Regularized Distances and Harmonic Measure in Co-Dimension Greater than One

Max Engelstein
(joint work with G. David (Allez les Bleus!) and S. Mayboroda)

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Definition of Harmonic Measure

$X \in \Omega \subset \mathbb{R}^n$. $E \subset \partial\Omega$. $\omega^X(E) =$ Probability a B.M. exits Ω first in E.

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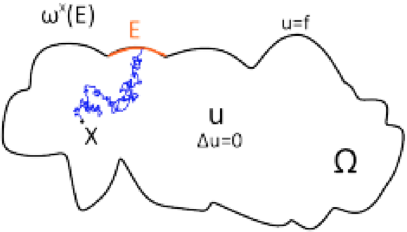

FIGURE: Brownian Motion exiting a domain (figure credit Matthew Badger)

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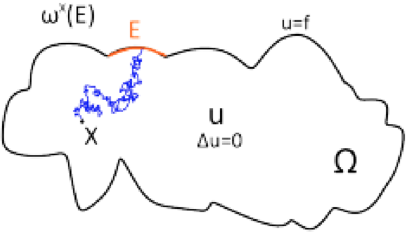

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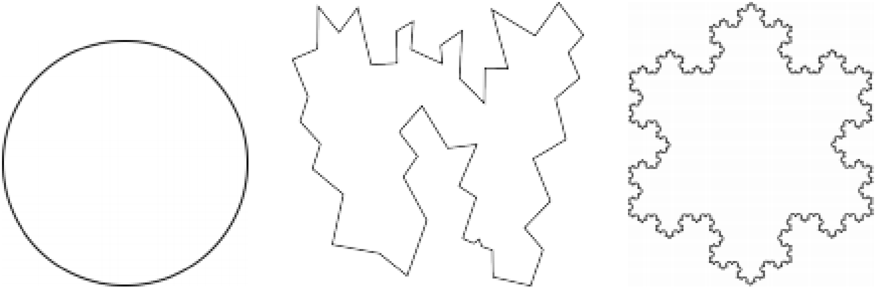
$$u_f(X) = \int_{\partial\Omega} f d\omega^X.$$

Three Examples: ω vs $\mathcal{H}^{n-1}|_{\partial\Omega}$



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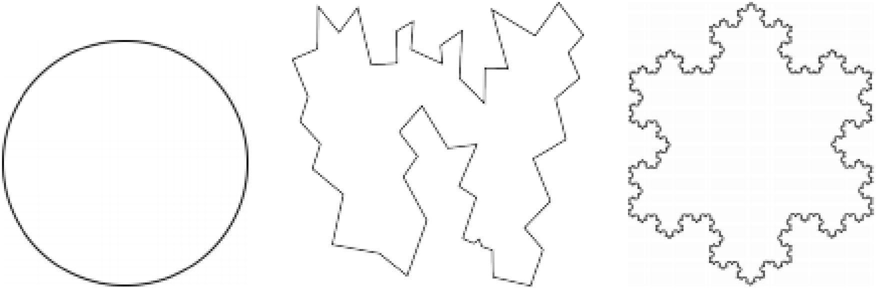

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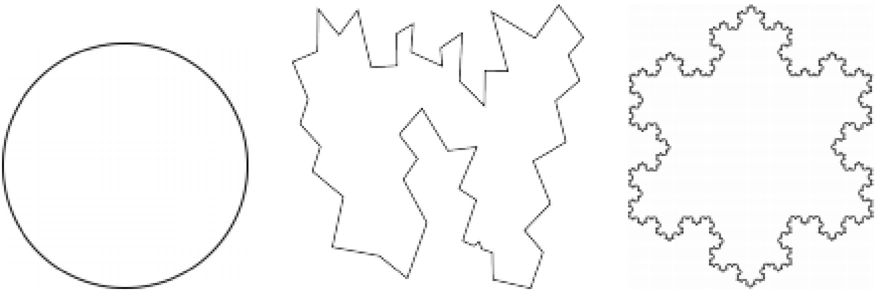
THREE EXAMPLES: ω VS $\mathcal{H}^{n-1}|_{\partial\Omega}$



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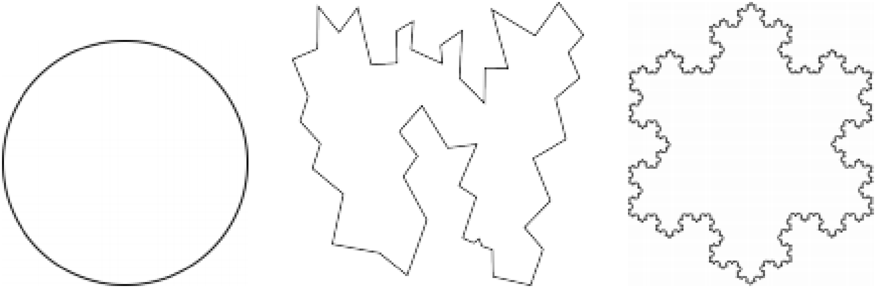

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For the snowflake $\omega^0 \perp \sigma$.

Relationship between Geometry and H.M.

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Connections to the Dirichlet problem, probability, potential theory...

Two Important Theorems

$\mathbb{R}^n \backslash E \equiv \Omega$, ω-harmonic measure of Ω, $\sigma = \mathcal{H}^{n-1}|_E$.
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Takeaway: Geometry is characterized by solutions of Laplacian in complement!

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Question: Geometry of E characterized by ω_L vs σ?

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Note: no topological assumptions needed. That is because E is so thin!

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- ∇D_α "sees" flatness of E (How....?)

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E is uniformly rectifiable if and only if $F_\alpha^2(x)\delta(x)^{-n+d}$ is a Carleson measure on $\mathbb{R}^n \backslash E$.
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Carleson Measure: $\int_{B(Q,R)} F^2 \delta^{-n+d} dx \leq CR^d$ for all $Q \in E$.

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Magic α!

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Takeaway: For magic α, $\frac{d\omega_\alpha}{d\sigma}$ doesn't control the regularity of E, and fails to do so in the most spectacular way possible!

What's up with "magic α"?

Can compute: see that for $\alpha = n - d - 2$ we have

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When α is magic $D_\alpha(x) = \left(\int_E \frac{1}{|x-y|^{n-2}} d\sigma\right)^{-1/\alpha}$. Note: $\frac{1}{|x|^{n-2}}$ is harmonic!

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2. Is this emblematic or pathological?
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   - Can we prove the converse for ω_β with β not magic?
3. What does $\alpha \mapsto D_\alpha$ look like?
   - The power $-\frac{1}{\alpha}$ makes this question harder.

Wild Speculation Part I: Energy

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Question: Does this phenomenon exist for other problems with energy? E.g. given a set E, can you come up with an obstacle type problem such that E is the contact set of the minimizer? Coefficients will be nasty.

Wild Speculation Part II: co-dimension 1

Let $E \subset \mathbb{R}^{n-1}$ be an $(n-1)$-Ahlfors regular set.

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Explicit Question: Does there exist an operator in $\mathbb{R}^2$ on the exterior of the four-corner Cantor set, C, such that $\omega_L \simeq \sigma$ on C?

Thank You For Listening!