Regularized Distances and Harmonic Measure in Co-Dimension Greater than One

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(joint work with G. David (Allez les Bleus!) and S. Mayboroda)

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Definition of Harmonic Measure

\( X \in \Omega \subset \mathbb{R}^n. \ E \subset \partial \Omega. \ \omega^X(E) = \text{Probability a B.M. exits } \Omega \text{ first in } E. \)
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\Delta u_f = 0 & x \in \Omega \\
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*Figure:* Brownian Motion exiting a domain (figure credit Matthew Badger)
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\[ u_f(X) = \int_{\partial \Omega} f d\omega^X. \]

**Figure:** Brownian Motion exiting a domain (figure credit Matthew Badger)
Three Examples: \( \omega \) vs \( \mathcal{H}^{n-1]|_{\partial\Omega} \)

A disk, Lipschitz domain and Snowflake (figure from Matthew Badger)
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For a Lipschitz domain, $\omega^0 \ll \sigma \ll \omega^0$ (in a scale invariant way!)
For the snowflake $\omega^0 \perp \sigma$. 
Two big questions ($\omega$ is H.M., $\sigma = \mathcal{H}^{n-1}|_{\partial \Omega}$):

Q1 (direct): Does smoothness of $\Omega$ imply smoothness of $d\omega$?

Q2 (free boundary): Does smoothness of $d\omega$ imply smoothness of $\Omega$? (some a priori topology)

Lots of work: Badger, Bortz, David, E., Feneuil, Hofmann, Kenig, Martell, Mayboroda, Tolsa, Toro, Uriarte-Tuero, Zhao...and that's just the people here at PCMI! Many more also!

Connections to the Dirichlet problem, probability, potential theory...
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\( \mathbb{R}^n \setminus E \equiv \Omega, \) \( \omega \)-harmonic measure of \( \Omega, \sigma = \mathcal{H}^{n-1}|_E. \)

\( E \) is \((n-1)\)-Ahlfors regular: \( \sigma(B(Q, r)) \simeq r^{n-1} \) for all \( Q \in E, r > 0. \)
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**Theorem (Hofmann-Martell & Azzam-Mourgoglou-Tolsa)**

\( E \) is \((n - 1)\)-uniformly rectifiable (+ weak topological assumption) if and only if \( \omega \) is (quantitatively) absolutely continuous (weak-\( A_\infty \)) w.r.t \( \sigma \).

**Theorem (Kenig-Toro)**

Some topological assumptions on \( \Omega \). Then \( \text{osc} \left( \hat{n} \right) \) and \( \text{osc} \, d\omega \, d\sigma \) control each other.

In particular, \( \omega = \sigma \iff \Omega \) is a half-space.

A few seconds on quantitative regularity (Ahlfors regular, \( A_\infty \), uniformly rectifiable etc...)

**Takeaway:** Geometry is characterized by solutions of Laplacian in complement!
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$$Lu = -\text{div} \left( \frac{A(x)}{\text{dist}(x, E)^{n-d-1}} \nabla u \right) = 0,$$

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**Question:** Geometry of $E$ characterized by $\omega_L$ vs $\sigma$?
Regularized Distance I: The Direct Result

**Problem:** $x \mapsto \text{dist}(x, E)$ is not a nice function. Hard to talk about $\omega_L$. 

David-Feneuil-Mayboroda: family of smoothed out distances, $D^\alpha(x)$.

$D^\alpha(x) \approx \text{dist}(x, E)$.

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Theorem (David-Feneuil-Mayboroda (for Lipschitz graphs), David-Mayboroda (in progress))

Let $E$ be a $d$-uniformly rectifiable set, then $\omega^\alpha \in A^\infty(d\sigma)$.

This is great! We have that nice geometry implies nice behavior of the (degenerate) elliptic measure (analogue of Hofmann-Martell for higher co-dimension).

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Regularized Distance II: Who is $D_\alpha$?

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- $\nabla D_\alpha$ “sees” flatness of $E$ (How....?)
Oscillation of $|\nabla D_\alpha|$ and the flatness of $E$

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**Theorem (David-E.-Mayboroda, in preparation)**

$E$ is uniformly rectifiable if and only if $F_\alpha^2(x)\delta(x)^{-n+d}$ is a Carleson measure on $\mathbb{R}^n \setminus E$.

$E$ is rectifiable if and only if $\lim_{Q \leftarrow x \in \Gamma_\eta(Q)} |\nabla D_\alpha(x)|$ exists for $\sigma$-a.e. $Q \in E$. 
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Carleson Measure: $\int_{B(Q,R)} F^2 \delta^{-n+d} \, dx \leq CR^d$ for all $Q \in E$. 

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\[ \nabla D_\alpha = -\frac{1}{\alpha} D_\alpha^{1+\alpha} \left( \int_E \frac{x - y}{|x - y|^{d+\alpha+1}} d\sigma \right) \]
Intuition/Proof: $|\nabla D| = c \Rightarrow \text{FLAT}$

Key Step: $|\nabla D_\alpha|$ is constant if and only if $E$ is an affine space.
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- **UR $\Rightarrow$ Carleson Condition**: compare $|\nabla D_\alpha|$ to $|\nabla D_\alpha|$ of best approximation. Use $\alpha$ numbers.
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- **Rectifiability \(\Rightarrow\) NT Limits:** blow-up! Get a constant at almost every point.
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Does the oscillation of $\frac{d\omega_\alpha}{d\sigma}$ control the regularity of $E$?
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NO!!!!
Let $E \subset \mathbb{R}^n$ be $d$-Ahlfors regular. And $\alpha = n - d - 2 > 0$.

**Theorem (David-E.-Mayboroda, in preparation)**

For any $E$ as above and $\alpha = n - d - 2$, $C^{-1}\sigma \leq \omega_\alpha \leq C\sigma$. 

**NOTE:** $E$ could be a fractal! $d$ could be a non-integer dimension!!!
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Recall the work of Kenig-Toro:

**Theorem (Kenig-Toro)**

$\Omega \subset \mathbb{R}^n$ with mild topological assumptions. $\partial\Omega$ is $(n - 1)$-Ahlfors regular. Then $\omega = \sigma$ if and only if $\Omega$ is a half-space. More generally, regularity of $\frac{d\omega}{d\sigma}$ controls regularity of $\partial\Omega$. 
Let $E \subset \mathbb{R}^n$ be $d$-Ahlfors regular. And $\alpha = n - d - 2 > 0$.

**Theorem (David-E.-Mayboroda, in preparation)**

For any $E$ as above and $\alpha = n - d - 2$, $C^{-1}\sigma \leq \omega_\alpha \leq C\sigma$. If $E$ is rectifiable then $\omega_\alpha \equiv c\sigma$.

**NOTE:** $E$ could be a fractal! $d$ could be a non-integer dimension!!!

Recall the work of Kenig-Toro:

**Theorem (Kenig-Toro)**

$\Omega \subset \mathbb{R}^n$ with mild topological assumptions. $\partial \Omega$ is $(n - 1)$-Ahlfors regular. Then $\omega = \sigma$ if and only if $\Omega$ is a half-space. More generally, regularity of $\frac{d\omega_\alpha}{d\sigma}$ controls regularity of $\partial \Omega$.

**Takeaway:** For magic $\alpha$, $\frac{d\omega_\alpha}{d\sigma}$ doesn’t control the regularity of $E$, and fails to do so in the most spectacular way possible!
Can compute: see that for $\alpha = n - d - 2$ we have

$$L_\alpha D_\alpha = -\text{div} \left( \frac{1}{D_\alpha^{n-d-1}} \nabla D_\alpha \right) = 0.$$ 

“The distance is a solution to the equation”
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When $\alpha$ is magic $D_\alpha(x) = \left( \int_E \frac{1}{|x-y|^{n-2}} d\sigma \right)^{-1/\alpha}$. Note: $\frac{1}{|x|^{n-2}}$ is harmonic!
Open Questions about the Magic $\alpha$

1. Why is magic $\alpha$ magic?
   - $D_\alpha$ satisfies an equation but what is really going on?
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Open Questions about the Magic $\alpha$

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3. What does $\alpha \mapsto D_\alpha$ look like?
   - The power $-\frac{1}{\alpha}$ makes this question harder.
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Wild Speculation Part I: Energy

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**Question:** Does this phenomenon exist for other problems with energy? E.g. given a set $E$, can you come up with an obstacle type problem such that $E$ is the contact set of the minimizer? Coefficients will be nasty.
Let $E \subset \mathbb{R}^{n-1}$ be an $(n-1)$-Ahlfors regular set.

Question: Under what conditions on $E$ can you find an elliptic operator, $L$ such that $\omega_L \simeq \mathcal{H}^{n-1}|_E$?
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Explicit Question: Does there exist an operator in $\mathbb{R}^2$ on the exterior of the four-corner Cantor set, $C$, such that $\omega_L \simeq \sigma$ on $C$?
Thank You For Listening!