The Sphere Theorem: Part 2 (Lecture 31)

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In this lecture, we will complete the proof of the sphere theorem.

Let us recall the situation. We are given an oriented, connected 3-manifold M and a $\pi_1 M$-invariant proper subgroup $N \subset \pi_2 M$. Our goal is to prove that there exists an embedded 2-sphere $S \subseteq M$ whose homotopy class does not belong to N.

Since $N \neq \pi_2 M$, there exists a map $f : S^2 \to M$ whose homotopy class does not belong to N. We may assume that f is in general position and (as we saw in the last lecture) an immersion. We will suppose that f has been chosen so as to minimize the number $t(f)$ of triple points of f.

In the last lecture, we argued as follows:

(1) If the map f has a simple double curve, then we can modify f so as to obtain a new map f' (whose homotopy class again does not belong to N) which either has fewer triple points ($t(f') < t(f)$) or the same number of triple points and fewer double curves. Since $t(f)$ is minimal, f' must have fewer double curves. Applying this procedure repeatedly, we can reduce to the case where f does not have any double curves.

(2) There exists a 3-manifold with boundary $\widetilde{M}$ (namely, the 3-manifold V_n at the top of the tower that we constructed in the last lecture) and an immersion $q : \widetilde{M} \to M$ with the following properties:

- (i) The map f lifts to a map $\widetilde{f} : S^2 \to \widetilde{M}$.
- (ii) The 3-manifold $\widetilde{M}$ is a regular neighborhood of $\widetilde{f}(S^2)$.
- (iii) The fundamental group $\pi_1 \widetilde{M}$ is finite. As we saw last time, this guarantees that the universal cover of $\widetilde{M}$ is a punctured sphere, so that $\pi_2 \widetilde{M}$ is generated (as a $\pi_1 \widetilde{M}$-module) by its boundary components.
- (iv) The map $\widetilde{f}$ is not an embedding (otherwise we were able to produce a simple double curve of f.

Let $\Sigma(\widetilde{f})$ denote the singular locus of the map $\widetilde{f}$. Condition (iv) guarantees that $\Sigma(\widetilde{f})$ is nonempty. Let X be a small neighborhood of $\Sigma(\widetilde{f})$ in $\widetilde{M}$. Since f is in general position, no point of M has more than 3 preimages under f. It follows that q must be injective on $\Sigma(\widetilde{f})$. Shrinking X, we may assume that q is injective on X. Let T denote the closure of $\widetilde{f}(S^2) - X$.

Let $x \in \Sigma(\widetilde{f})$. Since f is a general position map, $q(x)$ has at most 3 preimages under f. At least two of these are preimages of x under $\widetilde{f}$. There are two possibilities:

(a) The inverse image $f^{-1}(q(x)) = \widetilde{f}^{-1}(x)$. Then $q(x)$ does not intersect $q(T)$, so we can choose a neighborhood V_x of x such that $q(V_x) \cap q(T) = \emptyset$.

(b) The inverse image $f^{-1}(q(x))$ consists of $\widetilde{f}^{-1}(x)$ together with one additional point $s \in S^2$. Let $y = \widetilde{f}(s)$. Since q is injective on X, we must have $y \notin X$, so that $y \in T$. Since q is an immersion, there exists a neighborhood U of y in T on which q is injective. Then $q(x)$ does not intersect $q(T - U)$, so there is a neighborhood of V_x of x such that $q(V_x) \cap q(T) \subseteq q(U)$.

Let X_0 be a regular neigborhood of $\Sigma(\widetilde{f})$ which is contained in the open set $\bigcup V_x$. By construction, if $x \in X_0$ then there is at most one element $y \in \widetilde{f}(S^2)$ such that $x \neq y$ but $q(x) = q(y)$.

Let $X_1 \subset X_0$ be a slightly smaller regular neighborhood of $\Sigma(\widetilde{f})$. The map $\widetilde{f}$ is an embedding outside of X_1; let $S_1, \ldots, S_m$ be the connected components of its image. Then $\widetilde{f}(S^2)$ has a regular neighborhood of the form $X_1 \cup (S_1 \times [-1,1]) \cup \ldots \cup (S_m \times [-1,1])$. Shrinking $\widetilde{M}$ if necessary, we may assume that it coincides with this regular neighborhood.

Let $\widetilde{N}$ denote the inverse image of N in $\pi_2\widetilde{M}$. Since $\widetilde{N}$ does not contain the homotopy class of $\widetilde{f}$, it is a proper subgroup $\pi_1\widetilde{M}$-invariant subgroup of $\pi_2\widetilde{M}$. Using (*iii*), we deduce that $\widetilde{N}$ does not contain the homotopy class of some boundary component S of $\widetilde{M}$. Let $f' : S^2 \to \widetilde{M}$ be the inclusion of this boundary component. Then the image of f' is contained in

$$X_1 \cup (S_1 \times \{-1,1\}) \cup \ldots \cup (S_m \times \{-1,1\}).$$

Claim 1. *For each index $1 \leq i \leq m$, the image of f' cannot intersect both $S_i \times \{-1\}$ and $S_i \times \{1\}$.*

Proof. Otherwise, there exists a simple arc α on $f'(S^2)$ joining points $(x,-1)$ and $(y,1)$, where $x, y \in S_i$. Choose a path joining y to x in S_i, which determines a path β from $(y,1)$ to $(x,-1)$ in $S_i \times [-1,1]$. The composition $\alpha \circ \beta$ is a simple loop which meets $\widetilde{f}(S^2)$ transversely at exactly one point (belonging to S_i). It follows that $\alpha \circ \beta$ represents a nontorsion homology class in $\mathrm{H}_1(\widetilde{M}, \mathbf{Z})$, which contradicts our assumption that $\pi_1\widetilde{M}$ is finite. □

Using Claim 1, we can modify the map f' by an isotopy to obtain an embedding $f'' : S^2 \to \widetilde{M}$ whose image is contained in $X_0 \cup S_1 \cup S_2 \cup \ldots \cup S_m$. By construction, the homotopy class of f'' does not belong to $\widetilde{N}$, so the homotopy class of $q \circ f''$ does not belong to N. We will obtain a contradiction by showing that $t(q \circ f'')$ has fewer triple points than f''.

Let $x \in M$ be a triple point for $q \circ f''$. Since f'' is an embedding, we must have three distinct points $x_1, x_2, x_3 \in f''(S^2)$ such that $q(x_1) = q(x_2) = q(x_3) = x$. Note that $f''(S^2) \subseteq T \cup X_0$. Since q is injective on X_0, at most one element of $\{x_1, x_2, x_3\}$ belongs to X_0. However, if $x_i \in X_0$, then there is at most one element $y \in T$ distinct from x_i such that $q(y) = q(x_i)$. It follows that none of the elements x_1, x_2, and x_3 belong to X_0. Thus $x_1, x_2, x_3 \in T \subseteq \widetilde{f}(S^2)$, so that x is also a triple point of f. This proves that $t(q \circ f'') \leq t(f)$. To prove that the equality is strict, it suffices to show that f has at least one triple point x such that $q^{-1}\{x\}$ is not contained in T. For this, it suffices to show that the map $\widetilde{f}$ has a triple point. Assume otherwise. Then the singular locus $\Sigma(\widetilde{f})$ is a 1-dimensional submanifold of $\widetilde{M}$. This singular locus is nonempty (by (*iv*)), and therefore contains a circle C. This circle is a simple double curve of $\widetilde{f}$, so that $q(C)$ is a simple double curve of f, which contradicts (1).