PL vs. Smooth Fiber Bundles (Lecture 8)

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Our goal in this lecture is to begin to prove the following result:

Theorem 1. *Suppose given a commutative diagram*

$$\begin{array}{ccc} K & \xrightarrow{f} & M \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ L & \longrightarrow & N \end{array}$$

where K and L are polyhedra, M and N are smooth manifolds, and the horizontal maps are PD homeomorphisms. Assume that p is a submersion of smooth manifolds (so that q is a submersion of PL manifolds). Then p is a smooth fiber bundle if and only if q is a PL fiber bundle.

Since the horizontal maps are homeomorphisms, the morphisms p and q can be identified as continuous maps between topological spaces. It follows that p is proper if and only if q is proper. In the proper case, Theorem 1 is a consequence of the following:

Proposition 2. *Let $p : M \rightarrow N$ be a proper submersion between smooth (PL) manifolds. Then p is a smooth (PL) fiber bundle.*

In the smooth case, this result is elementary. We may assume without loss of generality that $N = \mathbb{R}^k$. Choose a Riemannian metric on M, which determines a splitting of the tangent bundle T_M into vertical and horizontal components $T_M \simeq T_M^v \oplus T_M^h$. Using the fact that p is proper, we deduce that for each $x \in M$ and each smooth path $h : p(x) \rightarrow y$, there exists a unique smooth path $\overline{h} : x \rightarrow \overline{y}$ lifting h such that the derivative of $\overline{h}$ lies in the horizontal tangent bundle T_M^h at every point. In particular, if we choose x to lie in the fiber $X_0 = p^{-1}\{0\}$ and h to be a straight line from $p(x) = 0$ to a point $y \in N$, then we can write $\overline{y}$ as a function $f(x, y)$. The function $f : M_0 \times N \rightarrow M$ is a diffeomorphism in a neighborhood of $M_0 \times \{0\}$, so that f is a submersion in a neighborhood of $0 \in N$.

We wish to give a proof which works also in the PL context. We note that we can assume without loss of generality that the base N is a simplex. We now introduce a bit of terminology:

Definition 3. Let $p : M \rightarrow \Delta^n$ be a map of polyhedra, let $x \in N$, and let $K \subseteq p^{-1}\{x\}$ be a compact subpolyhedron. We will say that p *has a product structure near* K if there exists an open subset $V \subseteq N$ containing x and open subset $U \subseteq M$ containing K such that U is PL homeomorphic to a product $U_0 \times V$ where U_0 is a PL manifold (and p is given by the projection to the second factor).

We note that p is a submersion if and only if it has a product structure near every point. If p is proper and has a product structure near the inverse image $M_x = p^{-1}\{x\}$, then we can take $U_0 = M_x$ so we get an open embedding $M_0 \times V \hookrightarrow p^{-1}(V)$. Using the properness of p, we deduce that this map is a homeomorphism (possibly after shrinking V).

It p has a product structure near a subset $K \subseteq p^{-1}\{x\}$, then it has a product structure near a larger polyhedron containing K in its interior. In particular, if p is a submersion, then it has a product structure near every simplex of some sufficiently fine triangulation of $p^{-1}\{x\}$. It now suffices to show:

Proposition 4. *Let $p : M \to \Delta$ be a map of polyhedra (where Δ denotes a simplex), let $0 \in \Delta$ be a point, let $M_0 = p^{-1}(0)$, and let $A, B \subseteq M_0$ be compact subpolyhedra. If p has a product structure near A and B, then p has a product structure near $A \cup B$.*

The proof will be based on the following nontrivial result of piecewise linear topology:

Theorem 5 (Parametrized Isotopy Extension Theorem)**.** *Let M be a piecewise linear manifold, let K be a finite polyhedron, and let Δ be a simplex containing a point 0. Let $f : K \times \Delta \to M \times \Delta$ be a PL embedding compatible with the projection to Δ, which we think of as a family of embeddings $\{f_t : K \to M\}_{t \in \Delta}$. Assume that f is* locally *extendible to family of isotopies of M: that is, we can embed K as a closed subset of another polyhedron U and extend f to an open embedding $U \times \Delta \hookrightarrow M \times \Delta$. Then there exists a PL homeomorphism $h : M \times \Delta \to M \times \Delta$ (which we can think of as a family of PL homeomorphisms $\{h_t : M \to M\}_{t \in \Delta}$) such that $h_0 = \mathrm{id}_M$ and $h(f_t(k)) = (f_0(k), t)$.*

Proof of Proposition 4. Shrinking Δ if necessary, we may assume that there are open sets $U, V \subseteq M_0$ containing A and B, respectively, and open embeddings $f : U \times \Delta \hookrightarrow M$, $g : V \times \Delta \hookrightarrow M$ such that $f|U \times \{0\}$ and $g|V \times \{0\}$ are the inclusions $U, V \subseteq M_0 \subseteq M$. Let K be a compact polyhedron contained in $U \cap V$ which contains a neighborhood of $A \cap B$. Shrinking Δ if necessary, we may assume that $f(K \times \Delta)$ is contained in $g(V \times \Delta)$, so we that $g^{-1} \circ f$ gives a well-defined map $q : K \times \Delta \to V \times \Delta$ such that $q_0 : K \to V$ is the identity. Using Theorem 5, we can find a map $h : V \times \Delta \to V \times \Delta$ such that h_0 is the identity and $h \circ q$ is the canonical inclusion $K \times \Delta \to V \times \Delta$. Replacing g by $g \circ h^{-1}$, we can assume that f and g agree on $K \times \Delta$. Let $U_0 \subseteq U$ and $V_0 \subseteq V$ be smaller open subsets containing A and B such that $U_0 \cap V_0 \subseteq K$. Then $f|U_0 \times \Delta$ and $g|V_0 \times \Delta$ can be amalgamated to obtain a map $e : W \times \Delta \to M$, where $W = U_0 \cup V_0$. Shrinking W and Δ if necessary, we can arrange that e is an open embedding, which provides the desired product structure near $A \cup B$. □

Let us now return to the general case of Theorem 1. We will concentrate on the "only if" direction (since this is what is needed for the purposes described in the last lecture). The problem is local on N, so we may assume that N consists of a single simplex Δ. We therefore have a trivial fiber bundle $p : M \times \Delta \to \Delta$ of smooth manifolds, a Whitehead compatible triangulation of $M \times \Delta$ such that p is a piecewise linear map, and we wish to show that p is a PL fiber bundle.

Choose a proper smooth map $f : M \to \mathbb{R}_{>0}$. Modifying f slightly, we may assume that $1, 2, \ldots \in \mathbb{R}$ are regular values of f, so that the subsets $M_i = f^{-1}[0, i]$ are compact submanifolds M with boundary $B_i = f^{-1}\{i\}$. Choose disjoint collar neighborhoods $U_i \simeq B_i \times \mathbb{R} \subseteq M$ such that $U_i \cap M_i \simeq B_i \times \mathbb{R}_{\leq 0}$.

Fix a point $0 \in \Delta$, so that $p^{-1}\{0\} \simeq M$ inherits a PL structure. Choose any Whitehead compatible triangulation of B_i, so that $B_i \times \mathbb{R}$ inherits a PL structure. The inclusion $f : B_i \times \mathbb{R} \hookrightarrow M$ need not be a PL homeomorphism. However, we saw in Lecture 5 that f can be approximated arbitrarily well by a PL homeomorphism $f' : B_i \times \mathbb{R} \to U$. In particular, we can assume that $C = f'(B_i \times (-\infty, 0])$ is a PL manifold with boundary of whose interior contains $B_i \times (-\infty, -1]$ and which is contained in $B_i \times (-\infty, 1]$.

We now require the following consequence of a special case of Theorem 1, which we will prove in the next lecture:

Lemma 6. *Let B be a smooth manifold. Suppose we are given a Whitehead compatible triangulation of $B \times \mathbb{R} \times \Delta$, where Δ is a simplex, such that the projection $p : B \times \mathbb{R} \times \Delta \to \Delta$ is a piecewise linear. Then there exists an open subset $E \subseteq B \times \mathbb{R} \times \Delta$ containing $B \times [-1, 1] \times \Delta$ such that the projection $E \to \Delta$ is a PL fiber bundle.*

Applying the Lemma in the case $B = B_i$, we deduce the existence of an open subset $V \subseteq B_i \times \mathbb{R}$ containing $B_i \times [-1, 1]$ and a PL homeomorphism $E \simeq V \times \Delta$. Shrinking Δ is necessary, we may assume that this homeomorphism carries $B \times \{-1\} \times \Delta \subseteq E$ into the interior of $C \times \Delta \subseteq V \times \Delta$. Let X_i denote union of the image of $(C \cap V) \times \Delta$ under this map with $(M_i - (B_i \times (-1, 0])) \times \Delta$. We now have an increasing filtration

$$X_1 \subseteq X_2 \subseteq \ldots \subseteq M \times \Delta$$

by compact subpolyhedra, and each of the projections $X_i \to \Delta$ is a submersion whose fibers are PL manifolds with boundary. It follows from a variant of Proposition 2 (allowing for the case of manifolds with boundary) that each of the maps $X_i \to \Delta$ is a PL fiber bundle, so we have PL homeomorphisms $h_i : X_i \simeq P_i \times \Delta$ for some PL manifold with boundary P_i. Using the parametrized isotopy extension theorem, we can adjust h_i so that the induced maps $P_{i-1} \times \Delta \to P_i \times \Delta$ are induced by embeddings $P_{i-1} \hookrightarrow P_i$. Taking P to be the direct limit of the P_i, we obtain a PL homeomorphism $P_i \times \Delta \to M \times \Delta$, which proves that the projection map $p : M \times \Delta \to \Delta$ is a fiber bundle in the PL category.