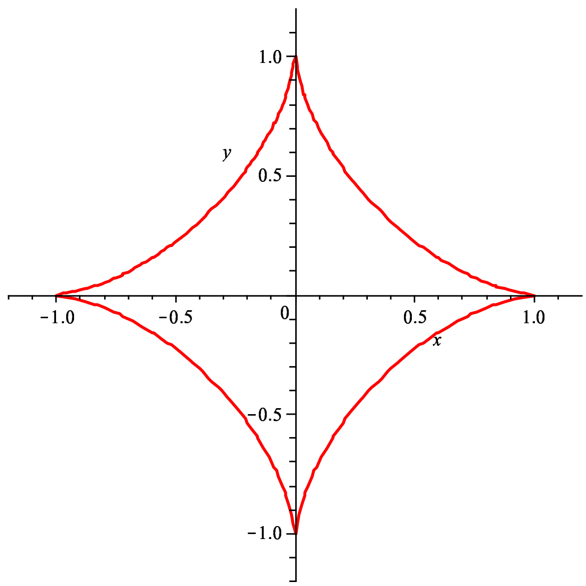
MATH 18.01 Problem Set 7 Solutions

Problem 1. (*8 pts: 2+2+2+2*) The curve $x^{2/3} + y^{2/3} = 1$ is known as a hypocycloid with 4 cusps.

a) Sketch the curve. Note that the cube root takes precedence over the square in the exponent, so $(-1)^{2/3} = \left((-1)^{1/3}\right)^2 = (-1)^2 = 1$. This means that $x^{2/3}$ is always positive. As a further hint, observe that the points $(\pm 1, 0)$ and $(0, \pm 1)$ are on the curve.

Solution. The given points define the corners. Note that $|x|^k + |y|^k = 1$ is a circle when $k = 2$, is a "diamond" when $k = 1$, and bows further inward for the situation here where $k = 2/3$. Another easy to find point is when $x = y$, which occurs at $\left(\left(\frac{1}{2}\right)^{3/2}, \left(\frac{1}{2}\right)^{3/2}\right)$. Finally, the derivatives calculated in part b) show that the corner points are "cusps", meaning that there are two opposing tangent lines that meet (the derivative is discontinuous at a cusp).



b) Use implicit differentiation to calculate the derivative $\frac{dy}{dx}$.

Solution. Evaluating the differential gives

$$\frac{2}{3}\, x^{-1/3}\, dx + \frac{2}{3}\, y^{-1/3}\, dy = 0,$$

so

$$\boxed{\frac{\boldsymbol{dy}}{\boldsymbol{dx}} = -\frac{\boldsymbol{y^{1/3}}}{\boldsymbol{x^{1/3}}}}\,.$$

c) Simplify $1 + \left(\frac{dy}{dx}\right)^2$ using part b) and the defining equation for the curve.

Solution. From the answer in part b),

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{y^{2/3}}{x^{2/3}}.$$

The equation for the hypocycloid can be rewritten as $1+\frac{y^{2/3}}{x^{2/3}}=\frac{1}{x^{2/3}}$, so

$$\boxed{\mathbf{1+\left(\frac{dy}{dx}\right)^2=\frac{1}{x^{2/3}}}}.$$

d) Calculate the total arc-length of the hypocycloid.

Solution. There is fourfold symmetry in the curve, so it is sufficient to calculate the arc-length in the first quadrant. This is

$$\int_{x=0}^{1}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_{x=0}^{1}\frac{1}{x^{1/3}}\,dx=\frac{3}{2}x^{2/3}\Big|_0^1=\frac{3}{2}.$$

Therefore the total arc-length is $4\cdot\frac{3}{2}=\boxed{\mathbf{6}}$.

As a quick reality check, this is less than $2\pi\sim 6.3$, which is the circumference of the circle of radius 1, and more than $4\sqrt{2}\sim 5.6$, which is the length of the diamond formed by straight lines.

Problem 2. (*4 pts: 2+2*) Two cylindrical tanks of radius 1m and depth 5m are full of water (density 1000kg/m^3)and must be drained.

a) In the first tank, the water is pumped through an inflexible tube that, once inserted, reaches all the way to the bottom. How much work is done in emptying the tank?

Solution. Since the outlet tube is at the bottom of the tank, all of the water must travel the full 5 meters. There is a total volume of $5\pi\,\mathrm{m}^3$, and thus a total mass of 5000kg. Thus the total work is $5\cdot 5000=\boxed{\mathbf{25,000}\mathrm{m}\cdot\mathrm{kg}}$.

This can also be set up as a very simple integral. The water may be viewed in thin horizontal slices of area $A=\pi$, volume $dV=A\,dh=\pi\,dh$, and mass $dm=1000\pi\,dh$. However, since all of the slices must travel the same distance, there is no dependence on the height, and the work is thus

$$W=\int dW=\int_{h=0}^{5}5\,dm=\int_{h=0}^{5}5\cdot 1000\cdot\pi=\boxed{\mathbf{25,000}}.$$

b) In the second tank, a flexible, buoyant hose floats on the top of the water as it drains. How much total work is done in emptying the tank?

Solution. Now the work depends on the height, as the water is pumped from the top. In particular, when the remaining water is a distance h from the top of the tank, the work done is $dW=h\,dm$. Thus

$$W=\int_{h=0}^{5}h\cdot 1000\pi\,dh=500\pi h^2\,\Big|_{h=0}^{5}=\boxed{\mathbf{12,500}\mathrm{m}\cdot\mathrm{kg}}.$$

Problem 3. (*3 pts: 2+1*) a) Verify that $F(x)=-2\cos(2x)$ and $G(x)=4\sin^2(x)$ are both anti-derivatives of the same function.

Solution. The derivatives are

$$F'(x) = -2\sin(2x) \cdot 2 = \boxed{\mathbf{4\sin(2}\boldsymbol{x}\mathbf{)}},$$

and by the sine double-angle formula,

$$G'(x) = 8\sin(x)\cos(x) = \boxed{\mathbf{4\sin(2}\boldsymbol{x}\mathbf{)}}.$$

b) What is the relation between $F(x)$ and $G(x)$?

Solution. The Mean Value Theorem implies that two functions with the same derivative must differ by a constant. In this case, the cosine double angle formula gives

$$F(x) = -2\cos(2x) = -2\left(\cos^2(x) - \sin^2(x)\right).$$

This simplifies further using $\cos^2(x) = 1 - \sin^2(x)$, yielding

$$\boxed{\boldsymbol{F(x)}} = -2\left(1 - 2\sin^2(x)\right) = \boxed{\boldsymbol{G(x)} - \mathbf{2}}.$$