The Structure of the Singular Set of a Two-Phase Free Boundary Problem for Harmonic Measure

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Singular Sets and Regular Sets

- Common theme in analysis, take an object and divide it into a “regular set” and a “singular set.”
- Hopefully “regular” set has smoothness and is relatively “large”
- Singular set should be relatively small and have structure.

Example (Minimal Hypersurfaces)

Area Minimizing Hypersurfaces: regular part $\cup$ singular part. The regular set is analytic. $\dim$ singular set $\leq n - 7$. Furthermore, is contained in a countable union of dimension $\leq n - 7$ Lipschitz submanifolds.

Plethora of other examples: zero sets of solutions to elliptic PDE (Cheeger-Naber-Valtorta ’15), support of uniform measures (Nimer ’15), solutions to the thin obstacle problem (Garofalo-Petrosyan ’09).
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How is this usually accomplished?

Traditional ingredients:
- Monotonicity formula
- Uniqueness of blowups (tangents)
- Control on rate of blowup (tangent)

What if you don’t know any of the above?

New Approach: (Badger-Lewis ’15 (inspired by Preiss ’87)) If you know all the possible (pseudo-) blowups of $A$, then understanding how these sets “fit together” can give information about $A$. Use this to understand two-phase problem for harmonic measure.
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Use this to understand two-phase problem for harmonic measure.
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**Ex:** For the unit disc $\omega^0 = \frac{\sigma}{2\pi}$. All points of the circle look identical.
What is Harmonic Measure?

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Given
\[
f \in C(\partial \Omega). \ \exists U_f \in C^2(\Omega) \cap C(\overline{\Omega}) \text{ which satisfies:}
\]
\[
\Delta U_f(x) = 0, \ x \in \Omega
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U_f(x) = f(x), \ x \in \partial \Omega.
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Harmonic measure at \( X \).

Credit Wikipedia
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For $X \in \Omega$ the harmonic measure $\omega^X$ is the Borel measure such that:

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\int_{\partial \Omega} f(Q) d\omega^X(Q) = U_f(X).
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Harnack inequality $\Rightarrow \omega^X << \omega^Y << \omega^X$. Will omit dependence on pole.
Two-phase Free Boundary Problems

$\Omega^\pm$ disjoint, NTA ("quantitatively connected") domains with $\omega^\pm$ harmonic measures. $\overline{\Omega^+ \cup \Omega^-} = \mathbb{R}^n$. Also $\Gamma \equiv \partial\Omega^+ = \partial\Omega^-$. 

Assume $\omega^+ \ll \omega^- \ll \omega^+$ on $\Gamma$. Let $h := \frac{d\omega^-}{d\omega^+}$. 

Question: What does the regularity of $h$ tell us about $\Gamma$?

Prior work: Kenig-Toro '06, Kenig-Preiss-Toro '09, Badger '11 '12 '13, E. '14, Azzam-Mourgoglou-Tolsa '16.

$\Omega \subset \mathbb{R}^2$, use complex analysis (Garnett and Marshall (chpt 6)).
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**Figure:** A typical two-phase setup. Picture by Matthew Badger

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![Figure: A typical two-phase setup. Picture by Matthew Badger](image)

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Picture Courtesy of Matthew Badger
KEY IDEA: Understand the (pseudo)-blowups of $\Gamma$. 
**Blowup Analysis**

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**Definition ((Pseudo)-Blowups)**

A set, $C$, is a **pseudo-blowup** of $\Gamma$ if there exists $Q_i \in \Gamma, r_i \downarrow 0$ such that

$$\frac{\Gamma - Q_i}{r_i} \equiv \Gamma_i \rightarrow C.$$ 

If $Q_i \equiv Q$, call it a **blowup**.

**Figure:** Blowing up at a point. Picture courtesy of Matthew Badger.
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**IMPORTANT**: May be multiple blowups at a point (for different $\{r_i\}$).
**Theorem (Kenig-Toro ’06)**

Let $\Omega^\pm \subset \mathbb{R}^n$ be complementary NTA with $\log(h) \in \text{VMO}(d\omega^\pm)$ (almost continuous) then every pseudo-blowup of $\Gamma$ is actually the zero set of a degree $\leq d_0$ harmonic polynomial, $p$. 

$h(x^2_1 + x^2_2 - x^2_3 - x^2_4)$ is a harmonic polynomial s.t. $\{h > 0\}$ and $\{h < 0\}$ are NTA.
Blowups of the Two-Phase Problem

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- $d_0$ depends on ambient dimension, NTA constants.
- $\{p > 0\}$ and $\{p < 0\}$ are **connected** (actually NTA).
- $p$ depends on $Q_i, r_i$ in the pseudo-blowup (not unique given a $Q \in \Gamma$!).
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$h(X) = x_1^2 + x_2^2 - x_3^2 - x_4^2$ is a harmonic polynomial s.t. $\{h > 0\}$ and $\{h < 0\}$ are NTA. Credit: Mathematica
The Main Theorem

**Theorem (Main Theorem, Badger-E.-Toro ('15))**

Let $\Omega^\pm \subset \mathbb{R}^n$ be complementary NTA domains and assume 
$\log(\frac{d\omega^-}{d\omega^+}) \in \text{VMO}(d\omega^+)$.  $\exists d_0 \in \mathbb{N}$ s.t. $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \ldots \cup \Gamma_{d_0}$, where:

- If $Q \in \Gamma_k$, then any blowup of $\Gamma$ at $Q$ is the zero set of a degree $k$ homogenous harmonic polynomial (not necessarily unique!).
- $\overline{\dim}_M \Gamma \setminus \Gamma_1 \leq n - 3$. $\Gamma \setminus \Gamma_1$ is the **singular set**.
- For any $k \leq d_0$: $\Gamma_1 \cup \Gamma_2 \cup \ldots \cup \Gamma_k$. is open inside of $\Gamma$
- For any $k \leq d_0/2$: $\dim_H \Gamma_2 \cup \Gamma_4 \cup \ldots \cup \Gamma_{2k} \leq n - 4$. 

These two examples $x_1^2 + x_2^2 - x_3^2 - x_4^2$ and $x_1^2(x_2^2 - x_3^2) + x_2^2(x_3^2 - x_1^2) + x_2^2(x_1^2 - x_2^2)$ show that the above dimension bounds are sharp. Credit: Mathematica and M. Badger.
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Method of Proof

**Kenig-Toro ’06:** all of the pseudo-blowups are zero sets of homogenous harmonic polynomials which split space into two NTA components.

**Badger-Lewis ’15:** main theorem follows if:

- **Detectability:** Let \( k \leq \ell \) and \( C, \delta > 0 \) be uniform constants. If \( p, h \) are harmonic polynomials of degree \( k, \ell \), respectively, \( \{ p = 0 \} \cap B(x, r) \) is within \( \delta r \) of \( \{ h = 0 \} \cap B(x, r) \), then for every \( s \in (0, 1) \) there is a degree \( k \) polynomial, \( p_s \), such that \( \{ p_s = 0 \} \cap B(x, rs) \) is within \( C r s^{1 + 1/k} \) of \( \{ h = 0 \} \cap B(x, rs) \).

- If you are close to a degree \( k \) polynomial at one scale, you get closer at smaller scales. ("improvement of flatness"-type result)

- **Dimension Estimates:** for every \( \delta > 0 \), \( \exists C > 0 \) such that for all harmonic polynomials, \( p \), of degree \( \leq d_0 \), \( \text{Vol}\left( \{ x \in B(0, 1/2) | p(x) = 0 \text{, dist}(x, S(p)) < r \} \right) \leq C r^{3 - \delta} \), where \( S(p) = \{ x_0 | p(x_0) = 0 = Dp(x_0) \} \) is the singular set of \( p \).
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Proof of Detectability

- **Key Tool**: Łojasiewicz inequality. Need to understand how harmonic polynomials grow near the zero set.
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- Compactness argument shows that if \( \{ p = 0 \} \) is very close to some degree \( k \) harmonic polynomial, then it must be near the first \( k \) terms of its Taylor series.
Theorem (Cheeger-Naber-Valtorta '15)

If \( u : B(0,1) \rightarrow \mathbb{R} \) is a harmonic function with \( u(0) = 0 \) and
\[
\int_{\partial B(0,1)} |\nabla u|^2 \, dx \leq \Lambda,
\]
then for every \( \eta > 0 \) and \( k \leq n - 2 \),
\[
\text{Vol}\left( \{ x \in B(0,1/2) | \text{dist}(x, S_k \eta, r(u)) \} \right) \leq C(n, \Lambda, \eta) r^{n-k-\eta}.
\]

- \( S_k \eta, r(u) \) are the points at which \( u \) "depends" on more than \( n-k \) variables at small scales (i.e. has \( k \) or more translational symmetries).

- "Regular points" depend on only one direction at infinitesimal scales.

So if \( k < n-1 \) we are looking at singular points.

- We show: Singular points \( S(p) \subset S(n-3, r) \) for all \( r \) and \( \eta \) small enough.

- Proof: Blow-up argument and no homogenous harmonic polynomial splits \( \mathbb{R}^2 \) into two connected components.
If $u : B(0, 1) \rightarrow \mathbb{R}$ is a harmonic function with $u(0) = 0$ and

$$\frac{\int_{B(0,1)} |\nabla u|^2 \, dx}{\int_{\partial B_1} u^2 \, d\sigma} \leq \Lambda,$$

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Singulär Punkte in Harmonischen Polynomen

**Singular Points in Harmonic Polynomials**

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Theorem (Cheeger-Naber-Valtorta ’15)

If \( u : B(0, 1) \rightarrow \mathbb{R} \) is a harmonic function with \( u(0) = 0 \) and
\[
\frac{\int_{B(0,1)} |\nabla u|^2 \, dx}{\int_{\partial B_1} u^2 \, d\sigma} \leq \Lambda,
\]
then for every \( \eta > 0 \) and \( k \leq n - 2 \),
\[
\text{Vol}(\{x \in B(0, 1/2) \mid \text{dist}(x, S^k_{\eta,r}(u))\}) \leq C(n, \Lambda, \eta) r^{n-k-\eta}.
\]

- \( S^k_{\eta,r}(u) \) are the points at which \( u \) “depends” on more than \( n - k \) variables at small scales (i.e. has \( k \) or more translational symmetries).
- “Regular points” depend on only one direction at infinitesimal scales. So if \( k < n - 1 \) we are looking at singular points.
- **We show:** Singular points \( S(p) \subset S^{n-3}_{\eta,r} \) for all \( r \) and \( \eta \) small enough.
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So if $k < n - 1$ we are looking at singular points.

• We show: Singular points $S(p) \subset S_{\eta, r}^{n-3}$ for all $r$ and $\eta$ small enough.

• Proof: Blow-up argument and no homogenous harmonic polynomial splits $\mathbb{R}^2$ into two connected components.
Some Open Questions/Future Work

1. Ongoing work: what if \( \log\left(\frac{d\omega^-}{d\omega^+}\right) \in C^{0,\alpha} \)?
   - We can prove uniqueness of blowup.

2. Does the regular set have locally finite measure (connected to the work of Azzam-Mourgoglou-Tolsa '16)?

3. Unique blowups at points?

4. Is \( \Gamma_k \) closed?

5. We need to understand better the zero sets of harmonic polynomials which split space into two NTA components.
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Thank You For Listening!