Diffeomorphisms of Hyperbolic Surfaces (Lecture 36)

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Let Σ be a compact, connected, oriented surface with $\chi(\Sigma) < 0$. Our goal in this lecture (and the next) is to describe the homotopy type of the diffeomorphism group $\mathrm{Diff}(\Sigma)$. We begin by observing that the universal cover $\widetilde{\Sigma}$ of $\Sigma - \partial\Sigma$ can be identified with the hyperbolic plane. It follows that Σ is an Eilenberg-MacLane space $K(\Gamma, 1)$, where Γ is a subgroup of $P\,\mathrm{SL}_2(\mathbb{R})$.

Lemma 1. *Let g be a nontrivial element of Γ. Then the centralizer of g is an infinite cyclic group, generated by an nth root of Γ for some $n \geq 1$.*

Proof. If Σ is closed, then g must be a hyperbolic element of $P\,\mathrm{SL}_2(\mathbb{R})$: without loss of generality, Σ corresponds to a fractional linear transformation of the form $z \mapsto \lambda z$. The centralizer of g in $P\,\mathrm{SL}_2(\mathbb{R})$ consists of linear fractional transformations of the form $z \mapsto \mu z$, where μ is a positive real number. It follows that the centralizer of g in Γ can be identified with a discrete subgroup of $(\mathbb{R}_{>0}, \times) \simeq (\mathbb{R}, +)$, and is therefore infinite cyclic.

If Σ has boundary, then g might be a parabolic element of $P\,\mathrm{SL}_2(\mathbb{R})$: in this case, we may assume without loss of generality that g is the linear fractional transformation $z \mapsto z + 1$. The centralizer of g in $P\,\mathrm{SL}_2(\mathbb{R})$ consists of linear fractional transformations of the form $z \mapsto z + t$. Consequently, the centralizer of g in Γ is a discrete subgroup of $(\mathbb{R}, +)$, and therefore infinite cyclic. □

Corollary 2. *The center of Γ is trivial.*

Proof. Let g be a nonzero element of the center of Γ. Lemma 1 implies that the centralizer of g is cyclic, so that Γ is cyclic (genereated by either a hyperbolic element of the form $z \mapsto \lambda z$ or a parabolic transformation of the form $z \mapsto z + 1$). In either case, $\Sigma - \partial\Sigma \simeq D/\Gamma$ is homeomorphic to an annulus, and has Euler characteristic zero. □

Let $\mathrm{Aut}(\Sigma)$ denote the monoid of self-homotopy equivalences of Σ, and let $\mathrm{Aut}_*(\Sigma)$ denote the monoid of self-homotopy equivalences of Σ that preserve a base point. Since Σ is a $K(\Gamma, 1)$, we deduce that $\mathrm{Aut}_*(\Sigma)$ is homotopy equivalent to the discrete space $\mathrm{Aut}(\Sigma)$ of automorphisms of the group Σ. We have a fiber sequence

$$\mathrm{Aut}_*(\Sigma) \to \mathrm{Aut}(\Sigma) \to \Sigma.$$

The long exact sequence of homotopy groups shows that $\pi_0 \mathrm{Aut}(\Sigma)$ can be identified with the group $\mathrm{Out}(\Gamma)\,\mathrm{Aut}(\Gamma)/\Gamma$ of outer automorphisms of Γ, the group $\pi_1 \mathrm{Aut}(\Sigma)$ can be identified with kernel of the map $\Gamma \to \mathrm{Aut}(\Sigma)$ (which vanishes by Corollary 2), and the groups $\pi_i \mathrm{Aut}(\Sigma)$ vanish for $i > 1$. In other words, $\mathrm{Aut}(\Sigma)$ homotopy equivalent to the discrete space $\mathrm{Out}(\Gamma)$.

Our goal in this lecture (and the next) is to prove the following:

Theorem 3. *Assume that Σ is closed. Then the obvious map $\mathrm{Diff}(\Sigma) \to \mathrm{Aut}(\Sigma) \simeq \mathrm{Out}(\Gamma)$ is a homotopy equivalence.*

We now describe the analogue of Theorem 3 in the case where Σ has boundary $C_1 \cup C_2 \cup \ldots \cup C_n$. Let $\gamma_1, \ldots, \gamma_n$ denote representatives for these loops in $\pi_1 \Sigma$. Let $\mathrm{Diff}_\partial(\Sigma)$ denote the group of diffeomorphisms of Σ that fix each C_i pointwise. Similarly, we let $\mathrm{Aut}_\partial(\Sigma)$ be the monoid of self-homotopy equivalences of the pair $(\Sigma, \partial\,\Sigma)$ which are the identity on the boundary. We have a fiber sequence

$$\mathrm{Aut}_\partial(\Sigma) \to \mathrm{Aut}(\Sigma) \to \mathrm{Map}(\partial\,\Sigma, \Sigma).$$

The base of this fibration can be identified with the nth power of $\mathrm{Map}(S^1, \Sigma)$, whose connected components can be identified with conjugacy classes in Γ where each connected component is a classifying space for the centralizer of the corresponding element of Γ. We obtain a group-theoretic description of $\mathrm{Aut}_\partial(\Sigma)$: it is homotopy equivalent to the discrete set $\mathrm{Out}_\partial(\Gamma)$ consisting sequences $(\phi; \phi_1, \ldots, \phi_n)$, where ϕ is an outer automorphism of Γ, and each ϕ_i is an automorphism of Γ representing ϕ such that $\phi_i(\gamma_i) = \gamma_i$.

Remark 4. To obtain this identification more precisely, we should be more careful about base points. Fix a point x_i on each C_i. A homotopy equivalence f of Σ which is the identity on $\partial\,\Sigma$ induces well-defined maps $\phi_i : \pi_1(\Sigma, x_i) \to \pi_1(\Sigma, x_i)$, each of which fixes the class γ_i represented by the loop C_i.

The analogue of Theorem 3 is the following:

Theorem 5. *Let Σ be a compact connected oriented surface with $\chi(\Sigma) < 0$. Then the obvious map $\mathrm{Diff}_\partial(\Sigma) \to \mathrm{Aut}_\partial(\Sigma) \simeq \mathrm{Out}_\partial(\Sigma)$ is a homotopy equivalence.*

We can break the assertion of Theorem 5 into two parts. Let $\mathrm{Diff}^0_\partial(\Sigma)$ denote the inverse image of the identity element of $\mathrm{Out}_\partial(\Sigma)$. We must show:

(1) The space $\mathrm{Diff}^0_\partial(\Sigma)$ is contractible.

(2) The map $\mathrm{Diff}_\partial(\Sigma) \to \mathrm{Out}_\partial(\Sigma)$ is surjective.

We will begin the proof of (1) in this lecture. The proof proceeds by induction on the complexity of Σ: we consider another surface Σ' to be simpler than Σ if either it has a smaller genus, or has the same genus and a smaller number of boundary components. The base case for the induction is when Σ is a pair of pants: a surface of genus zero with exactly three boundary components. We will treat this case (and assertion (2)) in the next lecture.

Assume therefore that Σ is more complicated than a pair of pants. If Σ has positive genus, then we can choose a simple nonseparating closed curve C in Σ such that cutting Σ along C decreases the genus. If Σ has genus 0 but $n > 3$ boundary components, then there exists a separating simple closed curve C which decomposes Σ into two components, each of which has fewer than n boundary components. In either case, we can choose the curve C to be smooth.

Proposition 6. *Let $\mathrm{Diff}_\partial(\Sigma, C)$ be the subgroup of $\mathrm{Diff}_\partial(\Sigma)$ consisting of those diffeomorphisms restrict to an orientation-preserving diffeomorphism of C, and let $\mathrm{Diff}'_\partial(\Sigma)$ be the subgroup of $\mathrm{Diff}_\partial(\Sigma)$ consisting of those elements which fix the conjugacy class in Γ represented by C. Then the inclusion $\mathrm{Diff}_\partial(\Sigma, C) \hookrightarrow \mathrm{Diff}'_\partial(\Sigma)$ is a homotopy equivalence.*

Proof. Let $X(\Sigma)$ denote the collection of all hyperbolic metrics on Σ with respect to which each boundary component is geodesic. Let Y be the collection of all smooth simple closed curves in Σ which are freely homotopic to C. Given a hyperbolic metric on Σ, the class $[C]$ has a unique geodesic representative: this determines a fibration $X(\Sigma) \to Y$, whose fiber is the subspace $X_0(\Sigma) \subseteq X(\Sigma)$ of hyperbolic metrics with respect to which C is a geodesic loop.

Let Σ' be the surface obtained by cutting Σ along C; and let $M(C)$ denote the collection of smooth metrics on C. We have a (homotopy) pullback diagram

$$\begin{array}{ccc} X_0(\Sigma) & \longrightarrow & X(\Sigma') \\ \downarrow & & \downarrow \\ M(C) & \longrightarrow & M(C) \times M(C). \end{array}$$

The space $M(C)$ is contractible, so $X_0(\Sigma) \to X(\Sigma')$ is a homotopy equivalence. Since $X(\Sigma')$ is contractible, we deduce that $X_0(\Sigma)$ is contractible. Since $X(\Sigma)$ is contractible, we conclude that Y is contractible. Finally, we have a fiber sequence

$$\mathrm{Diff}_\partial(\Sigma, C) \to \mathrm{Diff}'_\partial(\Sigma) \to Y,$$

which shows that $\mathrm{Diff}_\partial(\Sigma, C) \to \mathrm{Diff}'_\partial(\Sigma)$ is a homotopy equivalence. $\square$

Let $\mathrm{Diff}_{\partial \cup C}(\Sigma)$ be the group of diffeomorphisms of Σ which fix $\partial \Sigma \cup C$ pointwise. If $f \in \mathrm{Diff}_\partial(\Sigma)$ fixes C pointwise, then f determines an automorphism ϕ_C of $\pi_1(\Sigma, x)$ which fixes the class $\gamma \in \pi_1(\Sigma, x)$ represented by C. Let $\mathrm{Out}_{\partial,C}(\Gamma)$ denote the set of quadruples $(\phi, \phi_1, \ldots, \phi_n, \phi_C)$ where $(\phi, \phi_1, \ldots, \phi_n) \in \mathrm{Out}_\partial(\Gamma)$ and ϕ_C is as above. Note that since the centralizer of γ is isomorphic to the cyclic group generated by γ, we have an exact sequence

$$\mathbf{Z} \to \mathrm{Out}_{\partial,C}(\Gamma) \to \mathrm{Out}_\partial(\Gamma).$$

Similarly, we have a fiber sequence

$$\mathrm{Diff}_{\partial,C}(\Sigma) \to \mathrm{Diff}_\partial(\Sigma, C) \to \mathrm{Diff}^+(C),$$

fitting into a map of fiber sequences

$$\begin{array}{ccccc}
\Omega\,\mathrm{Diff}^+(C) & \longrightarrow & \mathrm{Diff}_{\partial,C}(\Sigma) & \longrightarrow & \mathrm{Diff}_\partial(\Sigma, C) \\
\downarrow & & \downarrow \psi & & \downarrow \\
\mathbf{Z} & \longrightarrow & \mathrm{Out}_{\partial,C}(\Gamma) & \longrightarrow & \mathrm{Out}_\partial(\Gamma).
\end{array}$$

Since the left map is a homotopy equivalence, we can identify fibers of the right map with fibers of the middle map. Consequently, to prove (1), it suffices to show that $\mathrm{Diff}^0_{\partial,C}(\Sigma) = \psi^{-1}\{e\}$ is contractible. Note that $\mathrm{Diff}_{\partial,C}(\Sigma)$ is homotopy equivalent to $\mathrm{Diff}_\partial(\Sigma')$. By the inductive hypothesis, $\mathrm{Diff}_\partial(\Sigma')$ is a union of contractible components. It therefore suffices to show that $\mathrm{Diff}^0_{\partial,C}(\Sigma)$ lies in a single one of these components. Unwinding the definitions, we must show that if f is a diffeomorphism of Σ fixing the boundary together with C and $\overline{f}$ is the corresponding diffeomorphism of Σ', then $\overline{f}$ induces the identity map on $\pi_1(\Sigma', x)$ for every point $x \in \partial \Sigma'$. To see this, it suffices to show that the map $\pi_1(\Sigma', x) \to \pi_1(\Sigma, x)$ is injective. There are two cases to consider:

- (a) The curve C is separating, so that $\Sigma' = \Sigma_1 \cup \Sigma_2$. The van Kampen theorem allows us to compute that $\pi_1\Sigma' \simeq \pi_1\Sigma_1 \star_{\pi_1 C} \pi_1\Sigma_2$. Since $\pi_1 C \simeq \mathbf{Z}$ is a subgroup of both $\pi_1\Sigma_1$ and $\pi_1\Sigma_2$ (this follows from Lemma 1), we conclude that the maps $\pi_1\Sigma_1 \to \pi_1\Sigma \leftarrow \pi_1\Sigma_2$ are injective.
- (b) We will discuss this case in the next lecture.