The Sphere Theorem: Part 1 (Lecture 30)

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In this lecture, we will begin to prove the following result:

Theorem 1 (The Sphere Theorem)**.** *Let M be an oriented connected 3-manifold and let $N \subset \pi_2 M$ be a $\pi_1 M$-invariant proper subgroup. Then there exists an embedded 2-sphere $S \hookrightarrow M$ whose homotopy class does not belong to N. In particular, M is not irreducible.*

Since N is a proper subgroup of $\pi_2 M$, we can choose a map $f : S^2 \to M$ representing a homotopy class which does not belong to N. We will follow a basic strategy similar to that of the loop theorem: we will repeatedly modify the map f until it becomes an embedding. To begin with, we may assume that f is in general position. The proof now proceeds in several stages:

(1) We may reduce to the case where f is an immersion.

To see this, we construct a tower similar to that appearing in our proof of the loop theorem. Namely, we define a sequence of maps $f_n : S^2 \to M_n$ by induction as follows:

- Set $M_0 = M$, and $f_0 = f$.
- Assume that we have constructed $f_n : S^2 \to M_n$. Let U_n be a regular neighborhood of $f_n(S^2)$ in M_n (a compact 3-manifold with boundary) If $\pi_1 f_n(S^2) \simeq \pi_1 U_n$ is finite, then we terminate the process. Otherwise, let M_{n+1} be the universal cover of U_n, and let $f_{n+1} : S^2 \to M_{n+1}$ be any map lifting f_n (such a map exists, since S^2 is simply connected).

As in the proof of the loop theorem, this process must eventually terminate at some stage n, so that $\pi_1 U_n$ is finite. It follows that $\mathrm{H}_1(U_n, \mathbf{Q}) = 0$. By Poincare duality, we have $\mathrm{H}_2(U_n, \partial U_n; \mathbf{Q}) = 0$. Using the long exact sequence

$$\mathrm{H}_2(U_n, \partial U_n; \mathbf{Q}) \to \mathrm{H}_1(\partial U_n; \mathbf{Q}) \to \mathrm{H}_1(U_n; \mathbf{Q})$$

we deduce that $\mathrm{H}_1(\partial U_n; \mathbf{Q}) = 0$, so that the boundary ∂U_n (which is an orientable 2-manifold) is a union of finitely many spheres. Let W be the universal cover of ∂U_n and let $\widehat{W}$ be the the 3-manifold obtained by capping off its boundary spheres. Since $\pi_1 U_n$ is finite, W is compact, so that $\widehat{W} \simeq S^3$ by the Poincare conjecture. It follows that W is obtained from S^3 by removing finitely many open disks, so that $\pi_2 W$ is generated by the classes represented by its boundary spheres. We deduce that $\pi_2 U_n \simeq \pi_2 W$ is generated (as a $\pi_1 U_n$-module) by the classes represented by boundary spheres.

Let N' be the inverse image of N in $\pi_2 U_n$. Since the homotopy class of f_n does not belong to N', we deduce that N' is a proper $\pi_1 U_n$-invariant subgroup of $\pi_2 U_n$. It follows that N' does not contain the class of some embedding $g : S^2 \hookrightarrow \partial U_n \subseteq U_n$. Let f' denote the composite map $S^2 \xrightarrow{g} U_n \to M$. Since g is an embedding, f' is an immersion. Replacing f by f', we can reduce to the case where f is itself an immersion.

Modifying f slightly, we may assume also that f is in general position: it may therefore have both double and triple points (but no branch points). Let $\Sigma(f)$ denote the singular locus of f (the subset of M consisting of those points $x \in M$ for which $f^{-1}(x)$ contains at least two points). Then $\Sigma(f)$ is a 1-dimensional subset of M, which is a submanifold except at a set of isolated points (the triple points of f). The inverse image

$f^{-1}\Sigma(f)$ is a 1-dimensional submanifold of S^2, which can be written as the union of finitely many circles. We will call the images of these circles under f *double curves* of M.

We now proceed by induction on the pair $(t(f), d(f))$, where $t(f)$ denotes the number of triple points of f and $d(f)$ the number of double curves of f. We order these pairs lexicographically: we consider another general position map $f' : S^2 \to M$ to be simpler than f if $t(f') < t(f)$ or if $t(f') = t(f)$ and $d(f') < d(f)$.

(2) Suppose that f has a simple double curve (i.e., there is a component of $f^{-1}\Sigma_f$ which embeds into M). Then we can replace f by a simpler map $f' : S^2 \to M$ which again represents a class in $\pi_2 M$ not belonging to N.

To see this, let $C \subseteq M$ be a simple double curve of M. Then $f^{-1}C$ consists of a few isolated points together with a double cover $\widetilde{C}$ of C. Since M and S^2 are oriented, the argument of the previous lecture shows that $\widetilde{C}$ must be disconnected, consisting of two circles $C_1, C_2 \subseteq S^2$. These circles bound disjoint disks $D_1, D_2 \subseteq S^2$. Let $h : D_1 \to D_2$ be a homeomorphism extending the identification $C_1 \simeq C \simeq C_2$. Let $f'_0 : S^2 \to M$ be the map given by the formula

$$f'_0(x) = \begin{cases} f(hx) & \text{if } x \in D_1 \\ f(h^{-1}x) & \text{if } x \in D_2 \\ f(x) & \text{otherwise,} \end{cases}$$

and let $f'_1 : D_1 \coprod_C D_2 \to M$ be the map given by amalgamating $f|D_1$ and $f|D_2$. Then:

(i) After replacing f'_0 by a small perturbation, we can arrange that f'_0 and f'_1 are general position maps, both simpler than the original map f (in both cases, we have either eliminated all triple points along the double curve C, or left the number of triple points constant while eliminating at least one double curve).

(ii) The homotopy class of f in $\pi_2 M$ belongs to the $\pi_1 M$-invariant subgroup generated by the homotopy classes of f'_0 and f'_1. Consequently, either $[f'_0]$ or $[f'_1]$ will not belong to the subgroup N.

This completes the proof of (2). Unfortunately, this is not yet enough to prove the sphere theorem, because the double curves of the map f will generally intersect themselves.

Lemma 2. *Let $q : \widetilde{M} \to M$ be a local homeomorphism of 3-manifolds, let $f : S^2 \to M$ be a general position map without branch points, and let $\widetilde{f} : S^2 \to \widetilde{M}$ be a lift of f. If $\widetilde{f}$ has a simple double curve C, then $q(C)$ is a simple double curve of f.*

Proof. It suffices to show that $q|C$ is injective. If not, then there exist points $x, y \in C$ such that $q(x) = q(y) = z \in M$. Then $f^{-1}M = \widetilde{f}^{-1}\{x\} \cup \widetilde{f}^{-1}\{y\}$ has at least four points, contradicting our assumption that f is in general position. □

We now try to exploit Lemma 2 using the tower

$$U_n \subseteq M_n \to U_{n-1} \to \cdots \to M_0 = M$$

constructed in (1) (for our given map f).

(3) Suppose that $f_n : S^2 \to M$ is an embedding. Then f has a simple double curve, and we may conclude by applying (2).

To prove (3), we first consider the group $\mathrm{H}_1(U_{n-1}; \mathbf{Z})$. If this group is finite, then the reasoning of step (1) implies that every boundary component of U_{n-1} is a sphere, so that the map $\pi_1 U_{n-1} \to \pi_1 M_{n-1}$ is injective by van Kampen's theorem. Since M_{n-1} is simply connected, we conclude that U_{n-1} is also simply connected, which contradicts our choice of n. Thus $\mathrm{H}_1(U_{n-1}, \mathbf{Z})$ is infinite. Let T denote the torsion subgroup of $\mathrm{H}_1(U_{n-1}, \mathbf{Z})$, and let $\widetilde{T}$ denote the inverse image of T in $\pi_1 U_{n-1}$; note that $\widetilde{T} \neq \pi_1 U_{n-1}$. Since the inclusion $f_{n-1}(S^2) \subseteq U_{n-1}$ is a homotopy equivalence, the inverse image of $f_{n-1}(S^2)$ in M_n is connected. This inverse image consists of all translates of the 2-sphere $S = f_n(S^2)$ by elements of $\pi_1 U_{n-1}$. It follows that the intersection

$$(\bigcup_{g \in \widetilde{T}} g(S)) \cap (\bigcup_{g' \notin \widetilde{T}} g'(S))$$

is nonempty, so that there exists an element $\tau \in \pi_1 U_{n-1} - \widetilde{T}$ such that $\tau(S) \cap S \neq \emptyset$.

By construction, the group element τ has infinite order. Let k be the largest integer such that $\tau^k(S) \cap S \neq \emptyset$, let Z denote the cyclic subgroup of $\pi_1 U_{n-1}$ generated by τ^k, and let $\widetilde{M} = M_n/Z$. We have a local homeomorphism $\widetilde{M} \to M$. Consequently, by Lemma 2, it will suffice to show that the composite map $\widetilde{f} : S^2 \xrightarrow{f_n} M_n \to \widetilde{M}$ has a simple double curve.

Since the map f is in general position, the spheres $\tau^k(S)$ and S must meet transversely in M_n. Let C be a connected component of their intersection. We claim that the image of C is a simple double curve of $\widetilde{f}$. To prove this, it suffices to show that the map $C \to \widetilde{M}$ is injective. Suppose otherwise: then there exist points $x, y \in C$ such that $x = \tau^{nk} y$ for some integer $n \geq 0$. Then $x \in S \cap \tau^{(n+1)k} S \neq \emptyset$, contradicting our choice of k. This completes the proof of (3).

It remains to treat the case where $f_n : S^2 \to M$ fails to be an embedding. We will return to this case in the next lecture.