Incompressible Surfaces (Lecture 32)

April 29, 2009

In this lecture, we will describe some applications of the loop theorem to the study of a 3-manifold M. For simplicity, we will restrict our attention to the case where M is connected, closed and oriented, though the ideas below generalize to the case of nonorientable manifolds with boundary.

Definition 1. An embedded two-sided surface $\Sigma \subseteq M$ is *compressible* if one of the following conditions holds:

(1) There exists an embedded loop $L \subseteq \Sigma$ which does not bound an embedded disk in Σ, but does bound an embedded disk D in M such that $D \cap \Sigma = \partial D$.

(2) The surface Σ is a 2-sphere which bounds a disk in M.

If Σ is not compressible, then we say that Σ is *incompressible*.

Lemma 2. *Let $\Sigma \subseteq M$ be a 2-sided surface of genus $g > 0$. Then Σ is incompressible if and only if the map $\pi_1 \Sigma \to \pi_1 M$ is injective.*

Proof. The "if" direction is clear: if $\pi_1 \Sigma \to \pi_1 M$ is injective, then any loop in Σ which bounds a disk in M (embedded or not) must be nullhomotopic in Σ, and therefore bound a disk.

Conversely, suppose that Σ is incompressible. If $\pi_1 \Sigma \to \pi_1 M$ is not injective, then there exists a nontrivial loop L in Σ which is the boundary restriction of a map $f : D^2 \to M$. We may assume without loss of generality that the map f is transverse to Σ, so that $f^{-1}\Sigma$ is a union of k circles for $k > 0$. We will assume that f has been chosen to minimize k.

Suppose first that $k = 1$, so that $f^{-1}\Sigma = \partial D^2$. Let M' be the 3-manifold with boundary obtained by cutting M along Σ. Then f lifts to a map $f' : D^2 \to M'$. Applying the loop theorem, we deduce that there exists an embedding $\widetilde{f'} : (D^2, S^1) \to (M', \partial M')$ representing a nontrivial homotopy class on the boundary. Then the composite map $D^2 \xrightarrow{\widetilde{f'}} M' \to M$ is an embedded disk in M, contradicting our assumption that Σ is incompressible.

If $k > 1$, then $f^{-1}\Sigma$ includes a circle C in the interior of D^2. We may assume that C is chosen innermost, so that it bounds a disk D' with $f^{-1}\Sigma \cap D' = C$. If $f|C$ is a nontrivial loop in Σ, then we can replace f by $f|D$ and thereby contradict the minimality of k. Otherwise, we may assume that $f|C$ is nullhomotopic, so that there exists another map $f_0 : D^2 \to M$ which agrees with f outside of D' and carries D' into Σ. Moving f_0 by a small homotopy on D', we obtain a new map $f_1 : D^2 \to M$ which agrees with f on the boundary and such that $f_1^{-1}\Sigma$ consists of $k - 1$ circles, again contradicting the minimality of k. □

Our next result guarantees the existence of a good supply of incompressible surfaces:

Proposition 3. *Let M be a closed connected oriented 3-manifold. Let X be a topological space containing an open subset homeomorphic to $Y \times (-1, 1)$, for some simply connected space Y (which we identify with $Y \times \{0\}$), and let $f : M \to X$ be a map. Assume that $\pi_2 Y \simeq 0$, and that π_2 vanishes for each component of $X - Y$. Then there exists a map $f' : M \to X$ satisfying the following conditions:*

(1) *The maps f and f' are homotopic when restricted to $M - F$, where F is a finite set (in fact, we can choose F to consist of only one point). In particular, f and f' induce the same map $\pi_1 M \to \pi_1 X$.*

(2) *The map f' is transverse to Y, and ${f'}^{-1} Y$ is a union of incompressible surfaces of M.*

Proof. Adjusting f by a small homotopy, we may assume that f is transverse to Y, so that $f^{-1}Y$ is a union of finitely many two-sided surfaces Σ_i in M, each having genus g_i. We will assume that these surfaces have been chosen to minimize $c(f) = \sum_i 3^{g_i}$. If each of these surfaces is incompressible, we are done. Otherwise, we will explain how to modify the map f to obtain a new map f' satisfying (1) with $c(f') < c(f)$; this will contradict the minimality of f.

Let Σ be a compressible component of $f^{-1}Y$. If Σ is a 2-sphere, then Σ bounds a disk D. Since $\pi_2 Y$ is equal to zero, there exists a map $f_0 : M \to X$ which agrees with f outside of D, and carries D into Y (moreover, this map is homotopic to f after removing a single point of D). Adjusting f_0 by a small homotopy, we obtain a map $f' : M \to X$ such that ${f'}^{-1}Y = f^{-1}Y - \Sigma$, so that $c(f') < c(f)$ as desired.

Suppose now that Σ is not a 2-sphere, so there exists a 2-disk $D \subseteq M$ such that $D \cap \Sigma = \partial D$ is a nontrivial loop in Σ. Choose a tubular neighborhood $D \times [-1, 1] \subseteq M$ such that $(D \times [-1, 1]) \cap M \subseteq \Sigma$. We may assume that $f(x, t) \in Y \times [0, 1]$ for $(x, t) \in D \times [-1, 1]$ near $\partial D \times [-1, 1]$.

We define a new map $f' : M \to X$ as follows:

(i) We let f' coincide with f outside of the interior of $D \times [-1, 1]$ (so that f' will be homotopic to f after removing a point of $D \times [-1, 1]$).

(ii) Since Y is simply connected, the loop $f | \partial D \times \frac{1}{2}$ extends to a map $g_+ : D \times \frac{1}{2} \to Y$; we let $f' | D \times \frac{1}{2} = g_+$. Define $f' | D \times \frac{-1}{2}$ similarly.

(iii) Using the assumption that each component of $X - Y$ has vanishing π_2, we can extend f' over $D \times [\frac{1}{2}, 1]$ and over $D \times [-1, \frac{-1}{2}]$, carrying the complement of $(D \times \{\pm\frac{1}{2}\}) \cup (\partial D \times [-1, 1])$ into $X - Y$.

(iv) Using the assumption that $\pi_2 Y = 0$, we can extend f' over $D \times [\frac{-1}{2}, \frac{1}{2}]$ so that $f'(D \times [\frac{-1}{2}, \frac{1}{2}] \subseteq Y$.

Adjust f' by a small homotopy which pushes $f'(D \times (\frac{-1}{2}, \frac{1}{2})$ into $Y \times (-1, 0)$. Then the inverse image ${f'}^{-1}Y$ can be identified with the surface obtained from $f^{-1}Y$ by doing surgery along the loop $L : \partial D$. There are two possibilities:

(a) The curve L is separating in Σ. Since L is nontrivial, we deduce that L surgery along L cuts Σ into two surfaces of positive genus g and g', where Σ has genus $g + g'$. Since $3^g + 3^{g'} < 3^{g+g'}$, we deduce that $c(f') < c(f)$.

(b) The curve L is nonseparating in Σ. Then surgery along L replaces Σ by a curve having smaller genus. Since $3^{g-1} < 3^g$ we deduce that $c(f') < c(f)$.

□

We now describe some applications of Proposition 3.

Corollary 4. *Let M be a closed connected oriented 3-manifold, and suppose that $\mathrm{H}_1(M; \mathbf{Q}) \neq 0$. Then M contains a two-sided incompressible surface.*

Proof. If $\mathrm{H}_1(M; \mathbf{Q}) \neq 0$, then $\mathrm{H}^1(M; \mathbf{Z}) \neq 0$. Choose a nontrivial cohomology class represented by a map $f : M \to S^1$. Applying Proposition 3, we may suppose that the inverse image of a point $x \in S^1$ is a union of incompressible surfaces in M. If $f^{-1}(x) = \emptyset$, then f is nullhomotopic. Otherwise, some component of $f^{-1}\{x\}$ is incompressible. □

Remark 5. If M is irreducible, then Corollary 4 must produce an incompressible surface Σ of positive genus. Let M' be the 3-manifold with boundary obtained by cutting M along Σ. Since not every boundary component of M' is a sphere, we must have $\mathrm{H}_1(M'; \mathbf{Q}) \neq 0$. Applying an analogue of Corollary 4 for 3-manifolds with boundary, we can produce another incompressible surface in M'. By repeatedly cutting M along incompressible surfaces in this way, it is possible to obtain a very good understanding of the 3-manifold M.

Corollary 6. *Let M be a closed oriented connected 3-manifold and suppose that $\pi_1 M \simeq G \star H$ is a free product of nontrivial groups G and H. Then M can be written as a connected sum $M_1 \# M_2$ where $\pi_1 M_1 \simeq G$ and $\pi_1 M_2 \simeq H$.*

Proof. Let BG and BH denote classifying spaces for G and H, and let X be the space $BG \coprod_{\{-1\}} [-1,1] \coprod_{\{1\}} BH$. Then X is a classifying space for $G \star H$, so there exists a map $f : M \to X$ which is the identity on $\pi_1 M$. Applying Proposition 3, we may suppose that f is transverse to $\{0\} \subseteq X$ and that $f^{-1}\{0\}$ is a union of incompressible surfaces Σ of M. If any such surface Σ has positive genus, then the map

$$\pi_1 \Sigma \to \pi_1 M \simeq G \star H$$

is injective (Lemma 2) which is a contradiction. Thus $f^{-1}\{0\}$ is a union of k spheres, for some k. Since G and H are both nontrivial, we must have $k > 0$. If $k = 1$, we obtain the desired connect sum decomposition of M. We will assume that f has been chosen to as to minimize k.

Assume that $k > 1$, and let α be a path in M joining two components of $f^{-1}\{0\}$. Then $f(\alpha)$ is a loop in X. Since $\pi_1 M \simeq \pi_1 X$, we can adjust the path α by composing with a loop in M to guarantee that $f(\alpha)$ is nullhomotopic. Adjusting α by a homotopy, we may assume that $\alpha : [0,1] \to M$ is transverse to $f^{-1}\{0\}$, so that α can be written as a composition

$$\alpha = \alpha_1 \circ \ldots \circ \alpha_m$$

where α_i lies in $f^{-1}(BG \coprod_{\{-1\}} [-1,0])$ for i odd (without loss of generality) and α_i lies in $f^{-1}([0,1] \coprod_{\{1\}} BH)$ for i even. We assume that m has been chosen as small as possible. Since $[f(\alpha)]$ vanishes, the structure of free products of groups guarantees that some $f([\alpha_i])$ must vanish. If α_i connects two different components of $f^{-1}\{0\}$, then we can replace α by α_i and reduce to the case $m = 1$. If α_i connects two points in the same component Σ of $f^{-1}M$, then we can replace α_i by a path α_i' in Σ. Adjusting the composite path

$$\alpha_1 \circ \ldots \circ \alpha_{i-1} \circ \alpha_i' \circ \alpha_{i+1} \circ \ldots \circ \alpha_m$$

by a small homotopy, obtain a new path having fewer intersections with $f^{-1}\{0\}$, again contradicting the minimality of M.

We may therefore assume that α is a path intersecting $f^{-1}\{0\}$ only in its endpoints. Let $K \simeq D^2 \times [0,1]$ be a tubular neighborhood of the image of α so that $K \cap f^{-1}\{0\} = D^2 \times \{0,1\}$. Using the assumption that $f(\alpha)$ is nullhomotopic, we can construct a new map $f' : M \to X$ which agrees with f outside of K (and therefore induces the same isomorphism $\pi_1 M \to \pi_1 X$) and carries $D' \times [0,1]$ into $\{0\}$, where D' is a slightly smaller disk in D^2. Adjusting f' by a small homotopy, we obtain a map such that ${f'}^{-1}\{0\}$ is obtained from $f^{-1}\{0\}$ by a surgery along the 0-sphere $\alpha | \partial([0,1])$: this surgery reduces the number of connected components which contradicts the minimality of k. □