(Q,\tau,Y,s) = 0$. So indeed the Martin boundary is equal to the whole boundary (after time $t_0$).

We will now prove, for a bounded solution $u$ to the adjoint-heat equation, $\mu_u << \hat{\omega}^{(X_0,t_0)}$ on any compact $K \subset\subset \{t > t_0\}$. To prove this, first assume that $u$ is positive (if not, add a constant to $u$ to make it positive). Let $(Q,\tau) \in \partial\Omega$ be such that $\tau > t_0$. Then there exists an $A \geq 100$ and an $r_0 > 0$ such that for all $r < r_0$ we have $(X_0, t_0) \in T^-_{A,r}(Q,\tau)$. Applying Lemmas 3.2.4 and 3.2.6 and observing that $\lim_{(X,t) \to (Q',\tau') \in \Delta_{r/4}(Q,\tau)} \omega^{(X,t)}(\Delta_{r/2}(Q,\tau)) = 1$ we can conclude that there is some constant $\gamma > 0$ such that

$$\hat{\omega}^{(X_0,t_0)}(\Delta_r(Q,\tau)) V^{(X_0,t_0)}(Q',\tau',A^-_r(Q,\tau)) \geq \gamma,$$

for all $(Q',\tau') \in \Delta_{r/4}(Q,\tau)$.

It follows that,

$$
\begin{aligned}
\frac{\mu_u(\Delta_{r/4}(Q,\tau))}{\hat{\omega}^{(X_0,t_0)}(\Delta_{r/4}(Q,\tau))} \leq & C\frac{\mu_u(\Delta_{r/4}(Q,\tau))}{\hat{\omega}^{(X_0,t_0)}(\Delta_r(Q,\tau))} \\
\leq & C\gamma^{-1}\int_{\Delta_{r/4}(Q,\tau)} V^{(X_0,t_0)}(Q',\tau',A_r^-(Q,\tau))d\mu_u(Q',\tau') \\
\leq & C\gamma^{-1}u(A_r^-(Q,\tau)) \leq C\gamma^{-1}\|u\|_{L^\infty}.
\end{aligned}
\tag{B.4.2}
$$

Therefore, there is a $g_u(P,\eta) = \frac{d\mu_u}{d\hat{\omega}^{(X_0,t_0)}}$ such that (by Martin's representation theorem)

$$
u(Y,s) = \int_{\partial\Omega} g_u(P,\eta)V^{(X_0,t_0)}(P,\eta,Y,s)d\hat{\omega}^{(X_0,t_0)}(P,\eta). \tag{B.4.3}
$$

Assume that $V^{(X_0,t_0)}(Q,\tau,Y,s) = K^{(X_0,t_0)}(Q,\tau,Y,s)$. Then equation (B.4.3) is equation (B.4.1). The existence of a non-tangential limit follows from a standard argument (see e.g. (HW68)) which requires three estimates. First, that $\omega^{(X_0,t_0)}$ is doubling, which we know is true after some scale for any point $(Q,\tau)$ with $\tau > t_0$. Second we need, for $(Q_0,t_0) \in \partial\Omega, r > 0$,

$$
\lim_{(X,t)\to(Q_0,t_0)} \sup_{(Q,\tau)\notin\Delta_r(Q_0,t_0)} K^{(X_0,t_0)}(Q,\tau,X,t) = 0.
$$

This follows from Harnack chain estimates and Lemma 3.2.1 (see the proof of Lemma 4.15 in (JK82) for more details). Finally, for some $\alpha > 0$, which depends on the flatness of $\Omega$, we want

$$
K^{(X_0,t_0)}(P,\eta,A_{4R}^-(Q,\tau)) \leq \frac{C2^{-\alpha j}}{\omega^{(X_0,t_0)}(\Delta_{2^{-j}R}(Q,\tau))},
$$

for all $(P,\eta) \in \Delta_{R2^{-j}}(Q,\tau)\backslash\Delta_{R2^{-j-1}}(Q,\tau)$ and for values of $R$ small. This follows from Lemmas 3.2.1 and Lemmas 3.2.4 (see (JK82), Lemma 4.14 for more details).

So it suffices to show that $V^{(X_0,t_0)}(Q,\tau,Y,s) = K^{(X_0,t_0)}(Q,\tau,Y,s)$. Fix $r > 0$, $(Q,\tau) \in \partial\Omega$ and consider the adjoint-caloric function $U(Y,s) = \hat{\omega}^{(Y,s)}(\Delta_r(Q,\tau))$. By the Martin representation theorem and equation (B.4.2) there is a function $g \equiv g_{Q,\tau,r}$ such that

$$U(Y,s) = \int V^{(X_0,t_0)}(P,\eta,Y,s)g(P,\eta)d\hat{\omega}^{(X_0,t_0)}. \tag{B.4.4}$$

We are going to show that $g(P,\eta) = \chi_{\Delta_r(Q,\tau)}$. If true then, by the definition of caloric measure, we conclude

$$\begin{aligned}\int_{\Delta_r(Q,\tau)} V^{(X_0,t_0)}(P,\eta,Y,s)d\hat{\omega}^{(X_0,t_0)}(P,\eta) &= \hat{\omega}^{(Y,s)}(\Delta_r(Q,\tau)) \\ &= \int_{\Delta_r(Q,\tau)} K^{(X_0,t_0)}(P,\eta,Y,s)d\hat{\omega}^{(X_0,t_0)}(P,\eta),\end{aligned}$$

for all surface balls. It would follow that $V = K$.

For a closed $E \subset \partial\Omega$, following the notation of (Mar41) Section 3, let $U_E$ be the unique adjoint-caloric function in $\partial\Omega$ given as the limit inferior of super adjoint-caloric functions which agree with $U$ on open sets, $\mathcal{O}$, containing $E$ and which are adjoint-caloric on $\Omega\backslash\overline{\mathcal{O}}$ with zero boundary values on $\partial\Omega\backslash\mathcal{O}$. In a $\delta$-Reifenberg flat domain (where the Martin boundary agrees with and has the same topology as the topological boundary) and if $E = \overline{\Delta_\rho(P,\eta)}$, for some $(P,\eta) \in \partial\Omega$ and $\rho > 0$, it is easy to compute that $U_E(Y,s) = \hat{\omega}^{(Y,s)}(E)$. By the uniqueness of distributions (Theorem III on page 160 in (Mar41)) it must be the case that

$$U_E(Y,s) = \int_E V^{(X_0,t_0)}(P,\eta,Y,s)g(P,\eta)d\hat{\omega}^{(X_0,t_0)},$$

where $g$ is as in equation (B.4.4). If $(Y,s) = (X_0,t_0)$ and $E = \overline{\Delta_\rho(P_0,\eta_0)} \subset \Delta_r(Q,\tau)$ then the above equation becomes

$$\hat{\omega}^{(X_0,t_0)}(\overline{\Delta_\rho(P_0,\eta_0)}) = \hat{\omega}^{(X_0,t_0)}(\Delta_\rho(P_0,\eta_0)) = \int_{\Delta_\rho(P_0,\eta_0)} g(P,\eta)d\hat{\omega}^{(X_0,t_0)}.$$

Letting $(P_0,\eta_0)$ and $\rho > 0$ vary it is clear that $g(P,\eta) = \chi_{\Delta_r(Q,\tau)}$ and we are done. $\square$

Our approach differs most substantially from the elliptic theory in the construction of sawtooth domains. In particular, Jones' argument using "pipes" ((Jon82), see also Lemma 6.3 in (JK82)) does not obviously extend to the parabolic setting. The crucial difference is that parabolic Harnack chains move forward through time, whereas elliptic Harnack chains are directionless. The best result in the parabolic context is the work of Brown (Bro89), who constructed sawtooth domains inside of $\mathrm{Lip}(1, 1/2)$-graph domains. Our argument below, which is in the same spirit as Brown's, works for $\delta$-Reifenberg flat domains. Before the proof we make the following observation.

**Remark B.4.2.** *Let $\Omega$ be a $\delta$-Reifenberg flat domain and let $(X,t) \in \Omega$. If $(P,\eta) \in \partial\Omega$ such that*

$$\|(P,\eta) - (X,t)\| = \delta_\Omega(X,t) := \inf_{(Q,\tau)\in\partial\Omega} \|(Q,\tau) - (X,t)\|$$

*then $\eta = t$. That is, every "closest" point to $(X,t)$ has time coordinate $t$.*

*Justification.* By Reifenberg flatness, for any $(P,\eta) \in \partial\Omega$ the point $(P,t)$ is within distance $\delta|t-\eta|^{1/2}$ of $\partial\Omega$. Then $\delta_\Omega(X,t) \leq |P - X| + \delta|t-\eta|^{1/2} < \|(P,\eta) - (X,t)\|$. □

We are now ready to construct sawtooth domains. Recall, for $\alpha > 0$ and $F \subset \partial\Omega$ closed, we define $S_\alpha(F) = \{(X,t) \in \partial\Omega \mid \exists (Q,\tau) \in F, \text{ s.t. } (X,t) \in \Gamma_\alpha(Q,\tau)\}$.

**Lemma B.4.3.** *Let $\Omega$ be a $(\delta^{10})$-Reifenberg flat parabolic NTA domain and $F \subset \partial\Omega \cap C_s(0,0)$ be a closed set. There is a universal constant $c \in (0,1)$ such that if $c > \delta > 0$ then $S_\alpha(F)$ is a parabolic $\delta$-Reifenberg flat domain for almost every $\alpha \geq \alpha_0(\delta) > 0$. Furthermore, if $(X,t) \in S_\alpha(F)$ then, on $F$, $\hat{\omega}^{(X,t)}_{S_\alpha(F)} << \hat{\omega}^{(X,t)} << \hat{\omega}^{(X,t)}_{S_\alpha(F)}$. Here $\hat{\omega}^{(X,t)}_{S_\alpha(F)}$ is the adjoint-caloric measure of $S_\alpha(F)$ with a pole at $(X,t)$.*

*Proof.* To prove that $S_\alpha(F)$ is $\delta$-Reifenberg flat first consider $(Q,\tau) \in F \subset \partial S_\alpha(F)$ and $\rho > 0$. Let $V$ be an $n$-plane through $(Q,\tau)$ containing a vector in the time direction such that $D[\partial\Omega \cap C_{2\rho}(Q,\tau), V \cap C_{4\rho}(Q,\tau)] \leq 2\delta^{10}\rho$. If $(X,t) \in \partial S_\alpha(F) \cap C_\rho(Q,\tau)$ then (for $\alpha \geq 1$) $(P,t) \in C_{2\rho}(Q,\tau) \cap \partial\Omega$ where $(P,t) \in \partial\Omega$ satisfies $\delta_\Omega(X,t) = \mathrm{dist}((P,t),(X,t))$. We can compute that

$$\operatorname{dist}((X,t),V) \leq \delta_\Omega(X,t) + \operatorname{dist}((P,t),V) \leq \frac{\rho}{1+\alpha} + 4\rho\delta^{10} \leq \rho\delta/11$$

for $\delta < 1/2$ and $1+\alpha \geq 20/\delta$.

One might object that the above is a one sided estimate, and that Reifenberg flatness requires a two sided estimate. However, Saard's theorem tells us that, for almost every $\alpha$, $S_\alpha(F)$ is a closed set such that $\mathbb{R}^{n+1}\backslash S_\alpha(F)$ is disjoint union of two open sets. Since this argument rules out the presence of "holes" in $S_\alpha(F)$, our one sided estimates are enough to conclude Reifenberg flatness. Hence, $D[C_\rho(Q,\tau)\cap\partial S_\alpha(F), C_\rho(Q,\tau)\cap V] \leq \rho\delta/11$ for almost every $\alpha \geq \frac{20}{\delta} - 1$.

We need to also show that $\partial S_\alpha(F)$ is flat at points not in $F$. Let $(Q,\tau) \in \partial S_\alpha(F)\backslash F$ and $R := \delta_\Omega(Q,\tau)$. Our proof of has four cases, depending on the scale, $\rho$, for which we are trying to show $\partial S_\alpha(F)$ is flat.

**Case 1:** $\rho \geq \frac{(1+\alpha)R}{10}$. Observe that $C_\rho(Q,\tau) \subset C_{11\rho}(P,\eta)$ for some $(P,\eta) \in F$. The computation above then implies that $D[C_\rho(Q,\tau)\cap\partial S_\alpha(F), C_\rho(Q,\tau)\cap\{L+(Q,\tau)-(P,\eta)\}] \leq \rho\delta$ where $L \equiv L(P,\eta,11\rho)$, is the plane through $(P,\eta)$ which best approximates $C_{11\rho}(P,\eta)\cap \partial\Omega$.

**Case 2:** $\frac{5}{\delta}R \leq \rho < \frac{(1+\alpha)R}{10}$. We should note that this case may be vacuous for certain values of $\alpha$ (i.e, if $1+\alpha < \frac{50}{\delta}$). Without loss of generality let $(Q,\tau) = (Q,0)$ and let $(0,0) \in \partial\Omega$ be a point in $\partial\Omega$ closest to $(Q,0)$ (which is at time zero by Remark B.4.2). If $L(0,0,4\rho)$ is the plane which best approximates $\partial\Omega$ at $(0,0)$ for scale $4\rho$, we will prove that $D[C_\rho(Q,0)\cap\partial S_\alpha(F), C_\rho(Q,0)\cap\{L(0,0,4\rho)+Q\}] \leq \delta\rho$.

We may assume $L(0,0,4\rho) = \{x_n = 0\}$. Note that $\delta_F(-)$ is a 1-Lipschitz function. Thus, if $(Z_1,t_1),(Z_2,t_2) \in C_\rho((Q,0))\cap\partial S_\alpha(F)$ then $|\delta_\Omega(Z_1,t_1)-\delta_\Omega(Z_2,t_2)| < \frac{2\rho}{1+\alpha} < \frac{R}{5}$. We may conclude that $\delta_\Omega(Y,s) < 2R$ for all $(Y,s) \in C_\rho(Q,0)\cap S_\alpha(F)$ and, therefore, if $(P,s) \in \partial\Omega$

is a point in $\partial\Omega$ closest to $(Y, s)$ then $(P, s) \in C_{2\rho}(0, 0)$. By Reifenberg flatness,

$$||y_n| - 4\rho\delta^{10}| \leq ||y_n| - |p_n|| \leq \delta_\Omega(Y, s) \leq 2R \leq \frac{2\rho\delta}{5} \Rightarrow |y_n| \leq \frac{3\rho\delta}{5}.$$

On the other hand $q_n \leq |Q| = R \leq \frac{\delta\rho}{5}$. Therefore, $|y_n - q_n| \leq |y_n| + |q_n| \leq \frac{3\rho\delta}{5} + \frac{\rho\delta}{5} \leq \rho\delta$, the desired result.

**Case 3:** $\delta^2 R \leq \rho < \frac{5}{\delta}R$. Let $(X, t)$ be the point in $\partial S_\alpha(F) \cap C_\rho(Q, \tau)$ which is furthest from $\partial\Omega$ and set $\delta_\Omega(X, t) = \tilde{R}$. We may assume that $(X, t) = (0, \tilde{R}, 0)$ and $(0, 0)$ is a point in $\partial\Omega$ which minimizes the distance to $(X, t)$. We will show that $D[C_\rho(Q, \tau) \cap \partial S_\alpha(F), C_\rho(Q, \tau) \cap \{x_n = \tilde{R}\}] < \rho\delta/2$, which of course implies that $D[C_\rho(Q, \tau) \cap \partial S_\alpha(F), C_\rho(Q, \tau) \cap \{x_n = q_n\}] < \delta\rho$.

First we prove that $L(0, 0, 4\rho)$, the plane which best approximates $\partial\Omega$ at $(0, 0)$ for scale $4\rho$, is close to $\{x_n = 0\}$. If $\theta$ is the minor angle between the two planes then the law of cosines (and the fact that $\delta_\Omega((0, \tilde{R}, 0)) = \tilde{R}$) produces

$$(\tilde{R} - 4\rho\delta^{10})^2 \leq L^2 + \tilde{R}^2 - 2L\tilde{R}\sin(\theta) \Rightarrow 2L\tilde{R}\sin(\theta) \leq L^2 + 8\rho\tilde{R}\delta^{10}$$

for any $L \leq 2\rho$. If $L \equiv \delta^4\tilde{R}$ then

$$2\delta^4\tilde{R}^2\sin(\theta) \leq \delta^8\tilde{R}^2 + 8\rho\tilde{R}\delta^{10} \overset{\delta\rho \leq 5\tilde{R}}{\Rightarrow} 2\delta^4\tilde{R}^2\sin(\theta) \leq \frac{3\delta^8\tilde{R}^2}{2}.$$

For small enough $\delta$, we conclude $\theta < \delta^4$.

Therefore, for any $(Y, s) \in C_\rho(Q, \tau) \cap S_\alpha(F)$ the distance between $(Y, s)$ and $L(0, 0, 4\rho)$ is $\leq y_n + 2\delta^4\rho$. On the other hand, $\Omega$ is $(\delta^{10})$-Reifenberg flat so that means $\delta_\Omega(Y, s) \leq y_n + 2\delta^4\rho + 4\delta^{10}\rho \leq y_n + 3\delta^4\rho$. Recall, from above, that $\delta_F(-)$ is a 1-Lipschitz function. Therefore, if $(Z_1, t_1), (Z_2, t_2) \in C_\rho((Q, \tau)) \cap \partial S_\alpha(F)$ then $|\delta_\Omega(Z_1, t_1) - \delta_\Omega(Z_2, t_2)| < \frac{2\rho}{1+\alpha} < \frac{\delta\rho}{10}$ if we pick $1 + \alpha \geq 20/\delta$. An immediate consequence of this observation is that $\tilde{R} < 2R$ (and thus

$\delta^4\tilde{R} \leq 2\rho$ above). This also implies that $\delta_\Omega(Y,s) \geq \tilde{R} - \frac{\delta\rho}{10}$. Therefore,

$$\tilde{R} - \frac{\delta\rho}{10} \leq y_n + 3\delta^4\rho \Rightarrow \tilde{R} - \frac{\delta\rho}{2} \leq y_n,$$

which is one half of the desired result.

On the other hand, it might be that there is a $(Y,s) \in C_\rho(Q,\tau) \cap \partial S_\alpha(F)$ such that $y_n > \tilde{R} + \rho\delta/2$. Arguing similarly to above we can see that $L(0,0,4\tilde{R}+4\rho)$ satisfies $D[C_1(0,0) \cap L(0,0,4\tilde{R}+4\rho), C_1(0,0) \cap \{x_n = 0\}] < 2\delta^4$. But if $(P,s)$ is the point on $\partial\Omega$ closest to $(Y,s)$ it must be the case that $(P,s) \in \partial\Omega \cap C_{2\tilde{R}+2\rho}(0,0)$. Therefore, $p_n < 4\delta^4(\tilde{R}+\rho) + 4\delta^{10}(\tilde{R}+\rho) < 5\delta^4(\tilde{R}+\rho) < 6\delta^2\rho$. Of course this implies that $\|(Y,s)-(P,s)\| \geq \tilde{R} + \rho\delta/2 - 6\delta^2\rho > \tilde{R}$ for $\delta$ small enough. This is a contradiction as no point in $C_\rho(Q,\tau) \cap \partial S_\alpha(F)$ can be a distance greater than $\tilde{R}$ from $\partial\Omega$.

**Case 4:** $\rho \leq \delta^2 R$. Again let $(X,t)$ be the point in $\partial S_\alpha(F) \cap C_\rho(Q,\tau)$ which is furthest from $\partial\Omega$. Let $\delta_\Omega(X,t) = \tilde{R}$ and without loss of generality, $(X,t) = (0,\tilde{R},0)$ and $(0,0)$ is the point in $\Omega$ closest to $(X,t)$. We will show that $D[C_\rho(Q,\tau) \cap \partial S_\alpha(F), C_\rho(Q,\tau) \cap \{x_n = \tilde{R}\}] < \rho\delta/2$, which of course implies that $D[C_\rho(Q,\tau) \cap \partial S_\alpha(F), C_\rho(Q,\tau) \cap \{\{x_n = 0\} + q_n\}] < \delta\rho$.

Assume, in order to obtain a contradiction, that there is a $(Y,s) \in C_\rho(Q,\tau) \cap \partial S_\alpha(F)$ such that $y_n < \tilde{R} - \rho\delta/2$ (we will do the case when $y_n \geq \tilde{R} + \rho\delta/2$ shortly). Examine the triangle made by $(Y,s), (0,\tilde{R},0)$ and the origin. The condition on $y_n$ implies that the cosine of the angle between the segments $\overline{(0,\tilde{R},0)(Y,s)}$ and $\overline{(0,\tilde{R},0)(0,0)}$ times the length of the segment between $(Y,s)$ and $(0,\tilde{R},0)$ must be at least $\rho\delta/2$. Consequently, by the law of cosines

$$|Y|^2 \leq 4\rho^2 + \tilde{R}^2 - \tilde{R}\rho\delta. \tag{B.4.5}$$

When $1+\alpha \geq 30/\delta$ and $\delta < 1/10$ it is easy to see that

$$\begin{aligned}(\tilde{R} - 2\delta^{10}\rho - \frac{2\rho}{1+\alpha})^2 &\geq (\tilde{R} - \frac{\delta\rho}{10})^2 \geq \tilde{R}^2 - \frac{\delta\rho\tilde{R}}{5} \\ &\geq \tilde{R}^2 - \frac{\delta\rho\tilde{R}}{5}(5 - 20\delta) = \tilde{R}^2 + 4\delta^2\rho\tilde{R} - \delta\rho\tilde{R}\end{aligned} \tag{B.4.6}$$

.

Equations (B.4.6) and (B.4.5) give $|Y| < (\tilde{R} - 2\delta^{10}\rho - \frac{2\rho}{1+\alpha})$. Hence, Reifenberg flatness implies $\delta_\Omega(Y,s) \leq \|(Y,s)-(0,s)\| + \delta_\Omega(0,s) \leq (\tilde{R} - 2\delta^{10}\rho - \frac{2\rho}{1+\alpha}) + 2\delta^{10}\rho \leq \tilde{R} - \frac{2\rho}{1+\alpha}$, which, as we saw in **Case 2**, is a contradiction.

Finally, it might be that there is a $(Y,s) \in C_\rho(Q,\tau) \cap \partial S_\alpha(F)$ such that $y_n > \tilde{R} + \rho\delta/2$. Let $(P,s)$ be the point in $\partial\Omega$ closest to $(Y,s)$ and note that $(Y,s) \in C_{3\tilde{R}}(0,0)$. Let $L(0,0,5\tilde{R})$ be the plane that best approximates $\partial\Omega$ at $(0,0)$ for scale $5\tilde{R}$. If $\theta$ is the angle between this plane and $\{x_n = 0\}$ then $\delta_\Omega((0,\tilde{R},0)) = \tilde{R}$ implies that

$$\tilde{R} \leq \tilde{R}\sin(\pi/2 - \theta) + 5\delta^{10}\tilde{R} \Rightarrow (1 - 5\delta^{10}) < \cos(\theta) \Rightarrow \theta < \delta^4.$$

Therefore, $p_n < 3\tilde{R}\delta^4 + 5\tilde{R}\delta^{10} < 4\tilde{R}\delta^4$. Thus, if $\beta$ is the angle between the segment from $Y$ to $P$ and the segment from $(0,\tilde{R})$ to $Y$ it must be that $\beta < \frac{\pi}{2} - \delta/4 + 10\delta^4$. The law of cosines (on the triangle with vertices $(0,\tilde{R},0),(P,0),(Y,0)$) gives

$$|(0,\tilde{R}) - P|^2 \leq 4\rho^2 + \tilde{R}^2 - 4\rho\tilde{R}\sin(\delta/4 - 10\delta^4) < 4\rho^2 + \tilde{R}^2 + 40\rho\tilde{R}\delta^4 - \delta\rho\tilde{R}/2. \quad \text{(B.4.7)}$$

Note that

$$8\rho + 80\tilde{R}\delta^2 + 8\tilde{R}\delta^{10} \leq \delta\tilde{R} \quad \text{(B.4.8)}$$

because $\rho \leq \delta^2\tilde{R}$ by assumption and we can let $\delta < 1/100$. With this in mind we can estimate

$$\begin{aligned} 4\rho^2 + \tilde{R}^2 + 40\rho\tilde{R}\delta^4 - \delta\rho\tilde{R}/2 &< (\tilde{R} - 2\delta^{10}\rho)^2 + (4\delta^{10}\rho\tilde{R} + 40\rho\tilde{R}\delta^4 + 4\rho^2 - \delta\rho\tilde{R}/2) \\ &\overset{eq.\ \text{(B.4.8)}}{\leq} (\tilde{R} - 2\delta^{10}\rho)^2 + (\rho/2)(\delta\tilde{R}) - \delta\rho\tilde{R}/2 = (\tilde{R} - 2\delta^{10}\rho)^2. \end{aligned} \quad \text{(B.4.9)}$$

Combine equation (B.4.9) with equation (B.4.7) to conclude that $|(0,\tilde{R}) - P| \leq (\tilde{R} - 2\delta^{10}\rho)$. On the other hand, by Reifenberg flatness, $(P,0)$ is distance $< 2\delta^{10}\rho$ from a point on $\partial\Omega$. Hence, by the triangle inequality, $\delta_\Omega((0,\tilde{R},0)) < \tilde{R}$ a contradiction.

Finally, we need to show the mutual absolute continuity of the two adjoint-caloric measures. In this we follow very closely the proof of Lemma 6.3 in (JK82). The maximum principle implies that $\hat{\omega}^{(X_0,t_0)}_{S_\alpha(F)} << \hat{\omega}^{(X_0,t_0)}$, for any $(X_0, t_0) \in S_\alpha(F)$. Now take any $E \subset F$ such that $\hat{\omega}^{(X_0,t_0)}_{S_\alpha(F)}(E) = 0$. First we claim that there is a constant $1 > C > 0$ (depending on $\alpha, n, \Omega$) such that $\hat{\omega}^{(Y,s)}(\partial\Omega \backslash F) > C$ for all $(Y, s) \in \partial S_\alpha(F)\backslash F$. Indeed, let $(Q, s) \in \partial\Omega$ be a point in $\partial\Omega$ closest to $(Y, s)$. Then there is a constant $C(\alpha, n) > 0$ such that $\hat{\omega}^{(Y,s)}(C_{\alpha\delta(Y,s)}(Q, s)) \geq C(\alpha, n)$, (see equation 3.9 in (HLN04)). As $C_{\alpha\delta(Y,s)}(Q, s) \cap F = \emptyset$ (by the triangle inequality) the claim follows.

Armed with our claim we recall that a *lower function* $\Phi(X, t)$, for a set $E \subset \partial\Omega$, is a subsolution to the adjoint heat equation in $\Omega$ such that $\limsup_{(X,t)\to(Q,\tau)\in\partial\Omega} \Phi(X, t) \leq \chi_E(Q, \tau)$. Potential theory tells us that $\hat{\omega}^{(Y,s)}(E) = \sup_\Phi \Phi(Y, s)$ where the supremum is taken over all lower functions for $E$ in $\Omega$. By our claim, $\Phi(X, t) \leq \hat{\omega}^{(X,t)}(E) \leq 1 - C$ for $(X, t) \in \partial S_\alpha(F)\backslash F$ for any lower function, $\Phi$, of $E$ in $\Omega$. In particular, $\Phi(X, t) - 1 + C$ is a lower function for $E$ inside of $S_\alpha(F)$. Therefore,

$$\sup_\Phi \Phi(X, t) - 1 + C \leq \hat{\omega}^{(X,t)}_{S_\alpha(F)}(E) = 0 \Rightarrow \Phi(X, t) \leq 1 - C, \ \forall (X, t) \in S_\alpha(F).$$

This in turn implies that $\hat{\omega}^{(Y,s)}(E) \leq 1 - C$ for every $(Y, s) \in S_\alpha(F)$. By Lemma B.4.1, the non-tangential limit of $\hat{\omega}^{(Y,s)}(E)$ must be equal to 1 for $d\hat{\omega}^{(X_0,t_0)}$-almost every point in $E$. Therefore, $\hat{\omega}^{(X_0,t_0)}(E) = 0$ and we have shown mutual absolute continuity. □

The representation theorem for solutions to the adjoint heat equation with integrable non-tangential maximal function follows as in the elliptic case.

**Lemma B.4.4.** *[Compare with Lemma A.3.2 in (KT03)] Let $\Omega, \delta$ be as in Lemma B.4.3 above, and let $u$ be a solution to the adjoint heat equation on $\Omega$. Assume also that for some $\alpha > 0$ and all $(X, t) \in \Omega$, $N^\alpha(u) \in L^1(d\hat{\omega}^{(X,t)})$. Then there is some $g \in L^1(d\hat{\omega}^{(X,t)})$ such that*

$$u(X, t) = \int_{\partial\Omega} g(P, \eta) d\hat{\omega}^{(X,t)}(P, \eta).$$

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