An enumerative relationship between maps and 4-regular maps

Michael La Croix

April 9, 2008
Outline

1. Background
   - Surfaces
   - Maps
   - Rooted Maps

2. Map Enumeration
   - A Counting Problem
   - A Remarkable Identity
   - Planar Maps
   - Non-Planar Maps

3. A Refinement
   - A Recurrence
   - Speculation
   - Refining the Conjecture
   - Structural Evidence
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   - Structural Evidence
Surfaces

Definition

A **Surface** is a compact connected 2-manifold without boundary.

This talk will focus on orientable surfaces.
An enumerative relationship between maps and 4-regular maps

Background

Surfaces

Theorem (Classification Theorem)

*Every orientable surface is an n-torus for some* \( n \geq 0 \).

\( n \) is the genus of the surface.
Surfaces can be represented by polygons with sides identified.
An enumerative relationship between maps and 4-regular maps

Maps

Definition

A **map** is a 2-cell embedding of a multigraph in a surface.

The graph is necessarily connected.
An enumerative relationship between maps and 4-regular maps

Maps

Definition

A **map** is a 2-cell embedding of a multigraph in a surface.

The embedding provides a cyclic order to edges at each vertex.
An enumerative relationship between maps and 4-regular maps

Maps

Definition

A map is a 2-cell embedding of a multigraph in a surface.

The embedding also defines faces.
Maps are considered up to topological deformations.
Maps

Definition

A **map** is a 2-cell embedding of a multigraph in a surface.

Deformations preserve faces and cyclic orders.
Maps on the Torus

Polygonal representations obfuscate structure.
Tiling the fundamental domain produces the universal cover,
Maps on the Torus

and reveals face structure.
The neighbourhood of a map defines a ribbon graph.
Ribbon Graphs and Flags

A ribbon graph determines the surface and embedding.
Ribbon Graphs and Flags

Vertex-edge intersections define flags.
Ribbon Graphs and Flags

Flags are permuted by map automorphisms.
Rooted Maps

Definition

A **rooted map** is a map together with a distinguished orbit of flags under the action of its automorphism group.
Rooted Maps

Rootings are indicated with arrows.

Note: A map with no edges has a single rooting.
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   - Structural Evidence
How Many Maps are There?

Denote the set of rooted orientable maps by $\mathcal{M}$.

- How many elements of $\mathcal{M}$ have genus $g$, $v$ vertices, $f$ faces, and $e$ edges?
How Many Maps are There?

Denote the set of rooted orientable maps by $\mathcal{M}$.

- How many elements of $\mathcal{M}$ have genus $g$, $v$ vertices, $f$ faces, and $e$ edges?

**Example**

Of the planar rooted maps with 2 edges, two have 3 vertices, five have 2 vertices, and two have 1 vertex.
How Many Maps are There?

The restriction of $\mathcal{M}$ to 4-regular maps is $\mathcal{Q}$.

- How many elements of $\mathcal{Q}$ have genus $g$, $v$ vertices, $f$ faces, and $e$ edges?
An enumerative relationship between maps and 4-regular maps

Map Enumeration

A Counting Problem

How Many Maps are There?

The restriction of $\mathcal{M}$ to 4-regular maps is $\mathcal{Q}$.

- How many elements of $\mathcal{Q}$ have genus $g$, $v$ vertices, $f$ faces, and $e$ edges?

Example

There are 15 maps rooted maps that are 4-regular with 2 vertices, 4 edges, 2 faces, and genus 1.
An enumerative relationship between maps and 4-regular maps

Map Enumeration

A Counting Problem

Generating Series

The genus series for rooted orientable maps is

$$M(u^2, x, y, z) = \sum_{m \in M} u^{2g(m)} x^{v(m)} y^{f(m)} z^{e(m)}.$$ 

The weights $g(m)$, $v(m)$, $f(m)$, and $e(m)$ are the genus, number of vertices, number of faces, and number of edges of $m$. 
The genus series for rooted orientable maps is

\[ M(u^2, x, y, z) = \sum_{m \in \mathcal{M}} u^{2g(m)} x^{v(m)} y^{f(m)} z^{e(m)}. \]

The corresponding series for 4-regular maps is

\[ Q(u^2, x, y, z) = \sum_{m \in \mathcal{Q}} u^{2g(m)} x^{v(m)} y^{f(m)} z^{e(m)}. \]

The weights \( g(m), v(m), f(m), \) and \( e(m) \) are the genus, number of vertices, number of faces, and number of edges of \( m \).
Jackson and Visentini derived the functional relation

\[
Q(u^2, x, y, z) = \frac{1}{2} M(4u^2, y + u, y, xz^2) + \frac{1}{2} M(4u^2, y - u, y, xz^2)
= \text{bis}_{\text{even}} u \quad M(4u^2, y + u, y, xz^2).
\]
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\[ = \text{bis}_{\text{even}} u \ M(4u^2, y + u, y, xz^2). \]

The right hand side is a generating series for a set \( \tilde{M} \).
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The right hand side is a generating series for a set \( \tilde{M} \).

- each handle is decorated independently in one of 4 ways
- an even subset of vertices is marked
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The right hand side is a generating series for a set \( \tilde{\mathcal{M}} \).

- each handle is decorated independently in one of 4 ways
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They conjectured that this bijection has a natural interpretation.
Conjecture (The $q$-Conjecture)

There is a natural bijection $\phi$ from $\tilde{M}$ to $Q$.

$\phi: \tilde{M} \rightarrow Q$

A decorated map with
- $v$ vertices
- $2k$ marked vertices
- $e$ edges
- $f$ faces
- genus $g$

A 4-regular map with
- $e$ vertices
- $2e$ edges
- $f + v - 2k$ faces
- genus $g + k$
Jackson and Visentin proved the identity indirectly.

Example (Encoding a Map)

Begin with a rooted map.
Jackson and Visentini proved the identity indirectly.

- Maps are decorated with edge labels and orientations.

Example (Encoding a Map)

\[ \epsilon = (1\ 1')(2\ 2')(3\ 3')(4\ 4')(5\ 5') \]

Decorate the edges.
Jackson and Visentin proved the identity indirectly.
- Maps are decorated with edge labels and orientations.
- Decorated maps are encoded as permutations.

Example (Encoding a Map)

The labels and cyclic orders give a vertex permutation.
Jackson and Visentin proved the identity indirectly.
- Maps are decorated with edge labels and orientations.
- Decorated maps are encoded as permutations.

Example (Encoding a Map)

\[
\epsilon = (1\ 1')(2\ 2')(3\ 3')(4\ 4')(5\ 5') \\
\nu = (1\ 2\ 3)(1'\ 4)(2'\ 5)(3'\ 5'\ 6)(4'\ 6') \\
\varphi = \nu \epsilon = (1\ 2'\ 5'\ 6'\ 4)(1'\ 4'\ 6\ 3)(2\ 3'\ 5)
\]

Multiplying produces the face permutation.
Jackson and Visentin proved the identity indirectly.
- Maps are decorated with edge labels and orientations.
- Decorated maps are encoded as permutations.
- The permutations are enumerated using character sums.

**Example (Encoding a Map)**

$\epsilon = (1\ 1')(2\ 2')(3\ 3')(4\ 4')(5\ 5')$

$\nu = (1\ 2\ 3)(1'\ 4)(2'\ 5)(3'\ 5'\ 6)(4'\ 6')$

$\varphi = \nu\epsilon = (1\ 2'\ 5'\ 6'\ 4)(1'\ 4'\ 6\ 3)(2\ 3'\ 5)$

Fixing 1' as the root, the encoding is $1 : 2^55!$. 
Jackson and Visentini proved the identity indirectly.
- Maps are decorated with edge labels and orientations.
- Decorated maps are encoded as permutations.
- The permutations are enumerated using character sums.
- Maps can be recovered using standard techniques.

Example (Encoding a Map)

$\epsilon = (1\ 1')(2\ 2')(3\ 3')(4\ 4')(5\ 5')$
$\nu = (1\ 2\ 3)(1'\ 4)(2'\ 5)(3'\ 5'\ 6)(4'\ 6')$
$\phi = \nu\epsilon = (1\ 2'\ 5'\ 6'\ 4)(1'\ 4'\ 6\ 3)(2\ 3'\ 5)$

Fixing $1'$ as the root, the encoding is $1 : 2^5 5!$. 
Using this encoding,

\[ M(u^2, x, y, z) = 2u^2z \frac{\partial}{\partial z} \ln R \left( \frac{x}{u}, \frac{y}{u}, \frac{zu}{2} \right) \]

\[ Q(u^2, x, y, z) = 2u^2z \frac{\partial}{\partial z} \ln R_4 \left( \frac{x}{u}, \frac{y}{u}, \frac{zu}{2} \right) \]

where \( R \) and \( R_4 \) are exponential generating series for edge-labelled not-necessarily-connected maps. The proof involved factoring \( R_4 \).
Using this encoding,

\[ M(u^2, x, y, z) = 2u^2 z \frac{\partial}{\partial z} \ln R \left( \frac{x}{u}, \frac{y}{u}, \frac{zu}{2} \right) \]

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where \( R \) and \( R_4 \) are exponential generating series for edge-labelled not-necessarily-connected maps. The proof involved factoring \( R_4 \).

\[ R_4(x, y, z) = R \left( \frac{1}{2}x, \frac{1}{2}(x + 1), 4z^2y \right) \cdot R \left( \frac{1}{2}x, \frac{1}{2}(x - 1), 4z^2y \right) \]
An enumerative relationship between maps and 4-regular maps

Map Enumeration

A Remarkable Identity

An Interpretive Bottleneck

It is difficult to interpret the factorization in terms of maps.

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The factorization is the key to the proof, but
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The factorization is the key to the proof, but

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- the factors lack a direct combinatorial interpretation,
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- the factors lack a direct combinatorial interpretation,
- the proof requires more refinement than the identity it proves,
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Map Enumeration

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\]

The factorization is the key to the proof, but

- it works at the level of edge-labelled maps,
- the factors lack a direct combinatorial interpretation,
- the proof requires more refinement than the identity it proves,
- it uses character sums.
The Planar Case

Evaluating the series at $u = 0$ restricts the sums to planar maps and gives

$$Q(0, x, y, z) = M(0, y, y, xz^2).$$
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$$Q(0, x, y, z) = M(0, y, y, xz^2).$$

Combinatorially, the number of 4-regular planar maps with $n$ vertices is equal to the number of planar maps with $n$ edges.
Evaluating the series at $u = 0$ restricts the sums to planar maps and gives

$$Q(0, x, y, z) = M(0, y, y, xz^2).$$

Combinatorially, the number of 4-regular planar maps with $n$ vertices is equal to the number of planar maps with $n$ edges. Tutte’s medial construction explains this bijectively.
An enumerative relationship between maps and 4-regular maps

The Medial Construction

Tutte’s medial construction explains the planar case.

Example
Tutte’s medial construction explains the planar case.

- Place a vertex on each edge.
The Medial Construction

Tutte’s medial construction explains the planar case.

- Place a vertex on each edge.
- Join edges that are incident around a vertex circulation.

Example
Tutte’s medial construction explains the planar case.

- Place a vertex on each edge.
- Join edges that are incident around a vertex circulation.
- The medials of planar duals are the same map.

Example
Properties of the Medial Construction

The construction has several properties that make it natural.

- Cut edges become cut vertices.
Properties of the Medial Construction

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- So do loops.
Properties of the Medial Construction

The construction has several properties that make it natural.

- Cut edges become cut vertices.
- So do loops.
- Faces and vertices of degree $k$ become faces of degree $k$. 

\[ \phi \]
Properties of the Medial Construction

The construction has several properties that make it natural.

- Cut edges become cut vertices.
- So do loops.
- Faces and vertices of degree $k$ become faces of degree $k$.
- Duality in $\mathcal{M}$ corresponds to reflection in $\mathcal{Q}$.
The medial construction extends to all surfaces.

- It produces all face-bipartite 4-regular maps.
- It preserves genus.

This gives an injection from undecorated maps to 4-regular maps.
The Medial Construction at Higher Genus

The medial construction extends to all surfaces.
- It produces all face-bipartite 4-regular maps.
- It preserves genus.

This gives an injection from undecorated maps to 4-regular maps.

**Conjecture**

*The medial construction is the restriction of ϕ to M.*
There is only one 4-regular map with one vertex on the torus.
What Else do we know?

There is only one 4-regular map with one vertex on the torus.

It is impossible to construct $\phi$ such that it preserves face degrees.
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   - Refining the Conjecture  
   - Structural Evidence
By considering root deletion, a refinement of $M$ can be shown to satisfy a combinatorially significant differential equation.

$$M(1, x, \vec{y}, z, \vec{r}) = r_0 x + z \sum_{i \geq 0} \sum_{j=1}^{i+1} r_j y_{i-j+2} \frac{\partial}{\partial r_i} M$$

$$+ z \sum_{i,j \geq 0} j r_{i+j+2} \frac{\partial^2}{\partial r_i \partial y_j} M$$

$$+ z \sum_{i,j \geq 0} r_{i+j+2} \left( \frac{\partial}{\partial r_i} M \right) \left( \frac{\partial}{\partial r_j} M \right).$$

Here $y_i$ marks non-root faces of degree $i$ and $r_i$ marks a root face of degree $i$. 
By considering root deletion, a refinement of $M$ can be shown to satisfy a combinatorially significant differential equation.

$$M(1, x, \vec{y}, z, \vec{r}) = r_0 x + z \sum_{i \geq 0} \sum_{j=1}^{i+1} r_j y_{i-j+2} \frac{\partial}{\partial r_i} M$$

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$$+ z \sum_{i, j \geq 0} r_{i+j+2} \left( \frac{\partial}{\partial r_i} M \right) \left( \frac{\partial}{\partial r_j} M \right).$$

Both $M$ and $Q$ are evaluations of this series.
The differential equations allows a proof of the following theorem within the realm of connected maps.

**Theorem**

*With* $N$ a positive integer and $\langle \cdot \rangle_e$ defined by

$$\langle f \rangle_e = \frac{\int_{\mathbb{R}^N} |V(\lambda)|^2 f(\lambda) \exp \left( \sum_{k \geq 1} \frac{1}{k} x_k p_k \sqrt{z}^k \right) e^{-\frac{1}{2} p^2(\lambda)} d\lambda}{\int_{\mathbb{R}^N} |V(\lambda)|^2 \exp \left( \sum_{k \geq 1} \frac{1}{k} x_k p_k \sqrt{z}^k \right) e^{-\frac{1}{2} p^2(\lambda)} d\lambda},$$

**evaluations of the map series are given by**

$$M(1, \vec{x}, N, z) = \sum_{k=0}^{\infty} x_k \sqrt{z}^k \langle p_k \rangle_e.$$
It also gives an integral recurrence for computing $M$. 
A Recurrence

It also gives an integral recurrence for computing $M$.

- The terms of the DE correspond to the three root types.

Border  Cut edge  Handle
A Recurrence

It also gives an integral recurrence for computing $M$.

- The terms of the DE correspond to the three root types.
- The number of edges of each type determines the number of decorations of a map.

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A Recurrence

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This suggests an inductive approach to identifying $\phi$. All that remains (!) is to determine how $\phi(m)$ and $\phi(m\backslash e)$ differ when $e$ is a root edge of each type.
An enumerative relationship between maps and 4-regular maps

Cut-Edges

For decorated maps, root edges come in two forms:

- Even cut edge have an even number of decorated vertices on each side of the cut.
Cut-Edges

For decorated maps, root edges come in two forms:

- Even cut edge have an even number of decorated vertices on each side of the cut.
- Odd cut edges have an odd number of decorated vertices on each side of the cut.
An enumerative relationship between maps and 4-regular maps

Cut-Edges

For decorated maps, root edges come in two forms:

- **Even cut edge** have an even number of decorated vertices on each side of the cut.
- **Odd cut edges** have an odd number of decorated vertices on each side of the cut.

An involution $\rho$ switches the form.

![Diagram](image-url)
The action of $\phi$, when the root edge is an even cut-edge, can speculated from the following commutative diagram.

$$\phi$$

$$\phi$$

$$\phi$$

$$\phi$$
An enumerative relationship between maps and 4-regular maps

Even Cut-Edges

The action of $\phi$, when the root edge is an even cut-edge, can speculated from the following commutative diagram.

\[
\begin{array}{c}
\text{Diagram}
\end{array}
\]

The induced product on $Q \times Q$ is genus additive.
Odd Cut-Edges

If $m$ is rooted at an odd cut-edge, then $m' = \rho(m)$ is rooted at an even cut-edge.
Odd Cut-Edges

If \( m \) is rooted at an odd cut-edge, then \( m' = \rho(m) \) is rooted at an even cut-edge.

\[
\begin{array}{cccc}
m \quad \xrightarrow{\rho} \quad m' \quad \xrightarrow{} \quad (m_1, m_2) \\
\downarrow \phi \quad \downarrow \phi \quad \downarrow \phi \otimes \phi \\
q \quad \longleftarrow \quad q' \quad \longleftarrow \quad (q_1, q_2)
\end{array}
\]
Odd Cut-Edges

If $m$ is rooted at an odd cut-edge, then $m' = \rho(m)$ is rooted at an even cut-edge.

$$m \xrightarrow{\rho} m' \xrightarrow{} (m_1, m_2)$$

$$\downarrow \phi \quad \downarrow \phi \quad \downarrow \phi \otimes \phi$$

$q \leftarrow q' \leftarrow (q_1, q_2)$

$\phi$ and $\rho$ induce a product $\pi$.

$$\pi: Q \times Q \rightarrow Q$$

$$(q_1, q_2) \mapsto q$$
An enumerative relationship between maps and 4-regular maps

A Refinement

Speculation

The Product $\pi$

$\pi$ is nearly genus additive.

\[
\begin{align*}
\pi &: Q \times Q \to Q \\
(q_1, q_2) &\mapsto q
\end{align*}
\]
The Product $\pi$

$\pi$ is nearly genus additive.

$$
\begin{align*}
\pi &: Q \times Q \to Q \\
(q_1, q_2) &\mapsto q
\end{align*}
$$

$$
\begin{align*}
\rho &: m \to m' \to (m_1, m_2) \\
\phi &: q \to q' \to (q_1, q_2)
\end{align*}
$$

The genus of $\pi(q_1, q_2)$ is determined by the genus of $q_1$, the genus of $q_2$, and how many of the root vertices of $m_1$ and $m_2$ are marked.
\( \pi \) is nearly genus additive.

\[
\begin{array}{ccc}
m & \xrightarrow{\rho} & m' \\
\downarrow{\phi} & & \downarrow{\phi} \\
q & \leftrightarrow & q'
\end{array}
\quad \pi : Q \times Q \rightarrow Q
\]

The genus of \( \pi(q_1, q_2) \) is determined by the genus of \( q_1 \), the genus of \( q_2 \), and how many of the root vertices of \( m_1 \) and \( m_2 \) are marked. \( \pi \) can be used to distinguish between marked and unmarked root vertices.
In arbitrary genus, the root vertex of a 4-regular map can be a cut-vertex in three distinct ways.
A Candidate For $\pi$

In arbitrary genus, the root vertex of a 4-regular map can be a cut-vertex in three distinct ways.

The first two cuts correspond to genus additive products.
An enumerative relationship between maps and 4-regular maps

A Candidate For \( \pi \)

The third corresponds to the product:

\[
\pi' : (\begin{array}{c}
\text{\scalebox{0.5}{\image}}
\end{array}, \begin{array}{c}
\text{\scalebox{0.5}{\image}}
\end{array}) \rightarrow \begin{array}{c}
\text{\scalebox{0.5}{\image}}
\end{array}
\]
The third corresponds to the product:

\[
\pi': ((\text{\raisebox{-1.0em}{\includegraphics{example1.png}}}), (\text{\raisebox{-1.0em}{\includegraphics{example2.png}}})) \rightarrow (\text{\raisebox{-1.0em}{\includegraphics{example3.png}}})
\]

\(\pi'\) is nearly genus additive.
A Candidate For $\pi$

The third corresponds to the product:

$$\pi' : (\bigcirc, \bigcirc) \mapsto \bigcirc$$

$\pi'$ is nearly genus additive. The correction term depends on how many factors have root edges that are face-separating, but $\pi'$ is never subadditive with respect to genus.
The qualitative similarities between $\pi'$ and $\pi$ suggest a relationship between decorated maps with a decorated root-vertex and 4-regular maps with a face-non-separating root-edge.
### A Numerical Surprise!

Constructing all maps with up to 5 edges, and all 4-regular maps with up to 5 vertices suggests that the sets are bijective.

<table>
<thead>
<tr>
<th>g = 0</th>
<th>$v = 1$</th>
<th>$v = 2$</th>
<th>$v = 3$</th>
<th>$v = 4$</th>
<th>$v = 5$</th>
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</table>

5-edge maps

<table>
<thead>
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<th>Total</th>
<th>Non-Sep</th>
<th>Sep</th>
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<tbody>
<tr>
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</tr>
<tr>
<td>56646</td>
<td>28674</td>
<td>27972</td>
<td></td>
</tr>
<tr>
<td>9450</td>
<td>9450</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

5-vertex, 4-regular maps

\[
2916 = 42 + 386 + 1030 + 1030 + 386 + 42
\]
\[
23979 = \left(\begin{array}{c} 2 \\ 2 \end{array}\right)1030 + \left(\begin{array}{c} 3 \\ 2 \end{array}\right)1030 + \left(\begin{array}{c} 4 \\ 2 \end{array}\right)386 + \left(\begin{array}{c} 5 \\ 2 \end{array}\right)42 + 4(420 + 1720 + 1720 + 420)
\]
\[
27972 = \left(\begin{array}{c} 4 \\ 4 \end{array}\right)386 + \left(\begin{array}{c} 5 \\ 4 \end{array}\right)42 + 4 \left(\begin{array}{c} 2 \\ 2 \end{array}\right)1720 + \left(\begin{array}{c} 3 \\ 2 \end{array}\right)420\right) + 16(483 + 483)
\]
\[
7920 = \left(\begin{array}{c} 1 \\ 1 \end{array}\right)386 + \left(\begin{array}{c} 2 \\ 1 \end{array}\right)1030 + \left(\begin{array}{c} 3 \\ 1 \end{array}\right)1030 + \left(\begin{array}{c} 4 \\ 1 \end{array}\right)386 + \left(\begin{array}{c} 5 \\ 1 \end{array}\right)42
\]
\[
28674 = \left(\begin{array}{c} 3 \\ 3 \end{array}\right)1030 + \left(\begin{array}{c} 4 \\ 3 \end{array}\right)386 + \left(\begin{array}{c} 5 \\ 3 \end{array}\right)42 + 4 \left(\begin{array}{c} 1 \\ 1 \end{array}\right)1720 + \left(\begin{array}{c} 2 \\ 1 \end{array}\right)1720 + \left(\begin{array}{c} 3 \\ 1 \end{array}\right)420\right)
\]
\[
9450 = \left(\begin{array}{c} 1 \\ 1 \end{array}\right)42 + 4\left(\begin{array}{c} 1 \\ 1 \end{array}\right)420 + 16\left(\begin{array}{c} 1 \\ 1 \end{array}\right)483
\]
Conjecture (Refined $q$-Conjecture)

If $Q_1$ is the restriction of $Q$ to maps rooted on face-separating edges, and $\hat{M}_1$ is the restriction of $\hat{M}$ to maps with undecorated root vertices, then

$$\phi(\hat{M}_1) = Q_1.$$
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$$\phi(\hat{M}_1) = Q_1.$$ 

In terms of generating series

$$Q_1(u^2, x, y, z) = \text{bis}_{u \text{ even}} \frac{y}{y + u} M(4u^2, y + u, y, xz^2),$$

and

$$Q_2(u^2, x, y, z) = \text{bis}_{u \text{ even}} \frac{u}{y + u} M(4u^2, y + u, y, xz^2).$$
Determining $Q_1$ and $Q_2$

The integral expression for $M$ does not allow a simultaneous refinement to track root-edge-type and vertex degrees.
Determining $Q_1$ and $Q_2$

David Jackson indirectly suggested an indirect approach to computing $Q_1$ and $Q_2$. 

\[ P(u_2, x, y, z) = x y Q_1(u_2, x, y, z) + x y u_2 Q_2(u_2, x, y, z) \]

\[ Q(u_2, x, y, z) = Q_1(u_2, x, y, z) + Q_2(u_2, x, y, z) \]
Determining $Q_1$ and $Q_2$

$M$ gives an expression for the generating series for $\mathcal{P}$, the set of maps that have a root vertex of degree 3, a vertex of degree 1, and are otherwise 4-regular.

$$P(1, x, N, 1) = x^2 \frac{\langle p_3 p_1 \exp(\frac{1}{4} p_4 x) \rangle}{\langle \exp(\frac{1}{4} p_4 x) \rangle}$$
Determining $Q_1$ and $Q_2$

$M$ gives an expression for the generating series for $\mathcal{P}$, the set of maps that have a root vertex of degree 3, a vertex of degree 1, and are otherwise 4-regular.

Root-cutting is a bijection from $\mathcal{Q}$ to $\mathcal{P}$.

\[ M \rightarrow \mathcal{Q} = \mathcal{Q}_1 + \mathcal{Q}_2 \]

\[ \mathcal{P} = x^2 \mathcal{Q}_1 + xy \mathcal{Q}_2 \]

The equations can be solved for $\mathcal{Q}_1$ and $\mathcal{Q}_2$. 

\[ \chi \rightarrow \mathcal{Q} \]
Determining $Q_1$ and $Q_2$

$M$ gives an expression for the generating series for $P$, the set of maps that have a root vertex of degree 3, a vertex of degree 1, and are otherwise 4-regular.

Root-cutting is a bijection from $Q$ to $P$.

\[
Q(u^2, x, y, z) = Q_1(u^2, x, y, z) + Q_2(u^2, x, y, z)
\]
\[
P(u^2, x, y, z) = \frac{x}{y} Q_1(u^2, x, y, z) + \frac{xy}{u^2} Q_2(u^2, x, y, z)
\]
Determining $Q_1$ and $Q_2$

$M$ gives an expression for the generating series for $\mathcal{P}$, the set of maps that have a root vertex of degree 3, a vertex of degree 1, and are otherwise 4-regular. Root-cutting is a bijection from $Q$ to $\mathcal{P}$.

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The equations can be solved for $Q_1$ and $Q_2$. 
Implications

Proving the enumerative portion of the refined $q$-Conjecture reduces to a factorization problem, similar to the existing proof of Jackson and Visentin.
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- One of the factors is the same.
- The other factor is messier.
- This work remains to be done.

A consequence would be the interpretation

$$P(u^2, x, y, z) = \frac{x}{u \text{ odd}} \text{bis}_u M(4u^2, y + u, y, xz^2).$$
As a special case of the refined conjecture, we get the concrete statement:

**Conjecture**

The bijection $\phi$ specializes to a bijection from planar maps with a decorated non-root vertex to 4-regular maps on the torus rooted at a face-non-separating edge.
As a special case of the refined conjecture, we get the concrete statement:

**Conjecture**

*The bijection $\phi$ specializes to a bijection from planar maps with a decorated non-root vertex to 4-regular maps on the torus rooted at a face-non-separating edge.*

This case avoids the product of 4-regular maps with face-non-separating root-edges.
A Special Case

The following cases occur.

- The root edge joins two marked vertices.
A Special Case

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- The root edge is a loop.
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A Special Case

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- The root edge joins two marked vertices.
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  - The marked vertex is inside the loop
  - The marked vertex is outside the loop
- The root edge joins a marked root-vertex to an unmarked non-root-vertex.
  - The unmarked vertex has degree 1.
A Special Case

The following cases occur.

- The root edge joins two marked vertices.
- The root edge is a loop
  - The marked vertex is inside the loop
  - The marked vertex is outside the loop
- The root edge joins a marked root-vertex to an unmarked non-root-vertex.
  - The unmarked vertex has degree 1.
  - The unmarked vertex has degree 2.
A Special Case

The following cases occur.

- The root edge joins two marked vertices.
- The root edge is a loop
  - The marked vertex is inside the loop
  - The marked vertex is outside the loop
- The root edge joins a marked root-vertex to an unmarked non-root-vertex.
  - The unmarked vertex has degree 1.
  - The unmarked vertex has degree 2.
  - The unmarked vertex has degree $\geq 3$. 
A Special Case

The following cases occur.

- The root edge joins two marked vertices.
- The root edge is a loop
  - The marked vertex is inside the loop
  - The marked vertex is outside the loop
- The root edge joins a marked root-vertex to an unmarked non-root-vertex.
  - The unmarked vertex has degree 1.
  - The unmarked vertex has degree 2.
  - The unmarked vertex has degree $\geq 3$.

I can inductively construct $\phi$ in all but the final case.
A Special Case

The following cases occur.

- **The root edge joins two marked vertices.**
- The root edge is a loop
  - The marked vertex is inside the loop
  - The marked vertex is outside the loop
- The root edge joins a marked root-vertex to an unmarked non-root-vertex.
  - The unmarked vertex has degree 1.
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I can inductively construct $\phi$ in all but the final case.
The Root Edge Joins Two Marked Vertices
The Root Edge Joins Two Marked Vertices

\[
\begin{align*}
\phi &\quad \sigma \quad \phi \\
g=0 &\quad g=0
\end{align*}
\]
An enumerative relationship between maps and 4-regular maps

A Refinement

Structural Evidence

The Root Edge Joins Two Marked Vertices
An enumerative relationship between maps and 4-regular maps

A Refinement

Structural Evidence

The Root Edge Joins Two Marked Vertices

\[ \phi \]

\[ \rho \]

\[ \phi \]

\[ \phi \]
The Root Edge Joins Two Marked Vertices

\[ \phi \]

\[ \rho \]

\[ \phi \]

\[ \phi \]
The Root Edge Joins Two Marked Vertices
An enumerative relationship between maps and 4-regular maps

A Refinement

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The Root Edge Joins Two Marked Vertices
An enumerative relationship between maps and 4-regular maps

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The Root Edge Joins Two Marked Vertices

Example
The Root Edge Joins Two Marked Vertices
An enumerative relationship between maps and 4-regular maps

- A Refinement
- Structural Evidence

The Root Edge Joins Two Marked Vertices

Example

[Diagram showing the root edge joining two marked vertices with multiple maps and 4-regular structures]
A special case

The following cases occur.

- The root edge joins two marked vertices.
- The root edge is a loop
  - The marked vertex is inside the loop
  - The marked vertex is outside the loop
- The root edge joins a marked root-vertex to an unmarked non-root-vertex.
  - The unmarked vertex has degree 1.
  - The unmarked vertex has degree 2.
  - The unmarked vertex has degree $\geq 3$. 
The Missing Case

The remaining maps have images with one of two root configurations.

It should be possible to treat them like contraction.
An enumerative relationship between maps and 4-regular maps

Extra Figures
An enumerative relationship between maps and 4-regular maps

Extra Figures
An enumerative relationship between maps and 4-regular maps

Extra Figures