$\beta$-Gaussian Ensembles and the Non-orientability of Polygonal Glueings

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Gaussian Ensembles

For $\beta \in \{1, 2, 4\}$ an element of the $\beta$-Gaussian ensemble is constructed as

$$A = G + G^*$$

where $G$ is $n \times n$ with i.i.d. Gaussian entries selected from $\{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$.

Motivating Question

What is the value of $E(f(A))$, when $f$ is a symmetric function of the eigenvalues of its argument?

Example

$$E(\text{tr}(A^4)) = \begin{cases} 5n + 5n^2 + 2n^3 & \beta = 1 \\ n + 2n^3 & \beta = 2 \\ \frac{5}{4}n - \frac{5}{2}n^2 + \frac{3}{4}n^3 & \beta = 4 \end{cases}$$
The eigenvalues of $A$ are all real with joint density proportional to

$$\prod_{1 \leq i < j \leq n} |\lambda_i - \lambda_j|^{\beta} \exp \left( -\frac{\beta}{2} \sum_{i=1}^{n} \frac{\lambda_i^2}{2} \right)$$

**Theorem**

For every $\theta$, $E(p_{\theta}(\lambda))_\beta$ is a polynomial in the variables $n$ and $b = \frac{2}{\beta} - 1$.

**Example**

$$E(p_4(\lambda))_\beta = (1 + b + 3b^2)n + 5bn^2 + 2n^3$$
A Recurrence behind the theorem

Set \( \Omega := e^{-\frac{1}{2(1+b)}p_2(x)}|V(x)|\frac{2}{1+b} \), so that \( \langle f \rangle = E(f(x)) = c_{b,n} \int_{\mathbb{R}^n} f \Omega \, dx \).

Integrate

\[
\frac{\partial}{\partial x_1} x_1^{j+1} p_\theta(x) \Omega = \frac{\partial}{\partial x_1} x_1^{j+1} p_\theta(x)|V(x)|\frac{2}{1+b} \, e^{-\frac{p_2(x)}{2(1+b)}}
\]

\[
= (j+1)x_1^j p_\theta(x) \Omega + \sum_{i \in \theta} im_i(\theta) x_1^{i+j} p_{\theta \setminus i}(x) \Omega + \frac{2}{1+b} \sum_{i=2}^{N} \frac{x_1^{j+1} p_\theta(x)}{x_1-x_i} \Omega - \frac{1}{1+b} x_1^{j+2} p_\theta(x) \Omega
\]

to get

An Algebraic recurrence

\[
\langle p_{j+2p_\theta} \rangle = b(j+1) \langle p_j p_\theta \rangle + \alpha \sum_{i \in \theta} im_i(\theta) \langle p_{i+j} p_{\theta \setminus i} \rangle + \sum_{l=0}^{j} \langle p_l p_{j-l} p_\theta \rangle.
\]
What’s being counted?

Example (for $\beta \in \{1, 2, 4\}$)

$$\text{tr}(A^4) = \sum_{i,j,k,l} A_{ij} A_{jk} A_{kl} A_{li}$$

$$E(\text{tr}(A^4)) = 1! \binom{n}{1} E(A_{11} A_{11} A_{11} A_{11}) + 2! \binom{n}{2} E(4 A_{11} A_{11} A_{12} A_{21})$$

$$+ 2! \binom{n}{2} E(A_{12} A_{21} A_{12} A_{21}) + 3! \binom{n}{3} E(2 A_{12} A_{21} A_{13} A_{31})$$

$$+ 4! \binom{n}{4} E(A_{12} A_{23} A_{34} A_{41}) + 3! \binom{n}{3} E(4 A_{11} A_{12} A_{23} A_{31})$$

$$+ 2! \binom{n}{2} E(2 A_{11} A_{12} A_{22} A_{21})$$
What's being counted?

Example (for $\beta \in \{1, 2, 4\}$)

\[
\text{tr}(A^4) = \sum_{i,j,k,l} A_{ij} A_{jk} A_{kl} A_{li}
\]

\[
E\left(\text{tr}(A^4)\right) = 1! \left(\binom{n}{1}\right) E(A_{11} A_{11} A_{11} A_{11}) + 2! \left(\binom{n}{2}\right) E(4A_{11} A_{11} A_{12} A_{21})
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\]
Expectations as Sums

Since the entries of $A$ are independent Gaussians,

$$E(A_{i_1j_1}A_{i_2j_2} \cdots A_{i_kj_k}) = \sum_m \prod_{(u,v) \in m} E(A_uA_v)$$

summed over perfect matchings of the multiset $\{i_1j_1, i_2j_2, \ldots, i_kj_k\}$

$$\sum_{p \text{ a painting}} \#\{\text{pairings consistent with } p\} = \sum_{m \text{ a matching}} \#\{\text{paintings consistent with } m\}$$
Count the polygon glueings in 2 different ways

<table>
<thead>
<tr>
<th>$n$</th>
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Identifying the edges of a polygon creates a surface.

Its boundary is a graph embedded in the surface.
Polygon Glueings = Maps
Extra polygons give extra faces (and possibly extra components)
### Definition
A **surface** is a compact 2-manifold without boundary. (Non-orientable surfaces are permitted.)

### Definition
A **graph** is a finite set of **vertices** together with a finite set of **edges**, such that each edge is associated with either one or two vertices. (It may have loops / parallel edges.)

### Definition
A **map** is a 2-cell embedding of a graph in a surface.
Equivalence of Maps

Two maps are equivalent if the embeddings are homeomorphic.

Homeomorphisms are more complicated than we might think

Dehn Twists

Y-Homeomorphisms
Not Present for Photo
Two maps are equivalent if the embeddings are homeomorphic.
Definition

The neighbourhood of the graph determines a **ribbon graph**, and the boundaries of ribbons determine flags.

Definition

Automorphisms permute flags, and a rooted map is a map together with a distinguished orbit of flags.
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The neighbourhood of the graph determines a ribbon graph, and the boundaries of ribbons determine flags.

Definition
Automorphisms permute flags, and a **rooted map** is a map together with a distinguished orbit of flags.
Twisted Ribbons allow Non-Orientable Maps

Example (A map on the Klein Bottle)
Twisted Ribbons allow Non-Orientable Maps

Example (A map on the Torus)
A rooted map with $k$ edges can be thought of as a sequence of $k$ maps. Consecutive submaps differ in genus by 0, 1, or 2, and these steps are marked by $1$, $b$, and $a = \frac{\alpha}{2}$ to assign a weight to a glueing.
Algebraic and Combinatorial Recurrences agree

An Algebraic Recurrence

\[
\langle p_{j+2} \rangle = b(j + 1) \langle p_j \rangle + \alpha \sum_{i \in \theta} m_i(\theta) \langle p_{i+j} \rangle + \sum_{l=0}^{j} \langle p_{l} \rangle .
\]

A Combinatorial Recurrence

It corresponds to a combinatorial recurrence for counting polygon glueings.

![Diagram](image-url)
The combinatorial interpretation has a two-parameter refinement. Is there a corresponding matrix question?

At $b = 0$, we obtain glueings in $2^g$: 1 correspondence with orientable glueings. Can this correspondence be made to preserve vertex degrees as well as face degrees?

A similar recurrence describes moments of the $\beta$-Laguerre distribution, with maps replaced by hypermaps.

For $\beta \in \{1, 2, 4\}$ we can refine the combinatorial model and compute moments of $X_A$. Is there a model for the $\beta$-Ensembles where this interpretation makes sense.

For $\beta = 1$ and $\beta = 2$, there is a natural duality between vertices and faces. What operation replaces it for $b$-weighted glueings?

The connection with Jack symmetric functions that needs to be explored.
Thank You
Example

is enumerated by \((x_2^3 x_3^2) (y_3 y_4 y_5) (z_2^6)\).

\[ \nu = [2^3, 3^2] \quad \phi = [3, 4, 5] \quad \epsilon = [2^6] \]
Explicit Formulae

The hypermap series can be computed explicitly when \( \mathcal{H} \) consists of orientable hypermaps or all hypermaps.

**Theorem (Jackson and Visentin - 1990)**

When \( \mathcal{H} \) is the set of orientable hypermaps,

\[
H_\mathcal{O}(p(x), p(y), p(z); 0) = t \frac{\partial}{\partial t} \ln \left( \sum_{\theta \in \mathcal{P}} t^{|\theta|} H_{\theta} s_{\theta}(x)s_{\theta}(y)s_{\theta}(z) \right) \bigg|_{t=1}.
\]

**Theorem (Goulden and Jackson - 1996)**

When \( \mathcal{H} \) is the set of all hypermaps (orientable and non-orientable),

\[
H_\mathcal{A}(p(x), p(y), p(z); 1) = 2t \frac{\partial}{\partial t} \ln \left( \sum_{\theta \in \mathcal{P}} t^{|\theta|} \frac{1}{H_{2\theta}} Z_{\theta}(x)Z_{\theta}(y)Z_{\theta}(z) \right) \bigg|_{t=1}.
\]
A Generalized Series

Jack symmetric functions, are a one-parameter family, denoted by \( \{ J_\theta(\alpha) \}_\theta \), that generalizes both Schur functions and zonal polynomials.

**b-Conjecture (Goulden and Jackson - 1996)**

The generalized series,

\[
H(p(x), p(y), p(z); b) := (1 + b) t \frac{\partial}{\partial t} \ln \left( \sum_{\theta \in \mathcal{P}} t^{\mid \theta \mid} J_\theta(x; 1 + b) J_\theta(y; 1 + b) J_\theta(z; 1 + b) \right) \bigg|_{t=1}
\]

\[
= \sum_{n \geq 0} \sum_{\nu, \phi, \epsilon \vdash n} c_{\nu, \phi, \epsilon}(b) p_\nu(x) p_\phi(y) p_\epsilon(z),
\]

has an combinatorial interpretation involving hypermaps. In particular

\[
c_{\nu, \phi, \epsilon}(b) = \sum_{h \in \mathcal{H}_{\nu, \phi, \epsilon}} b^{\beta(h)}
\]

for some invariant \( \beta \) of rooted hypermaps.
For general $\beta$, integrate over eigenvalues

**Definition**

For a function $f : \mathbb{R}^n \to \mathbb{R}$, define an expectation operator $\langle \cdot \rangle$ by

$$
\langle f \rangle_{1+b} := c_{1+b} \int_{\mathbb{R}^n} |V(\lambda)|^{\frac{2}{1+b}} f(\lambda) e^{-\frac{1}{2(1+b)} p_2(\lambda)} \, d\lambda,
$$

with $c_{1+b}$ chosen such that $\langle 1 \rangle_{1+b} = 1$.

**Theorem (Okounkov - 1997)**

*If $n$ is a positive integer, $1 + b$ is a positive real number, and $\theta$ is an integer partition of $2n$, then*

$$
\langle J_{\theta}^{(1+b)}(\lambda) \rangle_{1+b} = J_{\theta}^{(1+b)}(I_n)[p[2n]] J_{\theta}^{(1+b)}.
$$
With respect to the inner product defined by

$$\langle p_\lambda(x), p_\mu(x) \rangle_\alpha = \delta_{\lambda,\mu} \frac{|\lambda|!}{|C_\lambda|} \alpha^{\ell(\lambda)},$$

Jack symmetric functions are the unique family satisfying:

(P1) (Orthogonality) If $\lambda \neq \mu$, then $\langle J_\lambda, J_\mu \rangle_\alpha = 0$.

(P2) (Triangularity) $J_\lambda = \sum_{\mu \preceq \lambda} v_{\lambda\mu}(\alpha) m_\mu$, where $v_{\lambda\mu}(\alpha)$ is a rational function in $\alpha$, and ‘$\preceq$’ denotes the natural order on partitions.

(P3) (Normalization) If $|\lambda| = n$, then $v_{\lambda,[1^n]}(\alpha) = n!$. 
Jack Symmetric Functions

Jack symmetric functions, are a one-parameter family, denoted by \( \{ J_\theta(\alpha) \}_\theta \), that generalizes both Schur functions and zonal polynomials.

**Proposition (Stanley - 1989)**

*Jack symmetric functions are related to Schur functions and zonal polynomials by:*

\[
\begin{align*}
J_\lambda(1) &= H_\lambda s_\lambda, \\
J_\lambda(2) &= Z_\lambda,
\end{align*}
\]

\[
\begin{align*}
\langle J_\lambda, J_\lambda \rangle_1 &= H_\lambda^2, \\
\langle J_\lambda, J_\lambda \rangle_2 &= H_{2\lambda},
\end{align*}
\]

where \( 2\lambda \) is the partition obtained from \( \lambda \) by multiplying each part by two.
\[ \langle p_{j+2}p_\theta \rangle = b(j + 1) \langle p_j p_\theta \rangle + (1 + b) \sum_{i \in \theta} i m_i(\theta) \langle p_{i+j} p_\theta \setminus i \rangle + \sum_{l=0}^{j} \langle p_l p_{j-l} p_\theta \rangle \]

**Example**

\[ \langle 1 \rangle = 1 \]
\[ \langle p_0 \rangle = n \]
\[ \langle p_2 \rangle = b \langle p_0 \rangle + \langle p_0 p_0 \rangle = bn + n^2 \]
\[ \langle p_1 p_1 \rangle = (1 + b) \langle p_0 \rangle = (1 + b)n \]
\[ \langle p_4 \rangle = 3b \langle p_2 \rangle + \langle p_0 p_2 \rangle + \langle p_1 p_1 \rangle + \langle p_2 p_0 \rangle = (1 + b + 3b^2)n + 5bn^2 + 2n^3 \]
\[ \langle p_3 p_1 \rangle = 2b \langle p_1 p_1 \rangle + (1 + b) \langle p_2 \rangle + \langle p_0 p_1 p_1 \rangle + \langle p_1 p_0 p_1 \rangle = (3b + 3b^2)n + (3 + 3b)n^2 \]
\[ \langle p_2 p_2 \rangle = b \langle p_0 p_2 \rangle + 2(1 + b) \langle p_2 \rangle + \langle p_0 p_0 p_2 \rangle = 2b(1 + b)n + (2 + 2b + b^2)n^2 + 2bn^3 + n^4 \]
\[ \langle p_2 p_{1,1} \rangle = b \langle p_0 p_{1,1} \rangle + 2(1 + b) \langle p_{1,1} \rangle + \langle p_0 p_0 p_{1,1} \rangle = 2(1 + b)^2n + (b + b^2)n^2 + (1 + b)n^3 \]
\[ \langle p_1 p_3 \rangle = 3(1 + b) \langle p_2 \rangle = (3b + 3b^2)n + (3 + 3b)n^2 \]
\[ \langle p_1 p_{2,1} \rangle = 2(1 + b) \langle p_{1,1} \rangle + (1 + b) \langle p_0 p_2 \rangle = (2 + 4b + 2b^2)n + (b + b^2)n^2 + (1 + b)n^3 \]
\[ \langle p_{1,1,1,1} \rangle = 3(1 + b) \langle p_0 p_{1,1} \rangle = (1 + 2b + b^2)n^2 \]
\[ \langle p_{j+2}p_{\theta} \rangle = b(j + 1) \langle p_j p_{\theta} \rangle + (1 + b) \sum_{i \in \theta} i m_i(\theta) \langle p_{i+j}p_{\theta \setminus i} \rangle + \sum_{l=0}^{j} \langle plp_{j-l}p_{\theta} \rangle \]

Example

\[ \langle 1 \rangle = 1 \]
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\[ \langle p_1p_1 \rangle = (1 + b) \langle p_0 \rangle = (1 + b)n \]
\[ \langle p_4 \rangle = 3b \langle p_2 \rangle + \langle p_0p_2 \rangle + \langle p_1p_1 \rangle + \langle p_2p_0 \rangle = (1 + b + 3b^2)n + 5bn^2 + 2n^3 \]
\[ \langle p_3p_1 \rangle = 2b \langle p_1p_1 \rangle + (1 + b) \langle p_2 \rangle + \langle p_0p_1p_1 \rangle + \langle p_1p_0p_1 \rangle = (3b + 3b^2)n + (3 + 3b)n^2 \]
\[ \langle p_2p_2 \rangle = b \langle p_0p_2 \rangle + 2(1 + b) \langle p_2 \rangle + \langle p_0p_0p_2 \rangle = 2b(1 + b)n + (2 + 2b + b^2)n^2 + 2bn^3 + n^4 \]
\[ \langle p_2p_1,1 \rangle = b \langle p_0p_1,1 \rangle + 2(1 + b) \langle p_1,1 \rangle + \langle p_0p_0p_1,1 \rangle = 2(1 + b)^2n + (b + b^2)n^2 + (1 + b)n^3 \]
\[ \langle p_1p_3 \rangle = 3(1 + b) \langle p_2 \rangle = (3b + 3b^2)n + (3 + 3b)n^2 \]
\[ \langle p_1p_2,1 \rangle = 2(1 + b) \langle p_1,1 \rangle + (1 + b) \langle p_0p_2 \rangle = (2 + 4b + 2b^2)n + (b + b^2)n^2 + (1 + b)n^3 \]
\[ \langle p_1,1,1,1 \rangle = 3(1 + b) \langle p_0p_1,1 \rangle = (1 + 2b + b^2)n^2 \]
$b$ is ubiquitous

The many lives of $b$

<table>
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<tr>
<th></th>
<th>$b = 0$</th>
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<td>Hypermaps</td>
<td>Orientable</td>
<td>Locally Orientable</td>
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<tr>
<td>Symmetric Functions</td>
<td>$s_\theta$</td>
<td>$J_\theta(b)$</td>
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<td>Matrix Integrals</td>
<td>GUE</td>
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<td>Moduli Spaces</td>
<td>over $\mathbb{C}$</td>
<td>over $\mathbb{R}$</td>
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<td>Matching Systems</td>
<td>Bipartite</td>
<td>All</td>
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1. Orient and label the edges.
2. This induces labels on flags.
3. Clockwise circulations at each vertex determine $\nu$.
4. Face circulations are the cycles of $\epsilon\nu$.

$$
\epsilon = (1 \ 1')(2 \ 2')(3 \ 3')(4 \ 4')(5 \ 5')(6 \ 6') \\
\nu = (1 \ 2 \ 3)(1' \ 4)(2' \ 5)(3' \ 5' \ 6)(4' \ 6') \\
\epsilon\nu = \phi = (1 \ 4 \ 6' \ 3')(1' \ 2 \ 5 \ 6 \ 4')(2' \ 3 \ 5')
$$
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\]
Equivalence classes can be encoded by perfect matchings of flags.

Start with a ribbon graph.
Encoding all Maps

Equivalence classes can be encoded by perfect matchings of flags.

Ribbon boundaries determine 3 perfect matchings of flags.
Encoding all Maps

Equivalence classes can be encoded by perfect matchings of flags.

Pairs of matchings determine faces.
Encoding all Maps

Equivalence classes can be encoded by perfect matchings of flags.

Pairs of matchings determine, faces, edges,
Encoding all Maps

Equivalence classes can be encoded by perfect matchings of flags.

Pairs of matchings determine, faces, edges, and vertices.
Encoding all Maps

\[ M_v = \{ \{1, 3\}, \{1', 3'\}, \{2, 5\}, \{2', 5'\}, \{4, 8'\}, \{4', 8\}, \{6, 7\}, \{6', 7'\} \} \]

\[ M_e = \{ \{1, 2'\}, \{1', 4\}, \{2, 3'\}, \{3, 4'\}, \{5, 6'\}, \{5', 8\}, \{6, 7'\}, \{7, 8'\} \} \]

\[ M_f = \{ \{1, 1'\}, \{2, 2'\}, \{3, 3'\}, \{4, 4'\}, \{5, 5'\}, \{6, 6'\}, \{7, 7'\}, \{8, 8'\} \} \]
Hypermaps

Generalizing the combinatorial encoding, an arbitrary triple of perfect matchings determines a **hypermap** when the triple induces a connected graph, with cycles of $M_e \cup M_f$, $M_e \cup M_v$, and $M_v \cup M_f$ determining vertices, hyperfaces, and hyperedges.

Hypermaps both **specialize** and generalize maps.

Example

Hypermaps can be represented as face-bipartite maps.
Hypermaps

Generalizing the combinatorial encoding, an arbitrary triple of perfect matchings determines a **hypermap** when the triple induces a connected graph, with cycles of $M_e \cup M_f$, $M_e \cup M_v$, and $M_v \cup M_f$ determining vertices, hyperfaces, and hyperedges.

Hypermaps both specialize and **generalize** maps.

Example

Maps can be represented as hypermaps with $\epsilon = [2^n]$. 
Example

\[
\nu = [2^3]
\]

\[
\epsilon = [3^2]
\]

\[
\phi = [6]
\]