Parabolic NTA domains in $\mathbb{R}^2$

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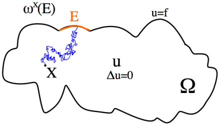

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Credit Wikipedia

Given
$f \in C(\partial\Omega)$. $\exists U_f \in C^2(\Omega) \cap C(\overline{\Omega})$ which satisfies:

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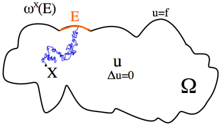

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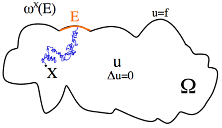

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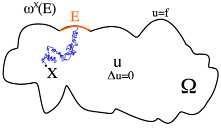

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Harnack inequality $\Rightarrow \omega^X << \omega^Y << \omega^X$. Will omit dependence on pole.

Simplest Problem for Harmonic Measure

Let Ω be an unbounded non-tangentially accessible (NTA) set, ω the harmonic measure with a pole at ∞, σ the surface measure.

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Also connection to Alt-Caffarelli '81 problem. Minimizers of the functional

$$\int |\nabla u|^2 + \chi_{\{u>0\}}$$

are solutions to the free boundary problem

$$\begin{aligned} \Delta u(x) =& 0, \ x \in \{u > 0\} \\ \partial_\nu u(x) =& 1, \ x \in \partial\{u > 0\}. \end{aligned}$$

What is an (Elliptic) NTA Domain?

Jerison and Kenig '82 (quantitatively open and path connected):

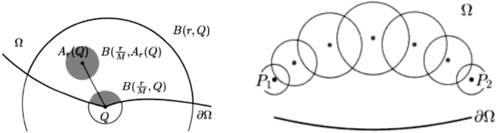

Figure: The Corkscrew and Harnack Chain Conditions. Figures from Capogna, Kenig and Lanzani 2005

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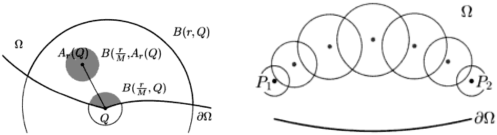

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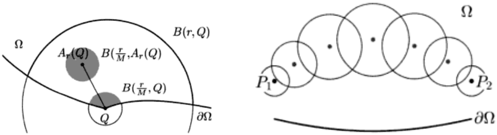

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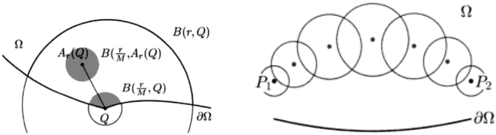

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So natural to restrict to Ω which are NTA.

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All this means they are a natural low-regularity setting to study elliptic PDE/free boundary problems

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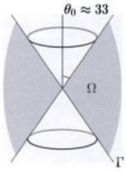

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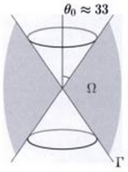

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Theorem (Alt-Caffarelli '81; Kenig-Toro '04)

If $\Omega \subset \mathbb{R}^n$ is a δ-Reifenberg flat (i.e. no cone points) NTA domain and $\omega \equiv \sigma$ then (after a possible rotation and translation) $\Omega = \{x_n > 0\}$. If $n = 2$, then a priori flatness assumption not necessary.

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Potential theory and geometry much less well understood!

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What if you assume flatness? (Partial results on this by Hofmann-Lewis-Nyström '04, Nyström 06, 12)

Theorem (Main Theorem, E' 15))

Let $\Omega \subset \mathbb{R}^{n+1}$ be a parabolic NTA domain with uniformly rectifiable boundary and assume that $\omega = \sigma$. Then if Ω is δ-Reifenberg flat (no cone points) it must be a half-plane.

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Parabolic NTA domains in $\mathbb{R}^2$

Theorem (E'16)

A parabolic NTA domain $\Omega \subset \mathbb{R}^2$ must be either a graph domain $\Omega = \{(x,t) \mid x > f(t)\}$ (after a possible reflection over the x-axis) or a "slab" domain $\Omega = \{(x,t) \mid f_1(t) > x > f_2(t)\}$. In other words, each connected component of $\partial\Omega$

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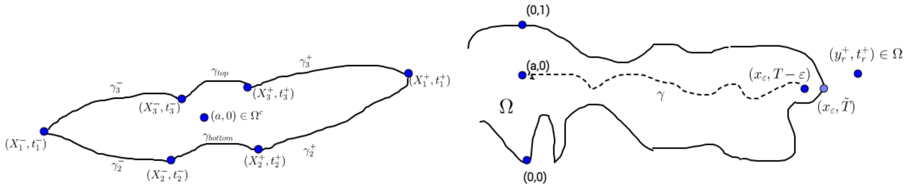

FIGURE: Pictorial Proof of the Theorem.

Parabolic Solutions

Corollary (E'16)

If Ω is a parabolic NTA domain in $\mathbb{R}^2$ with rectifiable boundary and $\omega \equiv \sigma$, then (after a possible reflection and translation), $\Omega = \{(x,t) \mid x > 0\}$.

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- What about when $n = 2$?
- Are there solutions to the parabolic problem which are not cylinders over solutions to the harmonic problem?

Connections to Geometry

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What do we even mean by a solution to (3)? (work by Caffarelli-Vazquez 96, Weiss '03, Andersson-Weiss '09 and others)

Some Open Questions/Future Work

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4. A different topological notion. (Each time slice convex? Related to ongoing work with Jerison and Mayboroda)

Thank You For Listening!