More on Mapping Class Groups (Lecture 37)

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Let us begin with a recap of the previous lecture. Let Σ be a compact, connected, oriented surface with $\chi(\Sigma) < 0$, and let Γ denote the fundamental group of Γ. We let $\mathrm{Out}(\Gamma) = \mathrm{Aut}(\Gamma)/\Gamma$ be the outer automorphism group of Γ. For any collection of embedded oriented loops $C_1, \ldots, C_n \subseteq \Gamma$, choose a base point x_i on each C_i, and let γ_i denote the homotopy class of C_i in $\pi_1(\Sigma, x_i) \simeq \Gamma$. We let $\mathrm{Out}_{C_1,\ldots,C_n}(\Sigma)$ denote the group of tuples $(\phi, \phi_1, \ldots, \phi_n)$ where $\phi \in \mathrm{Out}(\Gamma)$, and each ϕ_i is an automorphism of $\pi_1(\Sigma, x_i)$ which represents ϕ and fixes γ_i. The map

$$(\phi, \phi_1, \ldots, \phi_n) \to \phi$$

is a group homomorphism from $\mathrm{Out}_{C_1,\ldots,C_n}(\Sigma)$, whose image is the collection of outer automorphisms of Γ which fix the conjugacy classes of γ_i and whose kernel is the product of centralizers $\prod_{1 \leq i \leq n} Z(\gamma_i)$ Provided that each C_i is essential (that is, not nullhomotopic), these centralizers coincide with the cyclic group $\gamma_i^{\mathbf{Z}}$ generated by γ_i, and are canonically isomorphic to $\mathbf{Z}$.

In the special case where the collection C_i consist of all boundary components of Σ, we will denote $\mathrm{Out}_{C_1,\ldots,C_n}(\Gamma)$ by $\mathrm{Out}_\partial(\Gamma)$. If the collection C_i includes all boundary components together with one additional embedded loop C, we denote this group instead by $\mathrm{Out}_{\partial,C}(\Gamma)$.

Fix now an embedded loop C in Σ containing a point x, and let $\gamma \in \pi_1(\Sigma, x) \simeq \Gamma$ be the class represented by C. We let $\mathrm{Out}'_\partial(\Gamma)$ denote the subgroup of $\mathrm{Out}_\partial(\Gamma)$ consisting of outer automorphisms which fix the conjugacy class of γ. Let $\mathrm{Diff}_\partial(\Sigma)$ be the group of diffeomorphisms of Σ which fix the boundary pointwise, $\mathrm{Diff}_\partial(\Sigma, C)$ the subgroup consisting of diffeomorphisms which restrict to an orientation-preserving diffeomorphism of C, and $\mathrm{Diff}_{\partial,C}(\Sigma)$ the subgroup consisting of diffeomorphisms which fix C pointwise. In the last lecture, we saw that there is a homotopy pullback diagram

$$\begin{array}{ccccc}
\mathrm{Diff}_{\partial,C}(\Sigma) & \longrightarrow & \mathrm{Diff}_\partial(\Sigma, C) & \longrightarrow & \mathrm{Diff}_\partial(\Sigma) \\
\downarrow \psi & & \downarrow & & \downarrow \\
\mathrm{Out}_{\partial,C}(\Gamma) & \longrightarrow & \mathrm{Out}'_\partial(\Gamma) & \longrightarrow & \mathrm{Out}_\partial(\Gamma).
\end{array}$$

Moreover, $\mathrm{Diff}_{\partial,C}(\Sigma)$ is homotopy equivalent to $\mathrm{Diff}_\partial(\Sigma')$, where Σ' is the surface obtained by cutting Σ along C. Our ultimate goal is to prove that the vertical maps are homotopy equivalences. For the moment, we will be content to prove the following weaker statement:

($*$) In the above diagram, each of the vertical maps has a contractible kernel.

As we explained last time, the proof proceeds by induction. Since each square in the above diagram is a homotopy pullback, the kernels of the vertical maps are all homotopy equivalent. Consequently, it will suffice to show that the kernel of ψ is contractible. There are two cases to consider:

(1) The curve C is nonseparating. In this case, the surface Σ' is connected. Let $\psi' : \mathrm{Diff}_\partial(\Sigma') \to \mathrm{Out}_\partial(\Sigma')$ be the canonical map. Since Σ' is simpler than C, the inductive hypothesis guarantees that the kernel

$\ker(\psi')$ is contractible; in particular, the kernel of ψ' is the identity component of $\mathrm{Diff}_\partial(\Sigma')$. Since $\mathrm{Diff}_{\partial,C}(\Sigma)$ is homotopy equivalent to $\mathrm{Diff}_\partial(\Sigma')$, its identity component is also contractible. To prove that $\ker(\psi)$ is contractible, it suffices to show that $\ker(\psi)$ coincides with the identity component of $\mathrm{Diff}_{\partial,C}(\Sigma)$. Suppose otherwise: then there exists a diffeomorphism $f \in \mathrm{Diff}_{\partial,C}(\Sigma)$ which is not isotopic to the identity, such that f induces the identity map from $\pi_1(\Sigma, x_i)$ to itself, whenever x_i is a base point on C or some boundary component of Σ. Let f' be the induced diffeomorphism of Σ'. Then f' is not isotopic to the identity, so the image of $f' \in \mathrm{Out}_\partial(\Sigma')$ is nontrivial. It follows that for some base point y on some boundary component of Σ', f' induces a nontrivial automorphism $f'_* : \pi_1(\Sigma', y) \to \pi_1(\Sigma', y)$. We have a commutative diagram

$$\begin{array}{ccc} \pi_1(\Sigma', y) & \longrightarrow & \pi_1(\Sigma, y) \\ \downarrow{\scriptstyle f'_*} & & \downarrow{\scriptstyle f_*} \\ \pi_1(\Sigma', y) & \longrightarrow & \pi_1(\Sigma, y). \end{array}$$

Since f_* is the identity, we deduce that the horizontal maps are not injective.

On the other hand, we can compute $\pi_1\Sigma$ from $\pi_1\Sigma'$ using a generalization of van Kampen's theorem. Note that Σ is obtained from Σ' by gluing along a pair of boundary components B_0 and B_1 (having image C in Σ). Consider the following more general situation: let X' be a well-behaved connected topological space with a pair of disjoint, well-behaved connected closed subsets B_0 and B_1, and let X be the space obtained by gluing B_0 to B_1 along some homeomorphism h. The map h induces an isomorphism $\pi_1 B_0 \simeq \pi_1 B_1$; let us denote this common fundamental group by H. Let γ be a path in X' from a base point p of B_0 to the base point $h(p)$ of B_1, and take p to be a base point of X'. Then the inclusions of B_0 and B_1 into X' induce group homomorphisms $i, j : H \to G = \pi_1 X'$, where j is defined by carrying a loop α to $\gamma^{-1} \circ \alpha \circ \gamma$. Note that γ maps to a closed loop in X, and therefore determines a class $t \in \pi_1 X$. We have the following classical result:

Theorem 1. *The group $\pi_1 X$ is generated by $G = \pi_1 X'$ together with the element g, subject only to the relations $ti(h) = j(h)t$ for $h \in H$.*

In the special case where the maps i and j are injective, we say that $\pi_1 X$ is obtained from G by an *HNN-extension*. In this case, we can describe $\pi_1 X$ very explicitly. Choose a set C_+ of left coset representatives of $i(H)$ in G (including the identity) and set C_- of left coset representatives of $j(H)$ in G. Then every element of $\pi_1 X$ can be written uniquely in the form

$$g t^{n_1} c_1 t^{n_2} c_2 \ldots t^{n_k} c_k$$

where the n_i are nonnegative integers, $c_i \in C_+$ if $n_i > 0$, $c_i \in C_-$ if $n_i < 0$, and c_i is nonzero unless $n = k$. The image of G corresponds to those elements for which $k = 0$. This description shows that G injects into $\pi_1 X$.

In our case, the subsets B_0 and B_1 are inclusions of boundary components in the surface Σ'. We therefore have $\pi_1 B_0 \simeq \pi_1 B_1 \simeq \mathbf{Z}$, and the inclusion maps $i, j : \mathbf{Z} \to \pi_1\Sigma'$ are both injective. It follows that $\pi_1\Sigma' \to \pi_1\Sigma$ is injective, as desired.

(2) The curve C is separating. In this case, we can write Σ' as a disjoint union of two connected components $\Sigma_0 \cup \Sigma_1$, each of which contains C as a boundary curve. Let Γ_0 and Γ_1 be their fundamental groups. We have a map $\psi' : \mathrm{Diff}_\partial(\Sigma') \to \mathrm{Out}_\partial(\Gamma_0) \times \mathrm{Out}_\partial(\Gamma_1)$. The inductive hypothesis guarantees that $\ker(\psi')$ is contractible; in particular, it is the identity component of $\mathrm{Diff}_\partial(\Sigma')$. We conclude again that the identity component of $\mathrm{Diff}_{\partial,C}(\Sigma)$ is contractible. To complete the proof, it will suffice to show that this identity component coincides with $\ker(\psi)$. Assume otherwise; then we have a diffeomorphism $f \in \mathrm{Diff}_{\partial C}(\Sigma)$ which is not isotopic to the identity, but induces the identity on $\pi_1(\Sigma, x_i)$ for any base point x_i in $\partial\Sigma$ or in C. Let f' be the induced diffeomorphism of Σ'. Since f' does not lie in the boundary component of $\mathrm{Diff}_\partial(\Sigma')$, its image is nontrivial in either $\mathrm{Out}_\partial(\Gamma_0)$ or $\mathrm{Out}_\partial(\Gamma_1)$. It follows

that there exists a point y in some boundary component of Σ' such that $f'_* : \pi_1(\Sigma', y) \to \pi_1(\Sigma', y)$ is nontrivial. Since f_* is trivial on $\pi_1(\Sigma, y)$, we deduce that $\pi_1(\Sigma', y) \to \pi_1(\Sigma, y)$ is not injective. We will obtain a contradiction.

By van Kampen's theorem (in its usual form), the fundamental group $\pi_1 \Sigma$ can be recovered as an amalgamated product $\pi_1 \Sigma_0 \star_{\pi_1 C} \pi_1 \Sigma_1 = \Gamma_0 \star_{\mathbf{Z}} \Gamma_1$. Since the maps $\pi_1 C \to \pi_1 \Sigma_i$ are injective, this free product admits an explicit description: if we chose sets of left coset representatives C_0 and C_1 (including the identity) for $\mathbf{Z}$ in Γ_0 and Γ_1, then every element of $\pi_1 \Sigma$ can be written uniquely in the form

$$g c_0 c_1 c_2 \ldots c_k,$$

where $g \in \mathbf{Z}$ and the c_i are nontrivial elements of $C_0 \coprod C_1$ which alternate between C_0 and C_1. The uniqueness guarantees that the maps $\Gamma_0 \to \Gamma \leftarrow \Gamma_1$ are injective.

The inductive mechanism above reduces the proof of the main theorem to the case where Σ is the simplest possible hyperbolic surface: namely, a pair of pants. In this case, we let $\mathrm{Diff}(\Sigma, \partial)$ be the group of diffeomorphisms of Σ which restrict to orientation preserving diffeomorphisms of each boundary component. We have a fiber sequence

$$\mathrm{Diff}^+_\partial(\Sigma) \to \mathrm{Diff}(\Sigma, \partial) \to \mathrm{Diff}^+(S^1)^3.$$

(Here the notation Diff^+ indicates that we are restricting our attention to orientation-preserving diffeomorphisms.) Since $\mathrm{Diff}^+(S^1)$ is homotopy equivalent to the circle group, the fiber sequence gives rise to another fiber sequence in the homotopy category.

$$\mathbf{Z}^3 \to \mathrm{Diff}^+_\partial(\Sigma) \to \mathrm{Diff}(\Sigma, \partial).$$

This sequence fits into a commutative diagram

$$\begin{array}{ccccc} \mathbf{Z}^3 & \longrightarrow & \mathrm{Diff}^+_\partial(\Sigma) & \longrightarrow & \mathrm{Diff}(\Sigma, \partial) \\ \downarrow & & \downarrow \psi & & \downarrow \psi_0 \\ \mathbf{Z}^3 & \longrightarrow & \mathrm{Out}_\partial(\Sigma) & \longrightarrow & \mathrm{Out}(\Sigma). \end{array}$$

It follows that the right square is a homotopy pullback, so that $\ker(\psi)$ is homotopy equivalent to $\ker(\psi_0)$, which is a union of connected components of $\mathrm{Diff}(\Sigma, \partial)$. To complete the proof in this case, it will suffice to show that $\mathrm{Diff}(\Sigma, \partial)$ is contractible.

Let S^2 denote the 2-sphere, so that Σ can be identified with the surface obtained from S^2 by performing a real blow-up at three points $\{x, y, z\}$. Let $\mathrm{Diff}^+(S^2, \{x, y, z\})$ be the group of diffeomorphisms of S^2 that fix the points x, y, and z. Then the construction of the real blow-up induces a map $\mathrm{Diff}^+(S^2, \{x, y, z\}) \to \mathrm{Diff}(\Sigma, \partial)$. This map is a homotopy equivalence: it has a homotopy inverse (in the PL category, say) given by coning off the boundary components. Consequently, it suffices to prove that $\mathrm{Diff}^+(S^2, \{x, y, z\})$ is contractible.

Let X denote the open subset of $(S^2)^3$ consisting of triples of *distinct* points of S^2. We have a homotopy fiber sequence

$$\mathrm{Diff}^+(S^2, \{x, y, z\}) \to \mathrm{Diff}^+(S^2) \xrightarrow{a} X.$$

Consequently, we are reduced to proving that the map a is a homotopy equivalence. In a previous lecture, we saw that the group $\mathrm{PGL}_2(\mathbf{C})$ of biholomorphisms of $S^2 \simeq \mathbf{C}\mathrm{P}^1$ is homotopy equivalent to $\mathrm{Diff}^+(S^2)$. It therefore suffices to show that the action of $\mathrm{PGL}_2(\mathbf{C})$ on X determines a homotopy equivalence $\mathrm{PGL}_2(\mathbf{C}) \to X$. But this map is actually a homeomorphism: for every triple of distinct points $x, y, z \in \mathbf{C}\mathrm{P}^1$, there is a unique linear fractional transformation which carries (x, y, z) to $(0, 1, \infty)$.

To complete our understanding of mapping class groups, we would also like to know that the map $\psi : \mathrm{Diff}_\partial(\Sigma) \to \mathrm{Out}_\partial(\Gamma)$ is *surjective*. This assertion can formulated in group theoretic terms: for example, it implies that if Γ is a surface group given as an amalgamated free product $\Gamma_0 \star_{\mathbf{Z}} \Gamma_1$, then any automorphism of Γ which is trivial on the subgroup $\mathbf{Z}$ arises from automorphisms of Γ_0 and Γ_1. However, we will give a more direct geometric argument in the next lecture.