Triangulation in Families (Lecture 7)

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In the last lecture, we introduced the diagram of simplicial sets

$$\mathrm{Man}^m_{PL} \overset{\phi'}{\leftarrow} \mathrm{Man}^m_{PD} \overset{\phi}{\rightarrow} \mathrm{Man}^m_{\mathrm{sm}} .$$

Our goal in this lecture is to prove that the map ϕ is a trivial Kan fibration. In other words, we wish to show that every diagram

$$\begin{array}{ccc} \partial\,\Delta^n & \longrightarrow & \mathrm{Man}^m_{PD} \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & \mathrm{Man}^m_{\mathrm{sm}} \end{array}$$

can be completed by adding a suitable dotted arrow.

Let V be an infinite dimensional real vector space. Unwinding the definitions, we are given smooth submanifold $X \subseteq V \times \Delta^n$ which is a fiber bundle over Δ^n, a subpolyhedron $K_0 \subseteq V \times \partial\,\Delta^n$, and a PD homeomorphism $K_0 \rightarrow X \times_{\Delta^n} \partial\,\Delta^n$ such that the following diagram commutes:

$$\begin{array}{ccc} K_0 & \longrightarrow & X \\ \downarrow & & \downarrow \\ \partial\,\Delta^n & \longrightarrow & \Delta^n. \end{array}$$

We wish to find the following data:

- (i) A polyhedron K equipped with a PL map $\pi : K \rightarrow \Delta^n$ and a homeomorphism $K_0 \simeq Y \times_{\Delta^n} \partial\,\Delta^n$.
- (ii) A PD homeomorphism $K \rightarrow X$ which commutes with the projection to Δ^n.
- (iii) A lifting of π to a PL embedding $K \rightarrow V \times \Delta^n$ which extends the embedding already given on K_0.

Moreover, this data must satisfy the following condition:

- (iv) The projection map $K \rightarrow \Delta^n$ exhibits K as a fiber bundle in the PL category (automatically trivial, since Δ^n is contractible). In other words, there is a PL homeomorphism $Y \simeq \Delta^n \times N$, for some PL m-manifold N.

As we saw last time, the data of (iii) comes essentially for free, using general position arguments. We will focus on conditions (i) and (ii) for the time being, and return to (iv) at the end of the discussion.

Since $X \rightarrow \Delta^n$ is a fiber bundle in the smooth setting, we can identify X with a product $M \times \Delta^n$ for some smooth manifold M. The PD homeomorphism $K_0 \rightarrow X \times_{\Delta^n} \partial\,\Delta^n$ can be viewed as a providing a Whitehead compatible triangulation of $M \times \partial\,\Delta^n$ which is compatible with the polyhedron structure on $\partial\,\Delta^n$ (in other words, a PD homeomorphism $K_0 \rightarrow M \times \partial\,\Delta^n$ such that the composite map $K_0 \rightarrow \partial\,\Delta^n$ is PL. We wish to extend this to a Whitehead compatible triangulation of $M \times \Delta^n$ which is compatible with the projection $M \times \Delta^n \rightarrow \Delta^n$.

In the case $n = 0$, this reduces to the problem we solved in Lecture 4: namely, proving that every smooth manifold M admits a Whitehead compatible triangulation. For our present needs, we will require a more refined version of the same result:

Theorem 1. *Let M be a smooth manifold and let $f_0 : K_0 \to M \times \partial\,\Delta^n$ be a PD homeomorphism such that the composite map $K_0 \to \partial\,\Delta^n$ is PL. Then f can be extended to a PD homeomorphism $K \to M \times \Delta^n$, where the projection $K \to \Delta^n$ is PL.*

Proof. Write Δ^n as a union of two closed subpolyhedra L and L' whose interiors cover Δ^n, where L contains a neighborhood of $\partial\,\Delta^n$, $L' \cap \partial\,\Delta^n = \emptyset$, and there is a retraction $r : L \to \partial\,\Delta^n$. Let $\overline{K}_0 = K_0 \times_{\partial\,\Delta^n} L$. Then f_0 evidently extends to a PD embedding $\overline{f}_0 : \overline{K}_0 \to M \times \Delta^n$ with image $M \times L$.

For each $x \in M$, choose a smooth chart $i_x : \mathbb{R}^n \to M$ carrying 0 to x, and let U_x denote the image of the open ball. Since M is compact, we can choose a finite collection $\{x_1, \ldots, x_k\}$ such that the open balls $U_i = U_{x_i}$ cover M. We then have PD maps $\overline{f}_i : [-2, 2]^n \times L' \to M \times \Delta^n$ whose images cover $M \times L'$. We observe that each of the projections $\pi \circ \overline{f}_i$ is PL, where $\pi : M \times \Delta^n \to \Delta^n$ denotes the projection.

To produce the desired map $K \to M \times \Delta^n$, it will suffice to show that we can approximate the PD maps $\{\overline{f}_0, \ldots, \overline{f}_n\}$ by maps $\{\overline{f}'_i\}$ which are pairwise compatible, where $\overline{f}_0|K_0 = \overline{f}'_0|K_0$ and $\pi \circ \overline{f}_i = \pi \circ \overline{f}'_i$. We proceed as in Lecture 4 to define sequences of approximations $\{\overline{f}^j_0, \ldots, \overline{f}^j_j\}$ to $\{\overline{f}_0, \ldots, \overline{f}_j\}$ using induction on j. When $j = 0$, we set $\overline{f}^j_0 = \overline{f}_0$.

Suppose that we have already defined a sequence of pairwise compatible maps $\overline{f}^j_0, \ldots, \overline{f}^j_j$ which are close approximations to $\overline{f}_0, \ldots, \overline{f}_j$. These maps can therefore be amalgamated to product a single PD map $F : \overline{K}_j \to M \times \Delta^n$, where $\overline{K}_j$ is a polyhedron containing K_0 such that $F|K_0 = f_0$ and $\pi \circ F$ is PL. To complete the proof, it will suffice to show that we can choose close approximations F' to F with the same properties, so that F' is compatible with $\overline{f}_{j+1}$. To prove this, let $P \subseteq \overline{K}_j$ denote the inverse image of $\mathbb{R}^n \times (\Delta^n - \partial\,\Delta^n) \subseteq M \times \Delta^n$, and let $g : P \to \mathbb{R}^n$ denote the composition of F with the projection $\mathbb{R}^n \times \Delta^n \to \mathbb{R}^n$.

Let $P_0 \subseteq P$ be a compact subpolyhedron containing the inverse image of $[-3, 3]^n \times C$, where C is a closed neighborhood of L' in $\Delta^n - \partial\,\Delta^n$. Applying the main lemma from the last lecture, we can approximate g arbitrarily well by a map $g' : P \to \mathbb{R}^n$ whose restriction to X_0 is PL and which agrees with g outside a compact set. We can then define a map $F' : \overline{K}_j \to M \times \Delta^n$ by the formula

$$F'(x) = \begin{cases} F(x) & \text{if } x \notin P \\ (g'(x), \pi F(x)) & \text{if } x \in P. \end{cases}$$

If g' is a sufficiently good approximation to g, then ${F'}^{-1}[-2, 2]^n \times L' \subseteq P_0$, so that F' is PL on ${F'}^{-1}[-2, 2]^n \times L'$ and therefore compatible with $\overline{f}_{j+1}$. It is readily verified that F' has the desired properties. □

This completes the construction of a polyhedron K containing K_0 and a PD homeomorphism $f : K \to M \times \Delta^n$ such that $\pi \circ f$ is piecewise linear. To complete the proof, we need to verify (iv): that is, we need to show that $\pi \circ f$ exhibits K as a PL fiber bundle over the simplex Δ^n. We first establish a *local* version of this statement.

First, we need to introduce a bit of terminology:

Definition 2. Let $f : K \to L$ be a PL map of polyhedra. We will say that f is a *submersion* (of relative dimension n) if for every point $x \in K$, there exist open neighborhoods $U \subseteq K$ of x and $V \subseteq L$ of $f(x)$ and a PL homeomorphism $U \simeq V \times \mathbb{R}^n$ (such that f is given by projection onto the first factor).

Example 3. A polyhedron K is a piecewise linear manifold if and only if the unique map $K \to *$ is a submersion.

There is an analogous notion of submersion in the smooth category, which is probably more familiar: a map of smooth manifolds $M \to N$ is a submersion if its differential is surjective at every point. By the implicit function theorem, this is equivalent to the assertion that every point $x \in M$ has a neighborhood diffeomorphic to $V \times \mathbb{R}^n$, where V is an open subset of N.

The main result of lecture 3 admits the following relative version:

Theorem 4. *Suppose given a commutative diagram*

$$\begin{array}{ccc} K & \xrightarrow{f} & M \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ L & \longrightarrow & N \end{array}$$

where K and L are polyhedra, M and N are smooth manifolds, and the horizontal maps are PD homeomorphisms. Assume that p is a submersion of smooth manifolds. Then q is a submersion of PL manifolds.

If $L = N = *$, then the theorem reduces to the assertion that for any Whitehead compatible triangulation of a smooth manifold, the underlying polyhedron is a PL manifold. In the general case, we can use essentially the same argument. The assertion is local, so we can assume that M has the form $N \times \mathbb{R}^n$. We can then apply the "linearization" construction to the composite map

$$K \to M \to \mathbb{R}^n,$$

to approximate f arbitrarily well by maps $K \to L \times \mathbb{R}^n$ which are piecewise linear in a neighborhood of any given point in $x \in K$. Any sufficiently good approxmation will be a PL homeomorphism in a neighborhood of x, so that q is a submersion.

Of course, the condition of being a submersion is generally weaker than the condition of being a fiber bundle. To complete the verification of (iv) we will need the following technical result, whose proof will occupy our attention during the next few lectures:

Theorem 5. *Suppose given a commutative diagram*

$$\begin{array}{ccc} K & \xrightarrow{f} & M \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ L & \longrightarrow & N \end{array}$$

where K and L are polyhedra, M and N are smooth manifolds, and the horizontal maps are PD homeomorphisms. Assume that p is a submersion of smooth manifolds (so that q is a submersion of PL manifolds). Then p is a smooth fiber bundle if and only if q is a PL fiber bundle.

Remark 6. If the fiber dimensions are not equal to 4, then Theorem 5 can be deduced from the following result.

Theorem 7. *Let $p : M \to N$ be a submersion of smooth (PL) manifolds of relative dimension $\neq 4$, and assume that p is a fiber bundle in the category of topological manifolds. Then p is a fiber bundle in the category of smooth (PL) manifolds.*

Theorem 7 is false in relative dimension 4, but Theorem 5 is true in every dimension.