The Loop Theorem: Special Case (Lecture 29)

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In the last lecture, we reduced the proof of the Loop Theorem to the following special case:

Theorem 1. *Let M be a connected 3-manifold with boundary, let X be a connected 2-manifold with boundary contained in ∂M, and let N be a normal subgroup of $\pi_1 X$. Suppose we are given a map $g' : (D^2, S^1) \rightarrow (M, X)$ with the following properties:*

(1) *The map g' is an immersion without triple points (consequently, the singular locus of g' consists of closed double loops and double arcs which join double points belonging to X).*

(2) *The restriction $g'|S^1$ represents a class in $\pi_1 X$ which does not belong to N.*

Then there exists an embedding $g : (D^2, S^1) \rightarrow (M, X)$ which satisfies (2).

Our goal in this lecture is to prove Theorem 1. Let $X \subseteq D^2$ denote the locus consisting of points where g' is not an embedding. Since g' has only double points, X is a submanifold having codimension 1 in D^2: it therefore consists of finitely many closed curves and finitely many arcs whose endpoints lie in the boundary of the disk. Moreover, for every point $x \in X$ there is a unique point $y \neq x$ such that $f(x) = f(y)$. The construction $x \mapsto y$ is a fixed-point-free involution on X; let Y denote the quotient of X by this involution (it is a smooth submanifold of M).

Let $k = k(g')$ be the number of connected components of Y. We will prove Theorem 1 using induction on k. If $k = 0$, then the map g' is an embedding and we can take $g' = g$. Assume therefore that $k > 0$, so that X is nonempty. Our goal is to replace g' by another map g'' with $k(g'') < k(g')$ (in other words, g'' has fewer double curves than g').

First suppose that Y contains a closed curve $C \simeq S^1$. Let $\widetilde{C}$ denote the inverse image of C in X. There are two possibilities to consider:

(1) The curve $\widetilde{C}$ is connected. Without loss of generality, we may assume that $\widetilde{C}$ is a circle of radius $\frac{1}{2}$ in D^2, and that the involution on X restricts to the antipodal map on $\widetilde{C}$. We can then define a new map $g'' : D^2 \rightarrow M$ by the formula

$$g''(x) = \begin{cases} g(x) & \text{if } |x| \geq \frac{1}{2} \\ g(-x) & \text{if } |x| \leq \frac{1}{2}. \end{cases}$$

Modifying g'' by a small perturbation, we can arrange that g'' is injective along $\widetilde{C}$ (and elsewhere has the same singularities as g'). Since g'' and g' have the same restriction to ∂D^2, and that $k(g'') < k(g')$, we can conclude by the inductive hypothesis.

Remark 2. The analogue of case (1) will prove more troublesome in our proof of the sphere theorem. Consequently, it is worth noting now that (1) is impossible if the manifold M is orientable. More precisely, we have the following:

$(*)$ Let Σ be an oriented surface, M an oriented 3-manifold, and $f : \Sigma \rightarrow M$ a general position map. Suppose that $C \subseteq M$ is a closed double curve of f. Then the inverse image $\widetilde{C} \subseteq \Sigma$ of C is disconnected.

For suppose that $\widetilde{C}$ is connected. Since C is a circle, it has trivial tangent bundle; let v be a nowhere vanishing vector field on $\widetilde{C}$. Since Σ is orientable, the normal bundle N to $\widetilde{C}$ in Σ must also be trivial, so it has a nonzero section w over $\widetilde{C}$. Let σ denote the involution on $\widetilde{C}$. At every point $x \in \widetilde{C}$, the vectors $v_{f(x)}$, $df(w_x)$, and $df(w_{\sigma(x)})$ form an ordered basis for the tangent space $T_{M,f(x)}$, which depends continuously on x. However, if we replace x by $\sigma(x)$, then this ordered basis changes by an odd permutation. It follows that the orientation obstruction $w_1(M) \in \mathrm{H}^1(M; \mathbf{Z}/2\mathbf{Z})$ is nontrivial on $[C] \in \mathrm{H}_1(M; \mathbf{Z}/2\mathbf{Z})$.

Assume now that $\widetilde{C}$ has two connected components C_0 and C_1. These components bound disks D_0 and D_1. We next consider the special case:

(2) One of the disks D_i contains the other. Without loss of generality, we may assume that D_0 contains D_1. Choose a homeomorphism $h : D_0 \to D_1$ extending the homeomorphism $C_0 \simeq C \simeq C_1$. We can then define a new map $g'' : D^2 \to M$ by the formula

$$g''(x) = \begin{cases} g'(x) & \text{if } x \notin D_0 \\ g'(hx) & \text{if } x \in D_0. \end{cases}$$

It is easy to see that $k(g'') < k(g')$, and g'' has the same restriction to the boundary as g'; we may therefore conclude by the inductive hypothesis.

There is one other case to consider:

(3) Suppose that the disks D_0 and D_1 are disjoint. Choose a homeomorphism $h : D_0 \to D_1$ extending the homeomorphism $C_0 \simeq C \simeq C_1$ of their boundaries. We define a new map $g'' : D^2 \to M$ by the formula

$$g''(x) = \begin{cases} g'(hx) & \text{if } x \in D_0 \\ g'(h^{-1}x) & \text{if } x \in D_1 \\ g'(x) & \text{otherwise.} \end{cases}$$

Modifying g'' by a small perturbation, we again have $k(g'') < k(g')$, while g'' agrees with g' on ∂D^2, so we can conclude by induction.

Now suppose that g' has no closed double curves. Since $k(g') > 0$, g' must have a double arc $C \subseteq M$, which is doubly covered by a pair of arcs $C_0, C_1 \subseteq D^2$. We will identify D^2 with the product $[0,1] \times [-1,1]$. Without loss of generality, we may assume that $C_0 = \frac{1}{3} \times [0,1]$ and that $C_1 = \frac{2}{3} \times [-1,1]$. We have an identification $C_0 \simeq C \simeq C_1$, which we may assume without loss of generality is given by $(\frac{1}{3}, t) \mapsto (\frac{2}{3}, \pm t)$.

We define two new maps $g_0'', g_1'' : D^2 \to M$ by the following formulae:

$$g_0''(s,t) = \begin{cases} g'(\frac{2}{3}s, t) & \text{if } s \leq \frac{1}{2} \\ g'(\frac{2}{3}s + \frac{1}{3}, \pm t) & \text{if } s \geq \frac{1}{2} \end{cases}$$

$$g_1''(s,t) = \begin{cases} g'(s,t) & \text{if } s \leq \frac{1}{3} \\ g'(\frac{2}{3} - s, \pm t) & \text{if } \frac{1}{3} \leq s \leq \frac{2}{3} \\ g'(s,t) & \text{if } \frac{2}{3} \leq s \leq 1. \end{cases}$$

Note that $k(g_0'') < k(g')$ (since we have eliminated at least one double arc), and we will have $k(g_1'') < k(g')$ after replacing g_1'' by a small perturbation to ensure that it is in general position. To complete the inductive step, it will suffice to show that either g_0'' or g_1'' represents a class not belonging to the normal subgroup $N \subseteq \pi_1 X$. To prove this, it suffices to observe that $[g'|S^1]$ belongs to the normal subgroup of $\pi_1 X$ generated by $[g_0''|S^1]$ and $[g_1''|S^1]$ (this is clear if we draw some pictures which are not included in the notes).