

An Epiperimetric Approach to Isolated Singularities

MAX ENGELSTEIN

(joint work with Luca Spolaor, Bozhidar Velichkov)

We present a new technique for studying the infinitesimal behavior of energy minimizers near the points where the minimizer exhibits non-smooth behavior. To avoid further vaguaries, we focus our attention on minimizers to the functional,

$$(1) \quad \int |\nabla u|^2 + \chi_{\{u>0\}},$$

though the technique described below is quite general and has yielded similar results in the setting of (almost-)area minimizing currents (c.f. [7]).

We are interested in what is called the “free boundary”, $\partial\{u > 0\}$. In [2], Alt and Caffarelli proved the following dichotomy; let $x_0 \in \partial\{u > 0\}$, then either the free boundary in a neighborhood of x_0 can be written as the graph of an analytic function, or there is no $r > 0$ such that $B_r(x_0) \cap \partial\{u > 0\}$ is contained in a δr -neighborhood of an $(n - 1)$ -plane (for some $\delta > 0$). The former points are called **regular**, and the latter, **singular**.

We can rephrase this result in terms of parameterization; the free boundary near regular points is parameterized over an $(n - 1)$ -plane by a smooth function. The long-term goal of our investigation (and a central problem in the subject of regularity theory) is to extend this parameterization to singular points. The main theorem of this talk does this for a class of singular points:

Theorem 1. *[Main Theorem in [6]] Let $b \in W^{1,2}(B_1)$ be a 1-homogenous minimizer to (1), such that $\partial\{b > 0\}$ is smooth away from 0. Assume that u is a minimizer and $r_j \downarrow 0, x_0 \in \partial\{u > 0\}$ are such that $\frac{\partial\{u>0\}-x_0}{r_j} \rightarrow \partial\{b > 0\}$. Then $\lim_{r \downarrow 0} \frac{\partial\{u>0\}-x_0}{r} = \partial\{b > 0\}$ and there exists some $r_0 > 0$ such that $\partial\{u > 0\} \cap B_{r_0}(x_0)$ can be written as the $C^{1,\log}$ image of $B_{r_0}(0) \cap \partial\{b > 0\}$.*

Let us make three quick remarks; first if $\frac{\partial\{u>0\}-x_0}{r_j} \rightarrow S$ for some set S , we call S a **blow-up** of $\partial\{u > 0\}$ at x_0 . Alternatively, we can examine $u_{r_j, x_0}(x) \equiv \frac{u(r_j x + x_0)}{r_j}$ and refer to $\lim_j u_{r_j, x_0} = v(x)$ as the blowup. Note that $S = \partial\{v > 0\}$ in this scenario. The theorem above is an example of a “uniqueness of blow-ups” result, more on this below. Second, the assumption that b is 1-homogeneous is redundant; it is a result of Weiss [15] that if b is a blowup of a minimizer, then b must be 1-homogenous. The final remark is that the regularity of the parameterization depends on the “symmetries” of the cone b . Imprecisely, if the only deformations of b which preserve the energy (1) to second order are given by ambient isometries of the space, then we call b **integrable through rotations** and the parameterization given by Theorem 1 is $C^{1,\alpha}$. Otherwise, the stated $C^{1,\log}$ regularity is optimal. We note that the only known one-homogenous minimizers to (1) were constructed by De Silva and Jerison [5] and each of these are integrable through rotations.

As mentioned above, a central question is the uniqueness of blow-ups; the limit $\frac{u(r_j x + x_0)}{r_j}$ exists up to subsequence by compactness, but in order to parameterize

the free boundary over the blow-up it must be the case that the limit is independent of the subsequence $r_j \downarrow 0$. Theorem 1 is the only known uniqueness of blowups result in the setting of (1), but similar questions have been investigated for obstacle problems ([14], [9]), harmonic maps ([12]) and minimal surfaces ([12], [1], [13]). Uniqueness of blow-ups is not always true; Brian White constructs harmonic maps from $\mathbb{R}^4 \rightarrow N$ where N is a C^∞ four-manifold such that there is an isolated critical point with a continuum of blow-ups at that point, see [17].

The main tool in proving Theorem 1 is what is called an epiperimetric inequality. In [15], Weiss proved that

$$W(u, x_0, r) \equiv \frac{1}{r^n} \int_{B_r(x_0)} |\nabla u|^2 + \chi_{\{u>0\}} dx - \frac{1}{r^{n+1}} \int_{\partial B_r(x_0)} u^2 d\sigma,$$

is monotone increasing in r as long as u is a minimizer to (1) and that the difference, $W(u, x_0, r) - W(u, x_0, s)$, measures how far u is from being one-homogeneous in the annulus $B_r(x_0) \setminus B_s(x_0)$. Thus to prove a uniqueness of blowups result like Theorem 1, it suffices to bound the growth of $r \mapsto W(u, x_0, r)$ from above. This is the role of an epiperimetric inequality (see Theorem 2 below), which says, roughly, that the difference in energy between the one-homogeneous and minimizing extensions of a trace, $c \in L^2(\partial B_1)$, is proportional to the gap between c and the “closest” trace of a one-homogeneous minimizer (where the gap is measured by W).

Epiperimetric inequalities have been used to prove uniqueness of blow-ups and regularity in minimal surfaces [11, 13, 16] and free boundary problems [14, 8, 10]. Recently, the second and third authors, with Maria Colombo [3, 4], have pioneered the concept of a log-epiperimetric inequality, in which the gap (alluded to above) has non-linear dependence, which in turn gives a $C^{1, \log}$ rate of blow-up.

Before we state our epiperimetric inequality, let us briefly outline one critical way in which ours differs from those mentioned above. Our epiperimetric inequality is the first to treat blow-ups which are not integrable through rotations. In order for the minimizing extension to be quantitatively better than the homogeneous one, one often needs to identify which trace of a homogeneous minimizer is “closest” to the given trace. The condition of being integrable through rotations means that all the “nearby” traces of homogeneous minimizers are simply rotations of each other, which makes it easier to find the closest one through an implicit function theorem argument (see [16]).

In order to prove a log-epiperimetric inequality at singularities which are not integrable through rotations, we had to find nearby problematic traces by hand. To do so, we borrowed a powerful idea from L. Simon [12], and used a Lyapunov-Schmidt reduction to identify the closest “problematic” trace and used gradient flow to improve its energy. We then invoked the Łojasiewicz inequality to show that this energy improvement was quantitative.

Let us end with a statement of our epiperimetric inequality. For space considerations we take $x_0 = 0$ and $r = 1$ and refer to $W(f, x_0, r)$ simply as $W(f)$:

Theorem 2. *[Epiperimetric inequality in [6]] Let $b \in H^1(B_1)$ be a one-homogeneous minimizer of (1) with an isolated singularity at the origin. There exist constants*

$\epsilon = \epsilon(d, b) > 0, \gamma = \gamma(d, b) \in [0, 1)$ and $\delta_0 = \delta_0(d, b) > 0$, depending on b and on the dimension d , such that the following holds.

If $c \in H^1(\partial B_1, \mathbb{R}_+)$ is such that there exists $\zeta \in C^{2,\alpha}(\partial\{b > 0\} \cap \mathbb{S}^{n-1})$ such that $\partial\{c > 0\}$ is the graph (in the sphere) of ζ over $\partial\{b > 0\} \cap \mathbb{S}^{n-1}$ and

$$(2) \quad \|\zeta\|_{C^{2,\alpha}} \leq C_d \|\zeta\|_{L^2} < \delta, \quad \text{and} \quad \|c - b\|_{L^2(\partial B_1)} < \delta,$$

then there exists a function $h \in H^1(B_1, \mathbb{R}_+)$ such that $h = c$ on ∂B_1 and

$$(3) \quad W(h) - W(b) \leq \left(1 - \epsilon |W(z) - W(b)|^\gamma\right) (W(z) - W(b)),$$

where z is the 1-homogeneous extension of c to B_1 .

In the case where b is integrable through rotations, we can take $\gamma = 0$ in (3) above.

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